Choice of filters for the detection of gravitational waves from coalescing binaries II:

Detection in coloured noise

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Abstract

We discuss the problem of detecting gravitational wave signals embedded in coloured noise from coalescing binary systems. The signal is assumed to be Newtonian and matched filtering techniques are employed to filter out the signal. The problem is discussed at first for a general power spectral density of the noise and then specific numerical results are obtained for the standard recycling case. Since the signal parameters are unknown, a bank of filters is needed to carry out the signal detection. The number of filters in a bank, the spacing between filters etc. is obtained for different values of the minimum strength of the signal relative to the threshold. We also present an approximate analytical formula which relates the spacing between filters to the minimum strength. Finally we discuss the problem of detection probabilities given a data train.

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I. Introduction

The detection of gravitational waves has been an outstanding problem in experimental physics for over three decades now. Starting from the pioneering experiments of Weber using a bar detector [1], there has been a lot of effort in building detectors of higher sensitivity. In the recent years, several groups round the globe have been planning to build long baseline laser interferometric gravitational wave detectors, prototypes of which already exist Germany, Great Britain and USA. The aim of these interferometers is to improve the sensitivity to match the low amplitude of signals of galactic and extragalactic origin. In laser interferometric detectors the sensitivity is enhanced by one of several recycling techniques. The first one of such techniques to be invented [2] and implemented [3] goes by the name *standard recycling.* When standard recycling is implemented the detector noise in the relevant range of frequencies $\sim$ 100-2000 Hz, chiefly contributed by photon-shot noise, has been shown to possess a frequency dependent spectral density [4,5,6,7]. However, one can still assume that the noise is stationary and Gaussian. Even when the detector noise is coloured, as in the case of standard recycling, the method of maximum likelihood ratio and the associated matched filtering technique can be employed to detect gravitational waves buried in noisy data.

An important source of gravitational waves is a system of coalescing compact binary [4]. The wave from such a system is characterized by the masses of the component stars of the binary. Also, in the detection problem one does not know when a particular system will coalesce hence one does not know before hand the time of arrival of the signal. In addition, the phase of the signal will be unknown which depends on the history of the binary. These are the three unknown parameters of the signal and in order to apply the technique of matched filtering for its detection it is necessary to construct a bank of filters corresponding to different parameter values in the relevant range. The nature of such a bank of filters when the power spectral density of noise is frequency dependent will be different from that corresponding to white noise. In a recent paper [8] (henceforth referred to as paper-I) we have given an algorithm to make a choice of filters to detect gravitational waves from coalescing binaries assuming that the detector noise has a flat power spectrum i.e. the noise is white. In this paper we work out this algorithm for the case of coloured noise. Besides generalising the results of paper I to coloured noise, we derive an approximate analytical relation for the *distance* between filters in the parameter space in terms of the minimal strength. Such a relation does not exist for white noise. We further discuss the problem of detection probabilities. This problem arises since the statistic we use is the correlation which is a random variable and depends on the noise in the data.

The paper is organized as follows: In section II we briefly review the efforts in understanding the nature of the gravitational wave from coalescing binary systems, also known as the chirp wave form. Among other things, we also discuss in section II the normalized chirp wave form, the Fourier transform of chirp, power spectral density of noise in detectors that work with standard recycling, optimal filters for chirp wave forms buried in coloured noise and correlation of two wave forms that differ in their parameter values. In section III we present an algorithm for the construction of a lattice of filters in coloured noise. These results are then applied to the case of standard recycling. The threshold criterion for detection is then discussed. In section IV we consider the problem of detection probabilities. This needs some modification and reinterpretation of the results obtained in section III.

II. Chirp Wave Form and Optimal Filters

With the aid of point mass approximation Peter and Mathews [9] predicted the flux of gravitational waves from a system of binary stars and showed that the system radiates energy in the form of gravitational waves at an average rate that increases as the inverse fifth power of the distance between the two stars. The gravitational waves carry away the angular momentum of the system resulting in a slow coalescence of the two stars. As the two stars approach each other there will be an increase in orbital frequency and a consequent enhancement in the amplitude and frequency of the waves. Eventually, the two stars coalesce emitting a burst of gravitational waves with a very characteristic wave form. The nature of such a wave form was worked out by Clark and Eardley [10] in the quadrupole approximation using Newtonian orbits for the point mass stars. The analysis of the nature of the waves emitted by the system at the very last stages is, as yet an unsolved, fully general relativistic problem. There have been efforts to understand the nature of the late time signals using the post-Newtonian and the post-Minkowskian formalisms [11, 12, 13] . Several groups have tried to simulate the evolution of the system and estimate the associated emission of gravitational waves by numerical methods using fast computers [14]. However, the final word on the nature of the signal during the very last stages of evolution has not been said.

In spite of a lacking in our understanding of the wave form there is already an extensive discussion of the application of matched filtering technique to the detection of the Newtonian part of the wave form [4,15,16] as well as the first order post-Newtonian term [17]. Such an effort is to be expected in view of the proposals for building large interferometers. These analyses give us idea of the order-of-magnitude estimate of the average signal-to-noise ratio to be expected for such systems.

A. The Chirp Waveform

In the TT (transverse traceless) guage we describe the gravitational waves emitted by a coalescing binary system in terms of two polarisations usually denoted by $h_+(t)$ and $h_\times(t)$. However, the noise-free response of the detector is a linear combination of the two polarisation amplitudes with coefficients depending on the orientation of the detector relative to the direction of the wave. In paper I it has been shown that the effect of an arbitrary orientation of the detector and the inclination of the plane of the orbit of the binary is to *only alter the amplitude and the phase of the waveform.* Therefore for our purpose of constructing the matched filter it is enough to consider, say, the + polarisation which will be the noise-free response of the detector when the detector is optimally oriented and the plane of the orbit coincides with that of the sky. The wave form from such a system of total mass M and reduced mass μ located at a distance r is given by :

$$h_+(t) \equiv h(t) = N_h a(t)^{-\frac{1}{4}} \cos\left(2\pi \int_{t_a}^{t} f(t')\, dt' + \Phi\right), \tag{2.1}$$

where the quantities appearing above are defined as follows:

- t_a and Φ are respectively, the time-of-arrival and the phase of the signal when the instantaneous gravitational wave frequency of the signal reaches some fiducial frequency, say f_a.
- $a(t)$ is the time-dependent normalised distance between the stars (normalised to $a(t_a) = 1$),

$$a(t) = 1 - \frac{t - t_a}{\xi}, \tag{2.2}$$

- $f(t)$ is the instantaneous gravitational wave frequency given by

$$f(t) = f_a \left[1 - \frac{(t - t_a)}{\xi}\right]^{-3/8}. \tag{2.3}$$

- ξ is the time taken for the two stars to theoretically coalesce starting from a time when the instantaneous frequency is f_a,

$$\xi = 3.00 \left[\frac{\mathcal{M}}{M_\odot}\right]^{-5/3} \left[\frac{f_a}{100 \text{ Hz}}\right]^{-8/3} \text{sec.} \tag{2.4}$$

- $\mathcal{M} = (\mu^3 M^2)^{1/5}$ is called the mass parameter; the Newtonian waveform depends only on this parameter instead of the two individual masses of the stars.
- The constant N_h is given by,

$$N_h = 2.57 \times 10^{-23} \left[\frac{\xi}{3 \text{ sec}}\right]^{-1} \left[\frac{f_a}{100 \text{ Hz}}\right]^{-2} \left[\frac{r}{100 \text{ Mpc}}\right]^{-1}. \tag{2.5}$$

The coalescence time serves as a parameter to characterize the wave instead of $\mathcal{M}$. The amplitude and frequency of the wave are given by (2.2) and (2.3). They both diverge in the limit of $t \longrightarrow t_a+\xi$ but much before that the physical assumptions made in deriving them breaks down. They are valid only till a time when the velocities of the two stars are still non-relativistic which is true for orbital frequencies up to ~ 400 Hz or equivalently gravitational frequencies up to ~ 800 Hz.

