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THE HYDRODYNAMICS OF ATOMS OF SPACETIME: GRAVITATIONAL FIELD EQUATION IS NAVIER–STOKES EQUATION*

T. PADMANABHAN
Inter-University Centre for Astronomy and Astrophysics,
Post Bag 4, Ganeshkhind,
Pune 411 007, India
paddy@iucaa.ernet.in

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There is considerable evidence to suggest that field equations of gravity have the same conceptual status as the equations of hydrodynamics or elasticity. We add further support to this paradigm by showing that Einstein"s field equations are identical in form to Navier–Stokes equations of hydrodynamics, when projected on to any null surface. In fact, these equations can be obtained directly by extremizing of entropy associated with the deformations of null surfaces thereby providing a completely thermodynamic route to gravitational field equations. Several curious features of this remarkable connection (including a phenomenon of "dissipation without dissipation") are described and the implications for the emergent paradigm of gravity is highlighted.

Keywords: Horizon entropy; membrane paradigm; Lovelock gravity; black hole; Navier–Stokes.

1. Gravity: An Emergent Phenomenon

Theoretical investigations in the last decade have uncovered a deep and elegant relationship between gravitational dynamics and hydrodynamics of null surfaces, suggesting that gravity is an emergent phenomenon like fluid mechanics or elasticity.[1] A very direct argument in favor of this paradigm is the validity of principle of equipartition which allows one to use the classical field equations, along with the expression for Davies–Unruh temperature,[2,3] to determine the number density of microscopic degrees of freedom.[4,5] One finds that any null surface in the spacetime is endowed with a number density of microscopic degrees of freedom ("atoms of spacetime") which depends on the structure of the gravitational theory. In the simplest context of Einstein's theory, this density is a constant (equal to one degree of freedom per Planck area) while in a generic Lanczos–Lovelock model, it is proportional

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to the *spacetime entropy tensor* of the theory, $P^{abcd} = \partial L/\partial R_{abcd}$, where L is the Lanczos–Lovelock Lagrangian and R_{abcd} is the curvature tensor.[4,5] (In the case of Einstein's theory with $L \propto R$, we have $P^{ab}_{cd} \propto (\delta^a_c \delta^b_d - \delta^a_d \delta^b_c)$ and the relevant entropy will be proportional to the area of the surface; in general, however, the entropy will *not* be proportional to the area.)

This key "internal evidence" from the structure of gravitational theories — especially in Lanczos–Lovelock models — suggests that they may be viewed as the thermodynamic limit of the statistical mechanics of the microscopic degrees of freedom. This is reminiscent of the work by Boltzmann and others, who used the "internal evidence" of thermal phenomena to *deduce the existence* of atoms/molecules and determine the Avogadro number which quantifies their number density even without knowing what they are and having no direct observational support for their existence. The current results[1,4,5] can be thought of as the theoretical determination of the "Avogadro number of spacetime".

Such a connection opens the door for two further possibilities which we will now discuss.

2. Gravity as Hydrodynamics of Atoms of Spacetime

First, this result suggests that one must be able to obtain the gravitational field equations directly from a thermodynamic extremum principle, without treating the metric as a dynamical variable in the action. Instead, one considers deformations of null surfaces, which affect the causal connection between events and thus the accessibility of information for specific class of observers, and attributes an entropy density to the null surfaces. The extremization of such an entropy functional must lead to the relevant field equations, thereby providing a completely thermodynamic route to gravitational dynamics. This is indeed true and the Lanczos–Lovelock field equations can be obtained[8,9] by extremizing an entropy functional

$$S[n^a] = -\int_{\mathcal{V}} d^D x \sqrt{-g}\left(4P^{cd}_{ab}\nabla_c \ell^a \nabla_d \ell^b - T_{ab}\ell^a \ell^b\right), \tag{1}$$

