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GRAVITY WITHOUT METRIC



Abhijit K. Kshirsagar †

Inter-University Centre for Astronomy and Astrophysics

Post Bag 4, Ganeshkhind, Pune 411007

Abstract

A recent development in gravity theories is described. It is shown that a complete description of gravity is possible without explicit reference to metric at all, instead in terms of spin-connection variables. This represents a natural covariantisation of Ashtekar's new variables approach to gravity. The gravity without metric action is particularly suitable for a Yang-Mills type quantisation. I describe this theory in some detail, in a non-technical manner.

† e-mail address : abhi@iucaa.ernet.in

Marriage of gravity and quantum mechanics has been a long cherished dream of theoretical physics. A successful theory of quantum gravity will represent the ultimate understanding of all the four known interactions at all energy scales, and possibly that of a unification of all too. The principal reason for this is that the scale at which quantum gravitational effects manifest themselves is a combination of Newton's constant G_N , Planck's constant \hbar and the velocity light c , viz. the Planck scale ($\sim 10^{-33} \text{ cm}$ or 10^{19} GeV). Several approaches, with varying degrees of success and equally glaring failures have been proposed, to this problem for several decades now. One of them is to mimick the perturbative quantisation of electro-weak and strong interactions á la Yang-Mills, both in a continuum theory as well as on a lattice. Progress in the lattice approach has been slow, principally due to lack of computing techniques and power, but is still being vigorously pursued. Continuum perturbation theory faces the problem of being non-renormalisable. This is attributed to the fact that gravity is truly a nonlinear theory (in particular, the Einstein lagrangian is nonpolynomial in the basic variable, viz. metric) and more over the coupling constant G_N is dimensionful, unlike Yang Mills theories. Non-compactness of the 'gauge-group' of gravity is another problem faced. One possible way to cure the situation is to modify the Einstein lagrangian by adding extra terms, either by hand or by appealing to more sophisticated theories in higher dimensions etc. This can be clubbed into a collective title like 'Kaluza-Klein-Supergravity approach'. Superstring theories which became popular since the mid-80's have been shown explicitly to modify the Einstein action with the addition of terms proportional to higher powers of curvature.

On the other hand, one can take up the task of a non-perturbative quantisation of gravity. This approach is known as 'canonical quantisation'. The idea is to first construct a classical hamiltonian for the theory of gravity, postulate the canonical commutation relations (the Heisenberg algebra) for the operator version of the phase space conjugate variables, express the hamiltonian operator in terms of these and solve the corresponding "Schrödinger like" equation, to obtain the wavefunction for gravity. Straightforward as though it may sound, each step in this prescription

involves immense difficulties. They begin with the problem of isolation of true degrees of freedom describing gravity. Diffeomorphism invariance implies that there are constraints in the theory, which remain a fact of life in what follows. Operator ordering ambiguities plague the uniqueness of the hamiltonian operator. Choice of boundary conditions for the solution of the Wheeler-DeWitt equation (analog of the Schrödinger equation) is very hard to implement explicitly (with the possible exception of the 'no-boundary' proposal). The resulting equations are functional differential equations and are quite a formidable task to solve, in fact success has been achieved only in toy models (generally referred to as mini- or midi-superspace) where the degrees of freedom of gravity are brutally truncated to be only finite in number (in reality perhaps one or two !)

There is a lesson to be learnt from all the difficulties mentioned above, regarding the canonical quantisation of gravity, viz., perhaps an inappropriate set of variables is being used for analysing the problem. It is a well-known fact from mechanics that a canonical transformation on the phase space variables can lead to a set of variables, in terms of which it is trivial to solve for the equations of motion. So, maybe the space-space components of the metric (the space-time and 0 - 0 components are related to the solution of constraints) and their canonically conjugate momenta are not the best set of variables to do quantisation. This was precisely the realization of Abhay Ashtekar at Syracuse University, USA, who in 1986, proposed a set of canonical variables which are similar to those used in Yang-Mills gauge theories. The greatest advantage of his variables is in simplifying the constraint structure of the theory. In terms of Ashtekar variables, all the constraints are not only polynomial but they also form a closed algebra under Poisson bracket operation. i.e. all constraints are 'first class'.

The simplicity and beauty of Ashtekar's approach is most easily appreciated in 2 + 1 dimensions where the 'Einstein action' is

$$S = \int e^a \wedge R^{bc} \epsilon_{abc} \quad (1)$$

Here ϵ_{abc} is the totally antisymmetric object in three dimensions, e^a are the triads

(these are in a sense 'square-root' of the metric) and R^{ab} is the 'field-strength' of the Yang-Mills vector potential ω^{ab} , corresponding to the gauge group $SO(2,1)$. Equivalence to $(2+1)$ dimensional Einstein theory is obtained by solving for the metric in terms of e 's.