B. Coloured noise and the normalisation of filters

For the purpose of data analysis it is useful to deal with only the *normalized* response function which has in it all the time-dependence of the wave form (2.1) but whose amplitude is chosen so as to normalize the expectation value of the signal-to-noise ratio. There are several advantages of such a choice. We can express arbitrary chirp signals in terms of normalized wave forms and the coefficient will then directly be related to the signal-to-noise ratio. Such a choice simplifies the data analysis : one does not have to choose different threshold levels for filters which have different parameter values.

It is convenient to go over to the Fourier space to define the normalisation, since the power spectral density of the noise which is normally given as a function of frequency, plays the role of a weight function in the normalisation integral.

In the stationary phase approximation the Fourier transform of equation (2.1) is given by [18],

$$\tilde{h}(f) = \int_{-\infty}^{\infty} h(t) \exp\left(-2\pi i f t\right)\, dt = N_h \sqrt{\xi}\tilde{H}(f), \tag{2.6}$$

where,

$$\tilde{H}(f) = \sqrt{\frac{2}{3f}}\left[\frac{f_a}{f}\right]^{-7/6} \exp\left[i\psi(f)\right],$$

$$\psi(f) = 2\pi f t_a + 2\pi f_a \xi \alpha(f) + \Phi + \frac{\pi}{4}, \qquad \alpha(f) = \frac{1}{5}\left(8 - 3(\frac{f}{f_a})^{-5/3} - \frac{f}{f_a}\right).$$

The quantity $\tilde{H}(f)$ has been chosen to have unit normalisation, i.e.,

$$2\int_{f_a}^{\infty} |\tilde{H}(f)|^2\, df = 1, \tag{2.7}$$

and hence,

$$2\int_{f_a}^{\infty} |\tilde{h}(f)|^2\, df = N_h^2 \xi. \tag{2.8}$$

We make the following assumptions about the noise in the detector (A detailed discussion is to be found in [16]):

(i) The noise $n(t)$ is a random variable at each instant of time t with mean zero i.e. $< n(t) >= 0$; the angular brackets denote ensemble average.

(ii) The noise is stationary. This means that it can be described by the power spectral density $S_h(f)$ defined by the equation,

$$< \tilde{n}(f)\tilde{n}^*(f) >= S_h(f)\delta(f - f'), \tag{2.9}$$

where $\tilde{n}(f) = \int_{-\infty}^{\infty} n(t) \exp(-2\pi i f t)\, dt$.

(iii) Since the noise in the laser interferometric gravitational wave detector steeply rises below a certain freqency f_a mainly due to the seismic noise, we assume $S_h(f) = \infty$ for $f \leq f_a$. This is equivalent to introducing a lower frequency cutoff f_a since the signal will become *visible* in the detector only above f_a which we take to be 100 Hz. The future interferometric detectors intend to bring this down to ~ 10 Hz by using special seismic isolation techniques.

(iv) The noise is Gaussian.

For the present we make the assumptions (i), (ii) and (iii) while assumption (iv) will be used in setting up the threshold criterion. The matched filter produces the maximum signal-to-noise ratio among all linear filters. For stationary noise the matched filter $\tilde{q}(f)$ is given by,

$$\tilde{q}(f) = N_f \frac{\tilde{H}(f)}{S_h(f)}, \tag{2.10}$$

where N_f is a constant independent of f and is to be determined from the chosen normalisation of q.

Consider an output data stream $o(t)$ which consists of the two components: the noise $n(t)$ and the signal $s(t)$,

$$o(t) = n(t) + s(t), \tag{2.11}$$

where we have assumed that the noise is simply additive. The cross-correlation of the output of the detector with a filter $q(t)$ at a time-shift Δt is

$$C(\Delta t) = \int_{-\infty}^{\infty} o(t)q(t+\Delta t)\, dt. \tag{2.12}$$

For the purposes of calculation of the variance of C, the signal $s(t)$ is irrelevant since it does not affect the result. To save labour, we therefore ignore the signal, i.e. set $s(t) = 0$. This immediately implies that $\langle C \rangle = 0$ and the variance of C is just $\langle C^2 \rangle$. In Fourier space,

$$C(\Delta t) = \int_{-\infty}^{\infty} \tilde{n}(f)\tilde{q}^*(f)e^{2\pi i f \Delta t}\, df \tag{2.13}$$

where the $*$ denotes complex conjugation and we have used the relation $\tilde{q}^*(f) = \tilde{q}(-f)$ since the filters $q(t)$ are real valued functions. We may now compute $\langle C^2 \rangle$. Using the fact that C is real the following calculation ensues :

$$\langle C^2 \rangle = \langle CC^* \rangle = \int_{-\infty}^{\infty} df \int_{-\infty}^{\infty} df'\ \tilde{q}^*(f)\ \tilde{q}(f')\ e^{2\pi i (f-f')\Delta t}\ \langle \tilde{n}(f)\tilde{n}^*(f') \rangle. \tag{2.14}$$

Using equation (2.9) in (2.14), the variance of C is,

$$\langle C^2 \rangle = \int_{-\infty}^{\infty} |\tilde{q}(f)|^2 S_h(f)\, df. \tag{2.15}$$

We choose the normalisation of the filters so that the correlation has variance unity. If the noise is Gaussian then it follows [19] that C is a Gaussian random variable since it is obtained by a linear operation on the Gaussian random variables $n(t)$. With the above assumptions, and in the absence of a signal, C is a standard normal variate with mean 0 and standard deviation 1. Since $S_h(f) = \infty$ for $f \le f_a$, the integral in equation (2.15) can be converted from f_a to ∞ multiplied by a factor of two.

The normalisation of q is fixed by demanding,

$$2\int_{f_a}^{\infty} |\tilde{q}(f)|^2 S_h(f) df = 1. \tag{2.16}$$

In a large scale interferometric detector we take the power spectral density of the noise $S_h(f) \sim 10^{-48} Hz^{-1}$ [4]. Setting $S_0 = 10^{-48} Hz^{-1}$ we can express the power spectral density in units of S_0. Accordingly, we define the quantity J by,

$$J^2 = 2S_0 \int_{f_a}^{\infty} \frac{|\tilde{H}(f)|^2}{S_h(f)} df = \frac{4}{3} f_a^{4/3} S_0 \int_{f_a}^{\infty} \frac{df}{f^{7/3} S_h(f)}. \tag{2.17}$$

For white noise with the power spectral density equal to the constant S_0, we have $J = 1$. However, when the noise is not white and the power spectral density is near about S_0, $J \sim o(1)$.

The normalisation constant N_f is obtained from the equations (2.7), (2.10), (2.16) and (2.17). Thus,

$$N_f = \frac{\sqrt{S_0}}{J}. \tag{2.18}$$

The next task is to define the strength of a signal, say S, on the lines of paper I. A signal has strength S if the maximum value of the cross-correlation of the signal with the *normalised* matched filter has the value S. Thus using equations (2.6), (2.10), (2.17) and (2.18) we obtain,

$$\begin{aligned} S &= \max_{\Delta t} C(\Delta t) \\ &= N_h \sqrt{\frac{\xi}{S_0}} J. \end{aligned} \tag{2.19}$$

Substituting for N_h from equation (2.5) we obtain the strength of the signal in terms of r, ξ, and f_a,

$$S(r, \xi, f_a) = 44.5 \left[\frac{\xi}{3sec}\right]^{-\frac{1}{2}} \left[\frac{f_a}{100Hz}\right]^{-2} \left[\frac{r}{100Mpc}\right]^{-1} J. \tag{2.20}$$

We apply the above results to the specific case of standard recycling:

Standard recycling: The power spectral density of noise in detectors that employ Fabry-Perot cavities in their arms and use standard recycling to enhance the intensity of light in the cavities has been shown to be of the following form [4]:

$$S_h(f) = S_0 \left[1 + \frac{f^2}{f_k^2}\right], \tag{2.21}$$

where S_0 and f_k are constants depending on the parameters of the detector. The parameter f_k is the so called *knee frequency* which the experimenters can set by an appropriate choice of mirror reflectivities. In order to obtain maximum signal-to-noise ratio for coalescing binary signals there is a preferred value of f_k given by

$$f_k = 1.44\ f_a. \tag{2.22}$$

We observe that the power spectral density of noise in standard recycling is roughly a constant for frequencies below the knee frequency but rises quadratically above it. It therefore becomes increasingly harder to increase the signal-to-noise ratio by increasing the upper frequency cutoff. Such an analysis is made in reference [20] and here we merely mention the result that for the case under consideration, *viz* standard recycling in Fabry-Perot cavities, it is sufficient to sample coalescing binary wave forms up to about 400 Hz at which more than 98 % of the signal power can be extracted.