associated with the normals ℓ^a to the null surfaces in the spacetime. This functional depends on a tensor P^{abcd} which has the symmetries of the curvature tensor and is divergence-free in all indices. One can show that the thermodynamic extremum principle leads to the field equations of a Lanczos–Lovelock model with the same entropy tensor. The resulting field equations can be expressed in a suggestive form:

$$J_a \ell^a = \frac{1}{2} T_{ab}\ell^a \ell^b; \quad J_a q^a_m = \frac{1}{2} T_{ab}\ell^a q^b_m, \tag{2}$$

where $J_a = 2\mathcal{R}_{ab}\ell^b$ with $\mathcal{R}^a_b \equiv P_b^{\ ijk} R^a_{\ ijk}$ being the generalization of Ricci tensor to Lanczos–Lovelock models and q_{ab} is the transverse metric on the null surface.[10] When ℓ^a arises as a limit of the time-like Killing vector in the local Rindler frame, J_a can be interpreted as the Noether (entropy) current associated with the null surface. Then the gravitational field equations expressed as Eq. (2) describes the entropy balance equations across every null surface.

2.1. *Gravitational field equation is a Navier–Stokes equation*

Second, and more interestingly, we would expect the field equations resulting from such a thermodynamic approach to have the structure of equations of hydrodynamics rather the conventional (wave like) field equations. Remarkably enough, Eq. (2) can be recast as the Navier–Stokes (NS) equation for a (hypothetical) fluid built from the null normal. We will now explain how this result arises:

In a four-dimensional (4D) spacetime manifold $\mathcal{M}$ with a metric g_{ab}, a null surface with normal $\boldsymbol{\ell}$ will be a 3D sub-manifold $\mathcal{S}$ such that the restriction $\gamma_{\mu\nu}$ of the spacetime metric g_{ab} to the $\mathcal{S}$ is degenerate. We choose one of the coordinates such that x^3 = constant correspond to a set of null surfaces with, say, $x^3 = 0$ on $\mathcal{S}$. Let the intersection of $\mathcal{S}$ with a x^0 = constant surface (Σ_t) of the spacetime be a 2D surface $\mathcal{S}_t$ with transverse coordinates $x^A \equiv (x^1, x^2)$. The induced metric

$$ds^2 = q_{AB}(dx^A - v^A dt)(dx^B - v^B dt), \tag{3}$$

determines the *intrinsic* geometry of the surface. The *extrinsic* geometry of the null surface is determined by the derivative of the normal $\nabla_\mu \boldsymbol{\ell}$ along the tangential directions of $\mathcal{S}$. Because $\boldsymbol{\ell} \cdot \nabla_\mu \boldsymbol{\ell} = (1/2)\partial_\mu \boldsymbol{\ell}^2 = 0$, the covariant derivative of $\boldsymbol{\ell}$ along vectors tangent to $\mathcal{S}$ is orthogonal to $\boldsymbol{\ell}$ and hence is (also!) tangent to $\mathcal{S}$. Therefore $\nabla_\alpha \boldsymbol{\ell}$ is a vector which can be expanded using the coordinate basis $\boldsymbol{e}_\mu = \partial_\mu = (\partial_0, \partial_A) = (\boldsymbol{\ell} - v^A \boldsymbol{e}_A, \boldsymbol{e}_B)$ on $\mathcal{S}$. Writing this expansion with a set of coefficients (called Weingarten coefficients) $\chi^\alpha{}_\beta$ we have

$$\nabla_\alpha \boldsymbol{\ell} \equiv \chi^\beta_\alpha \partial_\beta = \chi^\beta_\alpha \boldsymbol{e}_\beta; \quad \nabla_\alpha \ell^\beta = \chi^\beta_\alpha. \tag{4}$$

It can be shown that the components of the 3×3 matrix $\chi^\beta{}_\alpha$ can be expressed in terms of q_{AB}, v_A and the quantities:

$$\Theta_{AB} = -\ell_m \Gamma^m_{AB}, \quad \omega_0 = \ell^m \Gamma^0_{m0}, \quad \omega_A = \ell^m \Gamma^0_{mA}. \tag{5}$$