This theory can be quantised exactly, if we choose the spatial components of the gauge potential $\omega_i (i = 1, 2)$, and their conjugate momenta E^i (which are just duals of the triads e_i). There are two constraints arising due to reparametrisation invariance :

$$D_i E^i = 0$$

and

$$R_{ij} = 0 \tag{2}$$

The first is analogous to Gauss law in electromagnetism and the second one implies that the Yang-Mills field strength for ω vanishes, i.e. it is flat. It is a well-known fact about reparametrisation invariant theories that the hamiltonian is simply a combination of constraints, in which case the Hilbert space of physically acceptable states is obtained by simply finding the states that are annihilated by all the constraints in operator form. The two constraints thus imply that the quantum states of $(2+1)$ dimensional gravity are gauge invariant functionals of flat $SO(2,1)$ connections.

When one tries similar things in four spacetime dimensions, the dynamical variables to be used are the spatial components of a $SO(3,1)$ (or equivalent) ω_i and their conjugate momenta E_i , from which the metric can be built. Ashtekar's formulation is in a sense a generalization of the Einstein formulation since the former is capable of handling degenerate metrics as well. The constraints of the theory now become

$$D_i E^i = 0$$

$$tr R_{ij} E^i = 0$$

and

$$tr R_{ij} E^i E^j = 0 \tag{3}$$

It is evident that the constraints are indeed polynomial. Besides extending the Einstein theory, Ashtekar action has another novel classical feature, viz., it admits $SU(2)$ instanton solutions on a curved background, in the Euclidean signature. In hindsight, this is to be expected, due to the similarity with Yang-Mills theory which indeed has a very rich 'instanton physics'. In the past couple of years, the quantization program of four dimensional Ashtekar action has been pushed quite far. The physical states of the theory are described in terms of what are called functionals of loop and strip variables. These are roughly the holonomies (analogous to Wilson loops or Aharonov-Bohm-Berry phases) of the gauge potential ω and the fields E^i around loops. It is beyond the scope of this article to summarize the tantalizing development following the invention of these variables, suffice will it to say that they offer a naturally discrete spacetime at Planck scale, electromagnetism and gravitons can be nicely described using them and quantum states describing the type II superconductors arise very naturally in this description.

To recapitulate, it is clearly advantageous to move away from the metric and adopt the gauge potential and its conjugate momentum as the basic variables in describing gravity. A natural question that arises now, is that whether it is possible to write the action purely in terms of the connection alone, where the metric can be constructed from the connection, if needed? This theory will clearly have the advantage of being akin to Yang-Mills theories and perhaps a much more standard path-integral quantisation using the Faddeev-Popov procedure can be adopted. It is somewhat surprising a priori, that the answer to this question is yes, although this form of the theory is just beginning to be explored. This formulation is known as 'gravity without metric'. In what follows, we shall look at this theory in some detail.

One begins by noting that in the four dimensional action

$$S = \int e \wedge e \wedge R \quad (4)$$

where we suppress all the indices, the wedge denotes the product of forms and R is the curvature of a $SL(2, C)$ connection. Also note that the tetrad one-form e

appear only in the combination $e \wedge e$, i.e. as a two-form. Is it then possible to write an action as a functional of the connection and two-forms alone? The answer is in the affirmative provided a constraint is imposed on the two-forms, ensuring that both the tetrad and two-form descriptions have the same number of degrees of freedom. The action then takes the form

$$S[\omega, \Sigma, \psi] = \int \Sigma^a \wedge R_a - \frac{1}{2} \psi_{ab} \Sigma^a \wedge \Sigma^b \quad (5)$$

where we explicitly introduce the lagrange multiplier matrix ψ which is symmetric and tracefree to ensure the above mentioned constraint. The physical origin of this constraint is interesting. The action (5) without the constraint term would correspond to some chiral fields coupled to the connection. The constraint then ensures that the entire spin-2 content of the theory (which represents the gravitational field) is expressible in terms of purely the spin-connection and not any other chiral fields. This is of course just what one needs in order to have a pure spin-connection description of gravity. At this juncture, we remark paranthetically that the action (5) can be trivially modified to include a cosmological constant as well as couplings to matter and gauge fields, which is necessary for a realistic situation. In this article we will concentrate on pure gravity alone.

It is evident that the action being quadratic, the equation of motion for the two form field Σ , is linear which at once allows to solve for it, rendering the action to be a functional of the spin-connection ω and the auxiliary field ψ only. The resulting action is

$$S[\omega, \psi] = \frac{1}{2} \int \psi^{-1ab} R_a \wedge R_b \quad (6)$$

Of course it is clear that this procedure hinges on the invertibility of the matrix ψ , which we shall assume hereafter. Suffices it to say that in the degenerate situation, both possibilities exist viz. the curvature being finite as well as blowing up. The matrix ψ being symmetric and tracefree, has five degrees of freedom, four of which can actually be fixed by usage of Bianchi identity, leaving thereby only one degree of freedom. It can be conveniently expressed in terms of another non-dynamical lagrange multiplier field (in general complex) μ which ensures the

tracelessness of ψ explicitly in the action. When the dust settles, we obtain finally the action for 'gravity without metric' (GWM) as

$$S[\eta, \omega] = \int \eta [\text{tr} M^2 - \frac{1}{2} (\text{tr} M)^2] \quad (7)$$

where the matrix M is quadratic in the curvature of the connection ω and η is proportional to the square-root of μ (scaled by the determinant of M). The action (7) is quartic in the curvature R . When one adds the cosmological constant and/or couplings to another fields, this polynomial feature is lost.