If we indentify the S_0 appearing in equation (2.17) with the one in equation (2.21) and make a change of variables to $x = f/f_a$ and $\gamma = f_k/f_a$, then the quantity J is given by,

$$J^2(\gamma) = \frac{4}{3}\gamma^2 \int_1^\infty \frac{dx}{x^{7/3}(x^2+\gamma^2)}. \tag{2.23}$$

For the optimum value of f_k given in equation (2.22), we have $\gamma = 1.44$. For this value of γ a numerical integration of equation (2.23) gives $J \sim 0.62$.

C. The correlation function for coloured noise

When a signal is present in the output of a detector it will have a definite set of values of the parameters. For such a signal the signal-to-noise ratio will be the largest when it is filtered using a template that matches all its parameters. But since the signal parameters are unknown we need a bank of filters closely spaced in the parameter space to guard against the possibility of a signal being missed. In a discrete lattice of filters it is very unlikely that there will be a filter that exactly matches all the parameters of the signal. In that case one gets for the signal the maximum signal-to-noise ratio with a filter that has the least mismatch with the actual parameters of the signal and this will in general be less than what one gets using a perfectly matched filter; the reduction in signal-to-noise on the average being less if one uses a larger number of filters. The statistic used is the correlation which is a random variable. The decision about the presence or absence of a signal can be made by comparing the maximum signal-to-noise ratio obtained using the bank of filters with a predetermined level called the *threshold* set by the maximum affordable *false alarm probability*. More details about this will be provided in the next section.

Consider the chirp $h(t,\xi,\Phi)$ and a filter $q(t,\xi+\Delta\xi,\Phi+\Delta\Phi)$ whose coalescence time and phase differ from that of the signal by $\Delta\xi$ and $\Delta\Phi$, respectively. The correlation function of these two wave forms is given by

$$C(\Delta t,\xi,\Delta\xi,\Phi,\Delta\Phi)=\int_{-\infty}^{\infty} h(t,\xi,\Phi)\; q(t+\Delta t,\xi+\Delta\xi,\Phi+\Delta\Phi)\; dt. \tag{2.24}$$

Going over to the Fourier domain via equation (2.6) and using the stationary phase approximation we have,

$$C(\Delta t,\Delta\xi,\Delta\Phi)=N_c\int_{f_a}^{\infty}\frac{\cos\left[2\pi f\Delta t+2\pi\alpha(f)f_a\Delta\xi+\Delta\Phi\right]}{f^{7/3}S_h(f)}df. \tag{2.25}$$

where $N_c=\frac{4}{3}f_a^{4/3}\xi^{1/2}N_hN_f$. As in the white noise case treated in paper I here also the correlation function depends only on the *differences* $\Delta t,\Delta\xi$ and $\Delta\Phi$ in the parameters of the two waveforms.

We note the following properties of C:

(i) The maximum value of C is $C(0,0,0)=N_h\sqrt{\frac{\xi}{S_0}}J$.

(ii) Reflection symmmetry about the maximum:

$$C(\Delta t,\Delta\xi,\Delta\Phi)=C(-\Delta t,-\Delta\xi,-\Delta\Phi) \tag{2.26}$$

as is obvious from (2.25). For our purposes it is important to note that the maximum of the correlation function for a fixed $\Delta\xi$ depends only on its modulus.

(iii) Finally, let us note that

$$C(\Delta t, \Delta\xi, \Delta\Phi) = C(\Delta t, \Delta\xi, 0)\ \cos\Delta\Phi + C(\Delta t, \Delta\xi, \pi/2)\ \sin\Delta\Phi, \tag{2.27}$$

which states that the correlation of a signal with a filter of arbitrary phase can be expressed as a linear combination of its correlation with two filters: one with phase equal to 0 and another with phase equal to $\pi/2$.

There is a word of caution about the statements made about the properties of the correlation function. The expression for the Fourier transform of the chirp is derived in the stationary phase approximation and it is this expression which has enabled us to show the simple dependence of the correlation function on its parameters. Therefore these properties are also approximate and as discussed in detail in paper-I they hold good only for values of coalescence time more than about 0.3 sec. It is important to remember this while generating a lattice of filters for low values of the coalescence time which correspond to high values of the mass parameter.

III. Choice of Filters

In this section we first discuss the nature of the probability density function of the correlation noise when the time series noise is coloured. Using a certain *false alarm probability* we obtain a *threshold* for filtered signals. We then introduce the idea of *minimal strength* . The minimal strength will then be related to the spacing of filters in the parameter space. Towards the end of this section we give an approximate analytical relation for the distance between filters as a function of the minimal strength.The details of the calculation are given in the appendix. This relation will be shown to hold good for values of minimal strength close to the threshold.

A. Threshold and minimal strength

For a stationary, Gaussian time-series noise we find that the correlation noise is also Gaussian. This enables us to find the threshold for filtered data.

As per the normalisation assumed, recall that the cross-correlation also is a Gaussian random variable with mean 0 and variance 1. Its probability density function is therefore given by,

$$p(C) = \frac{1}{\sqrt{2\pi}} e^{-C^2/2}. \tag{3.1}$$

The threshold η is set by the requirement that the number of times the statistic C exceeds the threshold in a given length of data, purely due to noise, is much smaller than the expected number of events in the data. Following paper-I, we consider a data segment for a one year period, and allow for just one false alarm in this period; *i.e.* the expected number of times C can cross η just due to noise in a year's time is one. Now the number of 'trials' in a years time, assuming say 100 to 1000 filters/second and a sampling rate of few kHz, is $\sim 10^{12}$ or 10^{13}. The false alarm probability should therefore be 10^{-12} or 10^{-13}. Using this fact, the threshold η is obtained from the solution to the equation,

$$\int_{\eta}^{\infty} p(C)\; dC \sim 10^{-13}, \tag{3.2}$$

which gives $\eta \sim 7$. If now C exceeds this threshold value of η we say that the signal is detected, otherwise it is not.

We now consider the problem of constructing a bank of filters which will pick out any signal whose strength is larger than a certain minimal strength which we denote by S_{min}. Since it is not possible to detect signals that have their peak correlation just above the threshold with filters whose parameters exactly match with the signal (this would require an infinite bank; the case $S_{min} = \eta$) we do the next best thing by choosing,

$$S_{min} = \kappa\eta, \tag{3.3}$$

where $\kappa > 1$, i.e. the minimum strength of the signal is κ times the threshold. This means that with a right filter (a filter whose parameters exactly match those of the signal) the correlation of a signal whose strength is larger than the minimal strength will be atleast $\kappa\eta$. The aim is to construct a bank of filters so that every signal of strength greater than or equal to the minimal strength S_{min} will be detected.

B. Bank of filters

We follow the same procedure as in paper I. The procedure is formally the same but the results are quantitatively different due the presence of a nonconstant $S_h(f)$. For the sake of completeness we give a brief description of the procedure.