Thus one can determine the covariant derivative $\nabla_\mu \boldsymbol{\ell}$ of the normal vector $\boldsymbol{\ell}$ — and thus the extrinsic geometry of $\mathcal{S}$ — completely in terms of these variables. Being a symmetric tensor, Θ_{AB} can be expressed in terms of its irreducible parts, viz. the trace $\theta \equiv \Theta^A_A$ and the trace-free part σ^A_B by $\Theta^A_B = \sigma^A_B + (1/2)\delta^A_B \theta$. It is also convenient to introduce the surface gravity of the null congruence through $\kappa \equiv \ell^\mu \omega_\mu$.

In terms of these variables, we can write[10] the second equation in Eq. (2) in a locally inertial frame (boosted by v_A in transverse directions) as the NS equation:

$$(\partial_0 + v^B \partial_B)\left(-\frac{\omega_A}{8\pi}\right) = \frac{1}{8\pi}\partial_B \sigma^B_A - \frac{1}{16\pi}\partial_A \theta - \partial_A\left(\frac{\kappa}{8\pi}\right) - T_{mA}\ell^m. \tag{6}$$

This has the *exact form* of a NS equation for a fluid with (i) momentum density $-\omega_A/8\pi$, (ii) pressure $(\kappa/8\pi)$, (iii) shear viscosity coefficient $\eta = (1/16\pi)$. (Note that in the conventional NS equation the viscous tensor $2\eta\sigma^A_B + \xi\delta^A_B\theta$ is defined with an extra factor 2 for shear viscosity.) (iv) bulk viscosity coefficient $\xi = -1/16\pi$ and (v) an external force $F_A = T_{mA}\ell^m$. We will now discuss several implications of this result.

2.2. *Features of hydrodynamics of "atoms of spacetime"*

To begin with note that our result is completely general and applicable to the projection of field equations on to any null surface. Several years back, similar results were derived in the *specific context* of black hole (BH) horizons,[11,12] leading to the membrane paradigm.[13] We do not assume any such on-shell restriction (e.g. like existence of a BH). In short, we have essentially proved that *Einstein's equations are identical to NS equations when projected on any null surface.* Previous results are very special cases of this.

Second, it is conceptually satisfying that the extremum condition for the entropy density of spacetime (directly related to null surfaces because they can act as local Rindler horizons), given by Eq. (2), has a natural interpretation as the NS equation on a null surface. This is unlike earlier results in which one first obtains the field equations by some variational principle *and then* recasts them as NS equation in the specific context of BHs. The gravitational entropy density — which is the part of the integrand $s_{\rm grav} \propto (-P_{ab}^{cd}\nabla_c\ell^a\nabla_d\ell^b)$ in Eq. (1) — leads to the "canonical momentum": $(\partial s_{\rm grav}/\partial(\nabla_c\ell^a)) \propto (-P_{ab}^{cd}\nabla_d\ell^b) \propto (\nabla_a\ell^c - \delta^c_a\nabla_i\ell^i)$. This term is analogous to the more familiar object $t^c_a = K^c_a - \delta^c_a K$ (where K_{ab} is the extrinsic curvature) that arises in the (1+3) separation of Einstein's equations and can be interpreted as a surface energy momentum tensor in the context of membrane paradigm because t_{ab} couples to δh^{ab} on the boundary surface when we vary the gravitational action. This result shows that the entropy density of spacetime is directly related to t^c_a and its counterpart in the case of null surface.

Third is the conceptual role of the "dissipative" viscous terms in the NS equation. At a fundamental level we do *not* expect to find irreversible dissipation to occur in a conservative system we are studying and yet, we have a viscous force term in the NS equation! How do we reconcile these two facts? The answer is fairly subtle which we will now describe.