It is worth pointing out at this point, that the tracelessness of the matrix ψ can be ensured in several inequivalent ways, in the action. All of these lead to the same dynamical content of the theory, because they give the same expressions for the constraints. This is another example of the well known phenomenon in classical mechanics where several inequivalent lagrangians lead to identical hamiltonian structure.

If we start with the action (7) for gravity, then it would be important to check its dynamical consistency with Ashtekar's formulation and/or the original Einstein formulation. In particular one should check the constraints of the theory, since they alone govern the quantum mechanics for gravity. At the classical level, this can be achieved by recalling the fact that gravity is a reparametrisation-invariant theory, and the energy -momentum tensor is the generator of these transformations, in particular the space-time components T_{0i} generate the spatial whereas the time time component generates temporal transformations. Invariance under these would imply therefore, the vanishing of these components. Thus the vector and scalar constraints (this terminology spills over from the original Arnowitt-Deser-Misner analysis of Einstein action) can be obtained by computing them and equating to zero. When this is done for the action (7) one indeed obtains the same constraints as those of Ashtekar. Moreover, the GWM action is a functional of connection and has gauge invariance incorporated into it. It leads to the usual analog of the Gauss law constraint from electromagnetism. One can complete the full circle too, viz.

start with the hamiltonian written as a sum of Ashtekar constraints and perform an inverse Legendre transform to obtain pure connection action, i.e. the GWM action. Indeed, perfect agreement is obtained.

Having confirmed and reconfirmed the validity of such GWM action at the classical level, one is naturally led to contemplating the quantisation of such a theory. Only initial steps have been taken in this direction and most remains to be explored yet.. As has been emphasized repeatedly, the similarity of GWM action to that of Yang-Mills is to be exploited. Therefore, one begins by writing down the formal definition of Euclidean path integral of the original two-form action involving the fields Σ, ψ and the spin-connection ω . We begin with first principles since the GWM action is in a sense a classically reduced action, i.e. equations of motion have been used, whereas a path-integral has to be performed over all configurations and not just over the classical ones. The path integral is

$$Z = \int D\omega D\psi D\Sigma e^{-\int \Sigma^a \wedge R_a + \frac{1}{2} \psi_{ab} \Sigma^a \wedge \Sigma^b - S_{gf} - S_{FP}} \cdot \delta(\text{tr}\psi) \quad (8)$$

The $\delta(\text{tr}\psi)$ is the constraint of tracelessness of ψ in the path integral. S_{gf} and S_{FP} are the terms in the action which arise from the gauge fixing of the connection part of the action and the resulting Faddeev-Popov determinants. Major efforts and technicalities are involved in getting these terms correctly. This is partly because the effective action for the connection (as will be argued here) as a result of integrating the other variables is non-polynomial.

The functional integral over the Σ field can be done exactly. (This is the analogous statement to being able to 'solve' for it, classically). Its result is to introduce a reciprocal determinant of the matrix ψ in the functional integral over the remaining fields ω and ψ . This determinant as well as the trace constraint on ψ can be promoted to being local terms in the action in the functional integral. One fact emerges right away, the path integral over ψ cannot be done exactly (the analogous classical statement would be that ψ cannot be solved in a 'closed form' in terms of the curvature of ω and μ) and has to be done by evolving a Feynman diagram technique for ψ . Feynman rules can indeed be set up to evaluate the contribution

to the effective action for the connection ω , by identifying the propagator, external source and self-interaction vertices. It turns out that we get vertices to all orders. This is to be expected since gravity is a nonlinear theory anyway. The effective action for ω then involves all powers of $R \wedge R$, rendering the GWM action non-polynomial. It is this action that should be quantised by appropriately choosing the gauge-fixing conditions and computing the Faddeev-Popov determinants. It remains to be seen, how explicit the results can be obtained in this approach.

Bibliographic acknowledgement

Owing to the non-technical nature of this article, it is sufficient to cite only the workers whose work has been touched upon, instead of giving complete references. The discussion about approaches to quantum gravity is a part of the by-now-standard folklore. Ashtekar theory has been profusely expounded upon by many, my favorite references are reviews by Ashtekar and Horowitz and the book by Ashtekar and Tate. The section on GWM theory is a result of the work of the Maryland group led by Jacobson, Peldán in Sweden and by the author and his colleagues at IUCAA, Pune, India.