As in the white-noise case of paper I, here too there exists a two-dimensional basis for the phase parameter: A chirp filter of arbitrary phase can be expressed as a linear combination of two filters one with phase equal to 0 and another with phase equal to $\pi/2$. Consequently the correlation of a given data set with a filter of arbitrary phase can be expanded in terms of the correlation of the same data set with the two basis filters. Moreover, the phase of the filter which maximizes the correlation can be found analytically once the correlations are calculated with the basis set: the maximum correlation (with respect to the phase) is just the square root of the sum of squares of the correlations obtained from the orthogonal basis set.

In constructing a lattice of filters for the mass parameter or equivalently for the coalescence time we make use of the symmetry of the correlation function namely that it depends only on the difference and not on the absolute values of the coalescence times of the signal and filter. We fix the phase to be zero i.e. $\Phi = 0$. The set of filters for $\Phi = \pi/2$ are going to consist of the same set of mass parameters as the set of filters for $\Phi = 0$. We start with the mass-parameter $\mathcal{M}_1$ at the minimum of the range. This corresponds to a coalescence time ξ_1. A typical value of $\mathcal{M}_1$ could be $0.5 M_\odot$ which corresponds to $\xi_1 \sim 9.54$ sec. This gives us the first filter in the set, namely $q(t, \xi_1, 0)$.

The next filter $q(t, \xi_2, 0)$ is obtained as follows: We consider the set of signals h_{min} which have minimal strength. Then by definition of minimal strength we have the relation,

$$\int_{t_a}^{t_a+\xi} h_{min}(t, \xi, \Phi) q(t, \xi, \Phi) dt = \kappa\eta. \tag{3.4}$$

Consider the signal $h_{min}(t, \xi_1, 0)$. If we now correlate $h_{min}(t, \xi_1, 0)$ with the first filter the correlation will be $\kappa\eta$. If we reduce the coalescence time in the signal slightly say to

$\xi = \xi_1 - \Delta\xi$ where $\Delta\xi > 0$ then the correlation will drop below this value. We now solve the following equation for $\Delta\xi$,

$$\max_{\Delta t, \Delta\Phi} \int_{t_a}^{t_a+\xi} h_{min}(t + \Delta t, \xi_1 - \Delta\xi, \Delta\Phi) q(t, \xi_1, 0) dt = \eta, \tag{3.5a}$$

where $\Delta\xi$ is taken to be positive. The maximisation with respect to Δt and $\Delta\Phi$ has been carried out to obtain the largest possible value of $\Delta\xi$ in order that the coarsest possible lattice is obtained with a minimum number of filters. The maximisation is necessary as there is a covariance between the parameters. Thus a minimum strength signal with $\xi = \xi_1 - \Delta\xi$, where $\Delta\xi$ is the solution of equation (3.5a) is just about picked up by the first filter. If $\xi < \xi_1 - \Delta\xi$ for a minimum strength signal then this signal is not picked up by the first filter. Therefore, there ought to be some other filter in the bank which should pick up this signal. We choose ξ_2 such that the maximum value of the correlation with respect to Δt and $\Delta\Phi$ of the filter $q(t, \xi_2, 0)$ with a minimum strength signal having $\xi = \xi_1 - \Delta\xi$ is η, i.e.,

$$\max_{\Delta t, \Delta\Phi} \int_{t_a}^{t_a+\xi} h_{min}(t + \Delta t, \xi_1 - \Delta\xi, \Delta\Phi) q(t, \xi_2, 0) dt = \eta, \tag{3.5b}$$

where $\Delta\xi$ is the solution of equation (3.5a). This equation can be solved using the symmetry of the correlation function in $\Delta\xi$. Thus, we have the relation $\xi_2 + \Delta\xi = \xi_1 - \Delta\xi$ i.e.

$$\xi_2 = \xi_1 - 2\Delta\xi. \tag{3.6}$$

The process is then repeated until the upper end of the range of the mass parameter is reached. Thus, the $k+1$th filter has the coalescence time :

$$\xi_{k+1} = \xi_1 - 2k\Delta\xi. \tag{3.7}$$

We observe that the filters in the ξ dimension are spaced with a constant spacing of $2\Delta\xi$. The numerical computations actually solve the equations (3.5a,b) without assuming this symmetry and bear this fact out except at the higher end of the mass parameter range, where the stationary phase approximation breaks down.

How many filters do we need? If we take the upper limit to the mass parameter as $\sim 10M_\odot$ then the corresponding $\xi \ll \xi_1$ and hence the number of filters with $\Phi = 0$ are just $\xi_1/2\Delta\xi$. But the full set of filters is obtained by including the filters with phase $\pi/2$ but with the same ξ_k. Therefore, the total number of filters n_f is,

$$n_f = \frac{\xi_1}{2\Delta\xi} \times 2 = \frac{\xi_1}{\Delta\xi} \tag{3.8}$$

From equation (2.20) we also obtain the furthest distance r_{max} upto which a coalescing binary could be detected with a bank of filters corresponding to the minimal strength $\kappa\eta$. Thus,

$$r_{max} = 630\kappa^{-1} J \left[\frac{\eta}{7}\right]^{-1} \left[\frac{\xi}{3sec}\right]^{-\frac{1}{2}} \left[\frac{f_a}{100Hz}\right]^{-2} \left[\frac{S_0}{10^{-48}Hz^{-1}}\right]^{-\frac{1}{2}} \text{Mpc} \tag{3.8}$$

In the limit of infinite number of filters, $\kappa = 1$, r_{max} tends to a maximal limit r_0 which is determined only by the threshold.

C. Bank of filters for the standard recycling case

In this section we present numerical results corresponding to various topics discussed in the previous section. Specifically we quote the number of filters and distance between them for different values of the minimal strength when the detector noise is that corresponding to standard recycling with the knee frequency $f_k = 144$ Hz. We obtain an approximate analytical relation for $\Delta\xi$, which is half the spacing between the consecutive filters in the bank. The details of the calculation are given in the appendix.

Using the algorithm developed earlier it is straightforward to find the constant distance between filters given a certain value of κ. It is enough to do one numerical computation, viz that in equation (3.5a), say, for a filter whose coalescence time is ξ_1, which is the starting point for interesting coalescence times. Having determined the distance between filters numerically one can construct filters of the lattice with the aid of equation (3.7). This procedure is accurate enough for most filters in the lattice. However, since the properties of the correlation function for low values of coalescence time (~ 0.3 sec, the exact value depending on the value of κ) may not strictly hold good, it is necessary to adopt the iterative method, discussed in detail in paper-I, for such values. Following such a procedure the bank of filters has been obtained for $\kappa^{-1} = 0.7, 0.8$ and 0.9. The filters labelled by their mass parameter have been presented in tables I (a), (b) and (c) respectively. We observe that the quantity $\Delta\xi$ is more or less constant for a given value of κ and grows larger for lower values of κ^{-1}. For $\kappa^{-1} = 0.7, 0.8$ and 0.9, $\Delta\xi \sim 0.031$, 0.019 and 0.010 secs. respectively. The number of filters n_f is related to $\Delta\xi$ as given in equation (3.8) which shows that it also depends on ξ_1 or $\mathcal{M}_1$. Table II gives for various values of κ^{-1}, $\Delta\xi$ for white noise and standard recycling noise and the number of filters n_f in the standard recycling case for $\mathcal{M}_1 = 0.25$ and $0.5M_\odot$. For comparison we have also given the values corresponding to the white noise case. Note that the number of filters required for standard recycling with $f_k = 144$ Hz is roughly half the number corresponding to the white noise case. For example, for $\kappa^{-1} = 0.8$ and $\mathcal{M}_1 = 0.5M_\odot$, the number of filters is 513 as compared to 1150 in the case of white noise.