2.3. *Dissipation without dissipation*

The emergent paradigm of gravity suggests that one needs to accept an intrinsic observer dependence in all thermodynamical variables. In particular, the Rindler observers and the freely falling observers at a given event will attribute completely different thermodynamical features to the same null surface (see Sec. 4.4 of Ref. 1). Therefore while projecting the field equations to a null surface and interpreting them as NS equation of a fluid living on that null surface, it is crucial to ask which part of the field equation survives in the freely falling frame.

Our analysis shows that the terms involving (i) the derivatives of the viscous stress tensor, (ii) the derivative of the pressure and the (iii) external momentum flux term remain nonzero in the freely falling frame because they involve *derivatives* of the Christoffel symbols. This is in spite of the fact that the viscous stress tensor and the pressure themselves vanish in the freely falling frame because these variables are proportional to Christoffel symbols. What is relevant for the equation to be interpreted as the NS equation is the existence of a viscous *force* on the fluid, arising from the nonzero *gradient* of the viscous tensor (and pressure), rather than a nonzero viscous tensor (or pressure) itself. Of course, this is very counter-intuitive compared to normal fluid mechanics. If flow of water exhibits viscosity, it will also have nonzero viscous tensor in the same frame of reference! But here, we are merely calling some combination of Christoffel symbols as viscosity tensor [see Eq. (5)] and their derivatives involving certain combinations of Ricci tensor as viscous force. Obviously, the former can vanish without the latter vanishing in a local inertial frame.

This feature has important implications for the characterization of viscous dissipation in the current context. In the conventional fluid dynamics based on NS equation, the viscous dissipation will be proportional to terms involving $\sigma_{AB}\sigma^{AB}$ and θ^2 and — in usual fluid mechanics — $\sigma_{AB}\sigma^{AB}$, θ and $\partial_B\sigma^{AB}$ will all be nonzero. In our case σ_{AB} and θ vanish in the freely falling frame and *the inertial observers in spacetime will not see any dissipation* — which is reassuring. But the force on the viscous fluid, which depends on the gradient $\partial_B\sigma^B_A$ and $\partial_A\theta$, does not vanish even in the freely falling frame showing that this force has an observer independent existence. It might appear, at first, paradoxical that a viscous force term exists for the fluid but no viscous dissipation. But, as we said above, algebraically this is no more paradoxical than the fact that effects due to curvature involving derivatives of Christoffel symbols can be present locally even when the Christoffel symbols vanish at a point. The result obtained above is just a translation of this well-known fact in the language of null surface dynamics. In fact the real paradox would be if there is an observer independent viscous dissipation in spacetime!

We stress that the NS equation derived here is applicable to *any* null surface in the spacetime, including patches of the local Rindler horizons. In the emergent paradigm of gravity one relies heavily on local Rindler horizons which are ordinary null surfaces in spacetime and not event horizons corresponding to some specific *solutions* of field equations. In fact, *to obtain the field equations* from a suitably defined entropy density of spacetime, one uses the local Rindler horizons as *off-shell* constructs in the theory and hence they cannot be linked to horizons arising in on-shell solutions. This is why we need to carefully distinguish generic features of interpretation which are valid for *any* null surface from some of the features which might have limited validity in the specific context of BH horizon. The dissipation and generation of entropy with accompanying irreversible thermodynamics may be acceptable in the case of physical processes involving black hole horizons, say, but such dissipation is difficult to interpret in terms of local Rindler horizons. So, again, it is welcome that $\sigma_{AB}\sigma^{AB}$ and θ^2 vanish in the freely falling frame but the NS equation for the fluid remains valid. From this point of view, it seems better to interpret the equations in the freely falling frame.

The relationship between thermodynamics of horizons and dynamics of gravity started out in the seventies as a tentative analogy. With the current results, it is a clear reflection of underlying reality — viz. *gravity is hydrodynamics.*

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