This can be understood in the following way: The noise rises very quickly once f increases beyond f_k, so that when $f \gtrsim 400$ Hz, the signal is basically drowned in the noise. This is reflected in the fact that the matched filter $\tilde{q}(f)$ decreases rapidly in amplitude as f increases beyond f_k so that the signal is effectively cut-off beyond ~ 400 Hz, so that, most of the contribution to the signal-to-noise ratio comes from $f \lesssim 400$ Hz. Now in the chirp waveform the rate of increase of frequency $\dot{f}$, where the dot represents derivative with respect to time, increases with time and mass parameter . This means that if a signal is cut off prematurely it is harder to determine the mass parameter since the final acceleration in the frequency which depends on the mass parameter is not seen. The effect is to make the peak of the correlation function blunt. This means that the correlation function will drop gradually with the growing mismatch of parameters, in particular, the mass parameter . This is depicted in Figure 1, where the correlation functions are plotted for white noise and standard recycling noise. We observe that in the standard recycling case the correlation function drops slower than in the white noise case, leading to a wider spacing between filters.

This has the following important implication: the computing time is considerably less than in the white noise case for a similar value of κ. We 'save' in two ways:

(i) Less number of filters are required to span the same range of mass parameter , by a factor of two or so, since the spacing between filters is widened by this factor.

(ii) The template can be cut off at about 400 Hz so that a lower sampling rate ~ 1 kHz is adequate compared to the 2 or 2.5 kHz rate we had used for white noise.

This should not be thought of as a saving in some real way since even in the case of white noise we could have chopped off the template at 400 Hz and got similar results. Here we have no alternative but to consider a frequency limited template. However, the real advantage with standard recycling is that the effective laser power is enhanced leading to a reduction in the overall noise i.e. in S_0. Another implication is that for a given computability a lower value of κ can be chosen than in the white noise case.

We also derive an approximate analytical formula for $\Delta\xi$ as a function of κ. This is achieved by Taylor expanding the correlation function about the peak and taking a 'slice' at $C = \kappa^{-1} C(0,0,0)$. The intersection is an ellipsoid in the parameter space from which the spacing between filters is obtained. The details of the computation are given in the appendix. We have the following results: Denoting this value by $\Delta\xi_{an}$ we have,

$$\Delta\xi_{an} = \frac{1}{2\pi f_a}\left[2(1-\kappa^{-1})\frac{\Gamma_{11}\Gamma_{33}-\Gamma_{13}^2}{\Gamma_{11}\Gamma_{22}\Gamma_{33}-(\Gamma_{11}\Gamma_{23}^2+\Gamma_{22}\Gamma_{13}^2+\Gamma_{33}\Gamma_{12}^2)+2\Gamma_{12}\Gamma_{23}\Gamma_{13}}\right]^{1/2}, \tag{3.9}$$

where the Γ-matrix is,

$$\Gamma = A\begin{pmatrix} \int_1^\infty \frac{x^2}{S(x)}dx & \int_1^\infty \frac{x a(x)}{S(x)}dx & \int_1^\infty \frac{x}{S(x)}dx \\ * & \int_1^\infty \frac{a^2(x)}{S(x)}dx & \int_1^\infty \frac{a(x)}{S(x)}dx \\ * & * & \int_1^\infty \frac{dx}{S(x)} \end{pmatrix}, \tag{3.10}$$

and,

$$S(x) = x^{7/3}(x^2+\gamma^2), \quad a(x) = \frac{8}{5} - \frac{3}{5}x^{-5/3} - x, \quad \text{and } A = \left[\int_1^\infty \frac{dx}{S(x)}\right]^{-1}. \tag{3.11}$$

For $\gamma = 1.44$ we have,

$$\Gamma = \begin{pmatrix} 3.07 & -1.02 & 1.58 \\ * & 0.54 & -0.34 \\ * & * & 1.00 \end{pmatrix} \tag{3.12}$$

The 'stars' in the matrices denote matrix elements which can be obtained by symmetry, since these matrices are symmetric.

The results are quoted in Table III. For values of κ^{-1} (column one) close to 1 there is indeed a good agreement between the analytical (Column two) and numerical (column three) results, when the quadratic approximation is expected to be adequate.

We comment that the matrix appearing in equation (3.10) is the so called Fischer information matrix and its inverse,

$$\gamma_{ij} = [\Gamma^{-1}]_{ij}, \tag{3.13}$$

is the expected covariance of errors of the various parameters of the signal, namely, t_a, ξ, Φ or more accurately, the scaled dimensionless parameters p_1, p_2, p_3 defined in the appendix. The square root of the diagonal elements of γ, namely, $(\gamma_{ii})^{\frac{1}{2}}$, represent the expected errors in the parameters p_i. A detailed discussion of this is being published elsewhere [18].

IV. Detection Probabilities

The foregoing analysis needs some modification if we are to apply it to a given output data train. The correlation output $C(\Delta t)$ with a given filter, ξ and Φ fixed, is a random variable for each value of Δt as seen from equation (2.12). However, in an actual data analysis problem we have to consider the fact that we have only one random output $C(\Delta t)$ to contend with and the decision whether the signal is present or absent has to be made based on this output. The analysis in the previous sections is valid only when we consider the expectation value of $C(\Delta t)$. Although in a given situation $< C(\Delta t) >$ may exceed the threshold for some filter, there is no guarantee that $C(\Delta t)$ will also exceed the threshold, as there is noise present in the output. So it may happen that at the instant when the $< C(\Delta t) >$ exceeds the threshold, a sufficiently large negative component of the noise 'pulls' down the $C(\Delta t)$ below the threshold level leaving the signal undetected, although it is actually present. The reverse may also take place: a correlation whose expectation value is below the threshold can get 'pushed up' above the threshold due to a positive noise component at that instant of time. There is also the added problem, that, considering just the maximum of the correlation may not be sufficient, since other values of the statistic which are not necessarily the maximum but lower, also have a chance of being 'pushed up' over the threshold by the noise and get detected. So that the problem involves considering the entire function $< C(\Delta t) >$ and not just the value at its maximum.

In this section we give a lower bound on the minimal strength of the signal that it be detected with a certain probability, called the detection probability , typically 95 %. We find that we need to modify the previous results to some extent. We also justify below that considering just the maximum of the correlation function is sufficient to decide a detection. This is done in subsection A. Subsection B deals with detection probabilities and the modified thresholds.

A. Covariance of the correlation at different instants of time

Consider a signal $h(t, \xi, \Phi)$ such that ξ is closest to the filter with $\xi = \xi_i$, i.e., $\xi = \xi - \Delta\xi_0$ where $0 < |\Delta\xi_0| < \Delta\xi$. Let us consider the case $\Delta\xi_0 > 0$. The argument for $\Delta\xi_0 < 0$ is analogous. The correlation with the filter with $\xi = \xi_i$ is given by,

$$C(\Delta t, \Delta\xi_0, \Delta\Phi) = h(t, \xi_i - \Delta\xi_0, \Phi) \circ q(t + \Delta t, \xi_i, \Phi + \Delta\Phi), \tag{4.1}$$

where the $\circ$ denotes the operation of correlation. Note that to obtain the filter for a general Φ a suitable linear combination of the filters for $\Phi = 0$ and $\Phi = \pi/2$ has to be taken. Let us denote the values of Δt and $\Delta\Phi$ at which C attains a maximum by Δt_m and $\Delta\Phi_m$ respectively, and consider the function $C(\Delta t, \Delta\xi_0, \Delta\Phi_m)$ for different values of Δt. In equation (2.25) we fix $\Delta\xi = \Delta\xi_0$ and $\Delta\Phi = \Delta\Phi_m$ and allow Δt to vary. Our aim is to find out the timescale in which the correlation drops to zero. If $S_h(f)$ is basically flat near the seismic cut-off frequency (as in the case of white noise or standard recycling), the steep wall near f_a, due to seismic noise and the rapid fall off in the power of the signal $\sim f^{-7/3}$ produces a Dirac-delta like function with a peak at $f \gtrsim f_a$ in the integral for C in equation (2.25). Therefore the correlation function C is approximately proportional to $\sim \cos(2\pi f_a \delta t)$ where $\Delta t = \Delta t_m + \delta t$. This leads to the correlation function dropping to zero for $\delta t \sim \pm\frac{1}{4f_a}$. For $f_a = 100 \cdot \text{Hz}$, $\delta t \sim \pm 2.5$ milli secs. This result agrees with the numerically obtained contour plot for the correlation function by Schutz [21].

We argue that this is the same timescale over which the correlation at different instants is correlated. It is not too hard to show that the covariance of the correlation between the instants Δt and $\Delta t + \delta t$ is given by,

$$
\begin{aligned}
&< C(\Delta t_m, \Delta\xi_0, \Delta\Phi_m) C(\Delta t_m + \delta t, \Delta\xi_0, \Delta\Phi_m) > \\
&- < C(\Delta t_m, \Delta\xi_0, \Delta\Phi_m) >< C(\Delta t_m + \delta t, \Delta\xi_0, \Delta\Phi_m) > = B \int_{f_a}^{\infty} \frac{\cos(2\pi f \delta t)}{f^{7/3} S_h(f)} df, \quad (4.2)
\end{aligned}
$$

where B is a constant. This equation again shows by the foregoing argument that the covariance $\propto \cos(2\pi f_a \delta t)$ and goes to zero over the timescale $\frac{1}{4f_a}$. The quantity δt is also called the de-correlation time [16] and basically gives a timescale over which the correlations computed at instants differing by a time interval greater than δt are uncorrelated.

The above considerations show that over the timescale when $< C(\Delta t) >$ is appreciable it is correlated to $C(\Delta t_m)$ and hence it is not unjustified if we base our conclusions on the statistic $C(\Delta t_m)$ to decide the presence or absence of a signal.

B. Detection Probabilities

Since the correlation is a random function there is a definite probability that when a signal is actually present a decision in favour of detection will be made. Let $p_1(C)$ be the probability density function of $C(\Delta t_m)$ when a signal is present. The detection probability Q_d is given by,

$$Q_d = \int_{\eta}^{\infty} p_1(C) dC \quad (4.3)$$

As assumed earlier the noise is Gaussian, then C is also a Gaussian variate with mean

$\tilde{\eta} = C(\Delta t_m, \xi_i - \Delta\xi_0, \Delta\Phi_m)$ and with the variance being set equal to unity. We then have,

$$p_1(C) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{(C-\tilde{\eta})^2}{2}) \tag{4.4}$$

For example if $\tilde{\eta}$ is just the threshold level η then by equation (4.3), $Q_d = 0.5$. This implies that there is an equal chance that the fluctuations due to noise may either bring C down or push it up above the threshold. It is desirable to have the detection probability high say $Q_d \sim 0.95$. In which case we must have $\tilde{\eta}$ fairly higher than η. Setting,

$$\Delta\eta = \tilde{\eta} - \eta \tag{4.5}$$

we have to solve the following equation for $\Delta\eta$ given a value for Q_d:

$$Q_d = \frac{1}{\sqrt{2\pi}} \int_{\eta}^{\infty} \exp(-\frac{(C-\tilde{\eta})^2}{2}) dC = \frac{1}{\sqrt{2\pi}} \int_{-\Delta\eta}^{\infty} \exp -(\frac{y^2}{2})\ dy \tag{4.6}$$

For example, if $Q_d \sim 0.975$, then $\Delta\eta \sim 1.96$ which means that $\tilde{\eta} \sim 9$. Therefore, the effect of the noise for a given data output is to increase the threshold level higher than the threshold level for $< C(\Delta t) >$ as evaluated in section III.

What happens to the bank of filters? We have seen that the above considerations only amount to shifting the threshold level from η to $\tilde{\eta}$ because if $< C >$ is atleast $\tilde{\eta}$ then C will exceed η with probability Q_d. This means that the minimum strength signals must be stronger i.e. $S_{min} = \kappa\tilde{\eta}$. Signals of this minimal strength will be detected with probability Q_d if a bank of filters corresponding to κ obtained in the previous section is used. We note that the spacing between filters and the number of filters is unaltered. Equation (3.8) can be reinterpreted suitably by substituting $\tilde{\eta}$ instead of η in the equation.

V. Conclusion

In this paper we generalise our results of paper I to coloured noise and specifically deal with the case for standard recycling. The filtering problem is first treated in general for coloured noise. For standard recycling with $f_k = 1.44 f_a, f_a = 100$ Hz banks of filters are obtained for various values of minimum strengths relative to the threshold. It is found that the spacing between filters is increased by a factor of two or more in general as compared to white noise. This is basically due to the fact that effectively the higher frequencies in the signal are cut off due to the high noise at these frequencies. Since the template is basically cut off at about 400 Hz , a lower sampling rate is adequate. Both these factors will contribute to saving in the computation time for a given κ. However, this is at the cost of the part of the signal being lost at high frequencies. Further, an approximate analytic formula is obtained for the spacing between filters in terms of κ. In the appendix an elegant geometrical construction is given for deriving this formula. This formula exists for the standard recycling case but not for the idealistic white noise where the correlation function is not sufficiently smooth at the maximum. Finally we discuss the case of a data output for which the problem of detection has to be reformulated. The statistic is the maximum of the correlation function and a threshold is set by demanding a high detection probability. However, the same bank of filters can be used here, although the results have to be modified to some extent as described in the text.

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Appendix

Analytical relation between minimal strength and distance between filters

In section III C a numerical method of determining the lattice spacing of filters was described which is inevitable for large values of $\Delta\xi$. However, for values of κ close to unity, *i.e.* when the distance between the filters is small, an analytical relation can be found. This analytical relation has been compared with numerical results and good agreement is obtained for $\kappa \sim 1$. The relation is basically derived by Taylor expanding the cross-correlation function about its maximum up to the second order and equating the result to the threshold level.

Let us write the correlation of two chirp wave forms that differ slightly in their parameter values as follows (cf. Eq (2.25)):

$$C(\Delta t, \Delta\xi, \Delta\Phi) = \tilde{A} \int_{f_a}^{\infty} \frac{\cos\left[2\pi f \Delta t + 2\pi\alpha(f) f_a \Delta\xi + \Delta\Phi\right]}{f^{7/3} S_h(f)}\, df, \tag{1}$$

where $\tilde{A}$ is an overall normalisation constant. It is convenient to use dimensionless quantities and to this end we set,

$$x = f/f_a, \qquad \gamma = f_k/f_a, \qquad a(x) = \frac{1}{5}(8 - 3x^{-5/3} - 5x), \tag{2}$$

$$p_1 = 2\pi f_a t, \qquad p_2 = 2\pi f_a \xi, \qquad p_3 = \Phi. \tag{3}$$

In terms of these new variables the correlation function takes the form

$$C(\Delta p_i) = A \int_{1}^{\infty} \frac{\cos\left[x\Delta p_1 + a(x)\Delta p_2 + \Delta p_3\right]}{S(x)}\, dx, \tag{4}$$

where $S(x)$ and A have already been defined in equation (3.11). The normalisation can be chosen arbitrarily as long as one chooses the threshold level accordingly. We choose the maximum of the C to be 1 and this occurs at $\Delta p_i = 0$. Thus,

$$C(\Delta p_i = 0) = 1. \tag{5}$$

Since the maximum of C is just κ times the threshold level, the threshold level is κ^{-1}. The problem therefore reduces to finding the maximum values of Δp_2 with the condition,

$$C(\Delta p_i) = \kappa^{-1}, \tag{6}$$

the other parameters Δp_1 and Δp_3 being otherwise free. To this end we Taylor expand C upto the second order about its maximum,

$$C(\Delta p_i) = C(\Delta p_i = 0) - \frac{1}{2}\Gamma_{ij}\Delta p_i \Delta p_j, \tag{7}$$

where

$$\Gamma_{ij} = -\left(\frac{\partial^2 C}{\partial \Delta p_i \partial \Delta p_j}\right)_{\Delta p_i = 0}. \tag{8}$$

In the above equation summation convention has been used: *i.e.* repeated indices are summed over. We observe that Γ_{ij} is a positive definite and symmetric matrix, since C has a maximum at the origin.

The above equations have been obtained in the noise-free case for which the quantities involved are ordinary functions. However, in the realistic case when noise exists the output of the detector is a random variable. Consequently, quantities like the cross-correlation C are also random variables. In particular, Γ_{ij} is a random tensor. But now the corresponding quantities will be the expectation values of these random variables which are expected to match with those in the foregoing noise-free treatment. We observe that the expectation value of Γ_{ij} is then just the *Fischer information matrix* [19].

Using (5) and (6) in (7) we have,

$$f(\Delta p_i) \equiv \Gamma_{ij}\Delta p_i \Delta p_j = 2(1 - \kappa^{-1}). \tag{9}$$

Geometrically, this is an equation of an ellipsoid in $(\Delta p_1, \Delta p_2, \Delta p_3)$ space. Further, in the 4-dimensional space spanned by $(\Delta p_i, C)$, the Γ_{ij} can be interpreted as curvatures of the cross-correlation hypersurface. The problem then is to find the maximum value of Δp_2 with the constraint described by equation (9). Geometrically, this amounts to the following construction: One may imagine a $\Delta p_2 =$ const. plane which is tangent to the ellipsoid. The distance of this plane from the $\Delta p_2 = 0$ plane is the required $\Delta p_{2\ \mathrm{max}}$ (see Figure 2). Since the normal to this tangent plane must be parallel to the Δp_2-axis, we have

$$\frac{\partial f}{\partial \Delta p_1} = \frac{\partial f}{\partial \Delta p_3} = 0. \tag{10}$$

Equations (10) written out explicitly in terms of the Fischer information matrix are,

$$\Gamma_{1i} \Delta p_i = 0, \tag{11a}$$

$$\Gamma_{3i} \Delta p_i = 0. \tag{11b}$$

We now eliminate Δp_1 and Δp_3 from (9) and (11a,b). This yields the result,

$$\Delta p_{2\ \mathrm{max}} = \left[2(1-\kappa^{-1}) \frac{\Gamma_{11}\Gamma_{33} - \Gamma_{13}^2}{\Gamma_{11}\Gamma_{22}\Gamma_{33} - (\Gamma_{11}\Gamma_{23}^2 + \Gamma_{22}\Gamma_{13}^2 + \Gamma_{33}\Gamma_{12}^2) + 2\Gamma_{12}\Gamma_{23}\Gamma_{13}} \right]^{1/2}, \tag{12}$$

where the Γ-matrix is,

$$\Gamma = A \begin{pmatrix} \int_1^\infty \frac{x^2}{S(x)} dx & \int_1^\infty \frac{x a(x)}{S(x)} dx & \int_1^\infty \frac{x}{S(x)} dx \\ * & \int_1^\infty \frac{a^2(x)}{S(x)} dx & \int_1^\infty \frac{a(x)}{S(x)} dx \\ * & * & \int_1^\infty \frac{dx}{S(x)} \end{pmatrix}. \tag{13}$$

The 'stars' denote matrix elements obtained by symmetry. This formula cannot be applied to white noise since many of the Γ_{ij} do not exist in this idealistic case.

Figure Captions

Figure 1. Correlation function of a standard recycling filter with the normalized response function drops much more slowly than that corresponding to white noise because the former cannot distinuguish between two chirp wave forms with differing coalescence times as good as the latter. This behaviour has an advantage in the case of standard recycling since the filters in this case can be more coarsely spaced than in the case of white noise filters.

Figure 2. Schematic diagram showing the ellipsoid of equation (9) in the appendix. The plane $\Delta p_2 = \Delta p_{2\ \mathrm{max}}$ touches this ellipsoid. The distance of this plane from the origin gives the lattice spacing in terms of p_2.

Table Captions

Tables I (a), (b) and (c)

These tables display banks of filters labelled by mass parameters for various values of κ^{-1}. The tables correspond to the following values of κ^{-1}:

(a) $\kappa^{-1} = 0.7$

(b) $\kappa^{-1} = 0.8$

(c) $\kappa^{-1} = 0.9$

Table II Distance between consecutive filters for power spectral density corresponding to white noise (Column 2) and noise in interferometers with standard recycling (Column 3) for different values of the parameter κ (Column 1). The distance between filters in the latter case is smaller since the template in that case filters only the lower frequency part of the signal where it is harder to distinguish between two chirp wave forms of different mass parameter values. Also quoted are the number of filters required in the case of standard recycling for two different ranges of the mass parameter: $\mathcal{M} \in [0.25,\ 20]\ M_\odot$ (Column 4) and $\mathcal{M} \in [0.5,\ 20]\ M_\odot$ (Column 5).

Table III Distance between consecutive filters in a particular lattice specified by κ^{-1} (Column 1) found by numerical methods (Column 2) and using the equations (3.36a,b). The analytical method becomes inaccurate for lower values of κ^{-1}.

$\kappa^{-1} = 0.7$

$\mathcal{M}$	$\Delta\xi$
$\mathcal{M} \leq 1.365$	30.3
$1.380 \leq \mathcal{M} \leq 1.676$	30.5
$1.700 \leq \mathcal{M} \leq 1.810$	30.6
$1.840 \leq \mathcal{M} \leq 2.422$	30.8
2.490	31.1
2.563	30.9
2.642	31.1
2.729	31.1
2.823	30.8
2.926	31.1
3.040	30.8
3.166	31.0
3.308	31.1
3.469	31.4
3.654	31.5
3.870	31.3
4.121	31.5
4.422	31.8
4.794	31.8
5.264	32.2
5.890	30.3
6.704	31.8
7.984	32.9
10.328	30.3

Table I (a)

$\kappa^{-1} = 0.8$

$\mathcal{M}$	$\Delta\xi$
$\mathcal{M} \leq 1.530$	18.6
$1.542 \leq \mathcal{M} \leq 2.375$	18.7
$2.413 \leq \mathcal{M} \leq 2.796$	18.8
2.856	19.0
2.920	18.9
2.988	19.0
3.060	19.0
3.138	19.2
3.221	19.2
3.311	19.2
3.408	19.0
3.512	19.2
3.626	19.1
3.750	19.2
3.886	19.2
4.036	19.0
4.202	19.2
4.388	19.2
4.598	19.2
4.839	19.2
5.116	19.5
5.444	19.5
5.835	19.8
6.319	19.8
6.928	19.9
7.729	19.2
8.790	18.6
10.298	18.5

Table I (b)

$\kappa^{-1} = 0.9$

$\mathcal{M}$	$\Delta\xi$
$\mathcal{M} \leq 2.004$	10.1
$2.016 \leq \mathcal{M} \leq 3.897$	10.2
$3.976 \leq \mathcal{M} \leq 4.929$	10.0
5.075	9.7
5.229	9.9
5.399	10.0
5.588	10.1
5.797	10.1
6.028	10.1
6.285	10.0
6.571	9.7
6.885	10.0
7.250	9.9
7.670	9.8
8.156	10.0
8.750	10.5
9.509	11.0
10.529	11.3
11.957	11.3
14.044	10.3

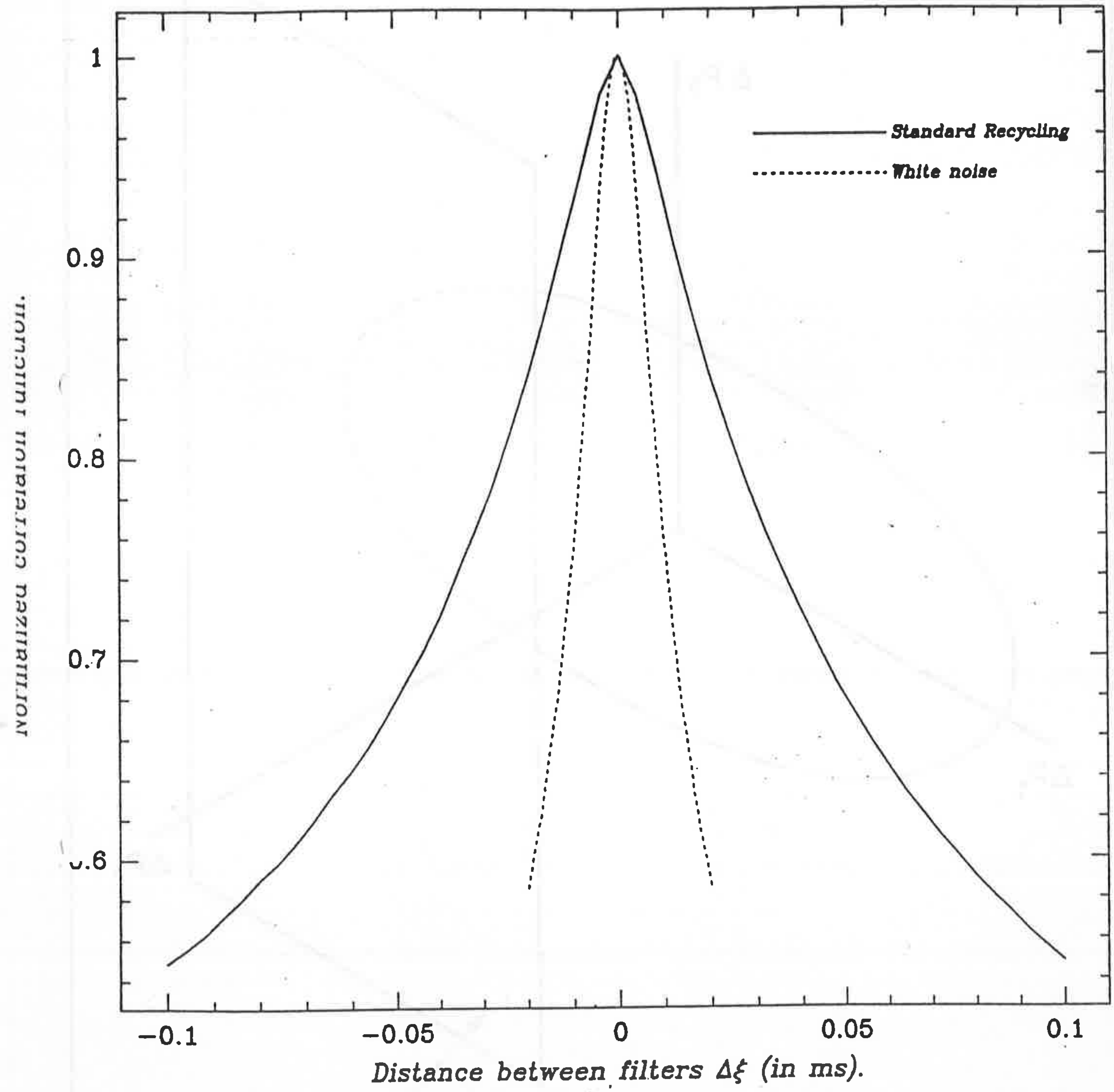
Table I (c)

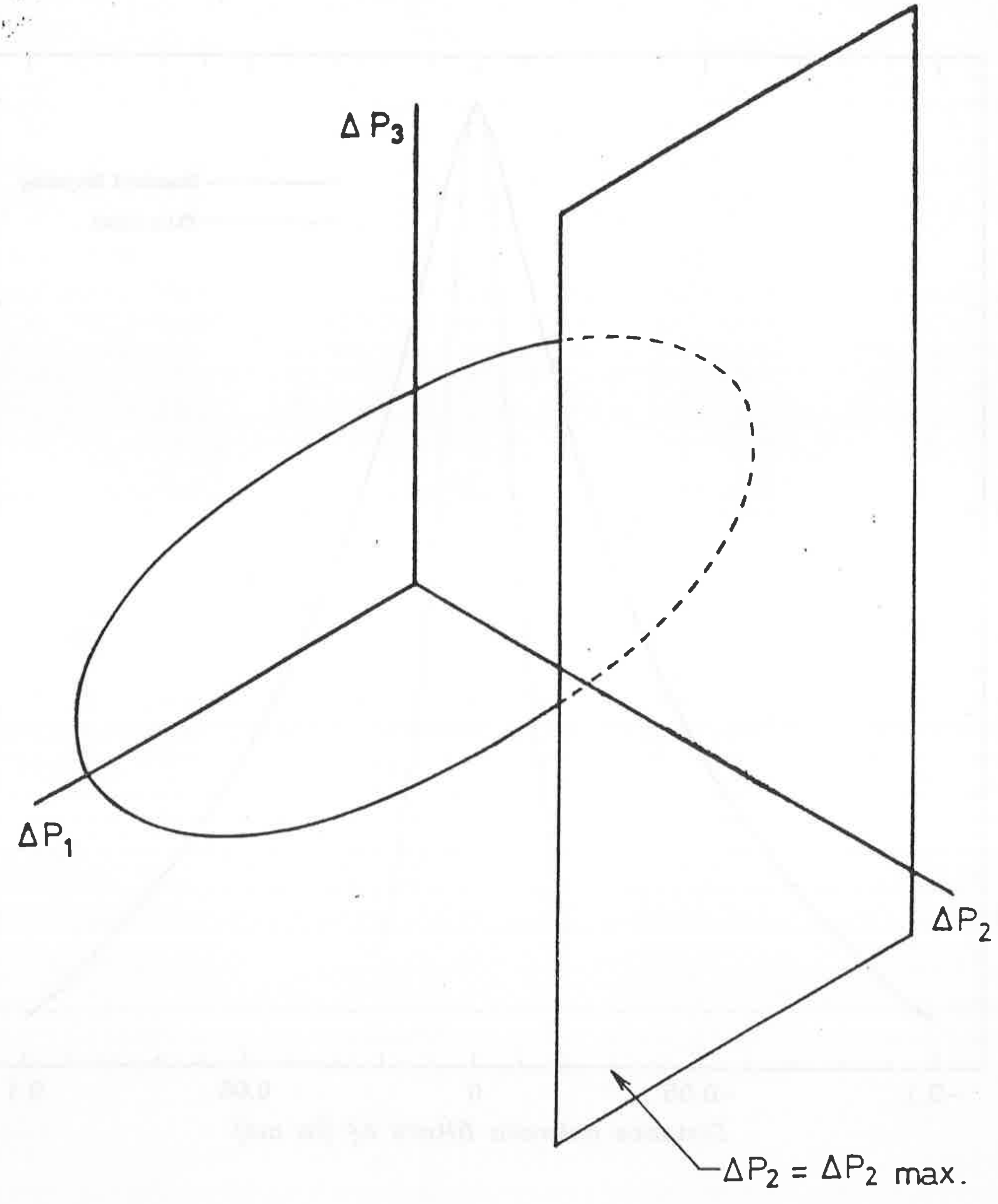
κ^{-1}	$\Delta\xi$ White-noise	$\Delta\xi$ SR-noise	n_f $\mathcal{M}_1 = 0.25$	n_f $\mathcal{M}_1 = 0.50$
0.95	3.4	6.2	4914	1548
0.90	4.9	10.1	2994	943
0.85	6.4	14.1	2152	678
0.80	8.2	18.6	1631	513
0.75	10.2	24.0	1261	397
0.70	12.2	30.4	995	313
0.65	15.4	38.2	793	249

Table II

κ^{-1}	$\Delta\xi$ Numerical	$\Delta\xi$ Analytical
0.99	2.49	2.15
0.98	3.54	3.04
0.97	4.47	3.73
0.96	5.35	4.30
0.95	6.19	4.81

Table III


Normalized correlation function.
1
0.9
0.8
0.7
0.6
Standard Recycling
White noise
−0.1
−0.05
0
0.05
0.1
Distance between filters Δξ (in ms).



ΔP3
ΔP1
ΔP2
ΔP2 = ΔP2 max.