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POSSIBLE QUANTUM INTERPRETATION OF CERTAIN POWER SPECTRA IN CLASSICAL FIELD THEORY

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In this paper we report an analogue for the vacuum state in *classical field theory* and its Planckian nature with respect to uniformly accelerated observers. When a real, monochromatic, mode of a scalar field is Fourier analyzed with respect to the proper time of a uniformly accelerating observer, the resulting power spectrum consists of three terms none of which have a simple classical meaning. Specifically, the three terms are (i) a factor (1/2) that is typical of the ground state energy of a quantum oscillator, (ii) a Planckian distribution $N(\Omega)$ and — most importantly — (iii) a term $\sqrt{N(N+1)}$, which is the root mean square fluctuations about the Planckian distribution. It is the appearance of the root mean square fluctuations that motivates us to attribute a "thermal" nature to the power spectrum. Such a power spectrum also arises when we Fourier analyze a real, monochromatic, plane electromagnetic wave in the frame of a uniformly accelerating observer. We also present a model of a detector whose response is the Fourier spectrum of the field with respect to its proper time, which illustrates that it should, in principle, be possible to physically measure the power spectrum we have obtained. These results show that some of the "purely" quantum mechanical results might have a classical analogue.

1. Introduction

It is well known that quantization of a field in Minkowski and Rindler coordinates are not equivalent.[1] It is also known that the response of a uniformly accelerating detector in the Minkowski vacuum is a thermal spectrum.[2,3] In both these situations, one obtains the thermal spectrum in the strict sense of the word: Not only that the mean occupation number in any mode is Planckian, but the fluctuations around the mean is also characterized by the standard thermal noise. These results suggest that quantum fluctuations in the vacuum appear as thermal fluctuations in an uniformly accelerated frame.

In contrast to quantum theory, classical field theory does not admit any intrinsic fluctuations. The absence of concepts such as vacuum and fluctuations in classical field theory may lead us to believe that nontrivial phenomena as the one mentioned

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in the above paragraph will not have any classical analogue. We shall show, however, that such is not the case. In this paper, we discuss a fairly nontrivial and interesting effect that arises purely in the context of *classical* field theory, which has a formal similarity with the quantum mechanical results mentioned above. We find that, when a *real*, monochromatic mode of a classical field is Fourier transformed with respect to the proper time of a uniformly accelerating observer, the resulting power spectrum consists of three terms none of which have a simple physical interpretation in terms of classical concepts. However, they closely resemble terms that have a definite *quantum mechanical* interpretation. More specifically, we show that the three terms which arise are: (i) a factor $(1/2)$ that is typical of the ground state energy of a quantum oscillator, (ii) a Planckian distribution $N(\Omega)$ and (iii) a term proportional to $\sqrt{N(N+1)}$, which is the root-mean-square fluctuations about the Planckian distribution in a quantum mechanical context. While one could have anticipated the second term N based on earlier results, the first and the third terms could not have been guessed from any previously known result. It is interesting — to say the least — that such terms arise in a situation where there is no genuine thermal phenomena, statistical steady state, thermal or quantum fluctuations, etc. The power spectrum has only the form of a thermal spectrum. Similar results are obtained when we consider a real, monochromatic, plane electromagnetic wave.

This paper is organized as follows. In Sec. 2, we calculate the power spectrum of a real, monochromatic mode of a scalar field as well as that of a plane electromagnetic wave in the frame of a uniformly accelerated observer. In Sec. 3, we generalize our result to different field configurations. Finally, in Sec. 4 we present a model of a detector which responds to the power spectrum of the field with respect to its proper time and also discuss the possible implications of our analysis.

2. Power Spectrum of a Real, Monochromatic Wave in an Accelerated Frame

2.1. *Power spectrum of a scalar field mode*

Consider a massless, minimally coupled, scalar field which satisfies the following Klein–Gordon equation:

$$\Box\Phi \equiv \Phi^{;\mu}{}_{;\mu} = 0\,. \tag{1}$$

In flat spacetime, the basis solutions to the above Klein–Gordon equation in the Minkowski coordinates $(t, \mathbf{x})$ can be taken to be plane waves labeled by the wave vector $\mathbf{k}$:

$$\Phi(t, \mathbf{x}) = \cos(\omega t - \mathbf{k}\cdot\mathbf{x})\,, \tag{2}$$

where $\omega = |\mathbf{k}|$. We now ask: Consider an observer who is moving on an arbitrary trajectory $(t(\tau), \mathbf{x}(\tau))$, parametrized by the proper time τ. How will this observer view the above Minkowski plane wave mode?

The moving observer will see the scalar field varying with respect to his (her) proper time in a manner determined by the function $\Phi[t(\tau), \mathbf{x}(\tau)]$. If the observer is in inertial motion then the monochromatic wave will appear to be another monochromatic wave with a Doppler shifted frequency. But, in general, for non-inertial trajectories, the wave will not appear to be monochromatic for the moving observer but will prove to be a superposition of waves with different frequencies. To determine the exact decomposition of the wave, we should Fourier analyze the Minkowski mode in the frame of the observer. The Fourier transform of the Minkowski plane wave with respect to the proper time τ of the observer in motion is described by the integral

$$\tilde{\Phi}(\Omega) = \int_{-\infty}^{\infty} d\tau\, e^{-i\Omega\tau}\Phi[t(\tau), \mathbf{x}(\tau)]\,. \tag{3}$$

This expression gives the amplitude of a component with frequency Ω (as defined by the moving observer) present in the original monochromatic wave. Given a particular plane wave, we can always align the coordinates such that the wave is traveling along the x-axis, i.e. the wave vector is given by $\mathbf{k} = (k, 0, 0)$. Then the plane wave mode (2) reduces to

$$\Phi(t, \mathbf{x}) = \cos(\omega t - kx) \tag{4}$$

and its Fourier transform is given by the integral

$$\tilde{\Phi}(\Omega) = \int_{-\infty}^{\infty} d\tau\, e^{-i\Omega\tau} \cos[\omega t(\tau) - kx(\tau)]\,. \tag{5}$$

We shall now specialize to the case of an observer who is accelerating uniformly with respect to the Minkowski coordinates. We shall assume that the observer is accelerating along the x-axis. Let us also assume that the observer is moving with a proper acceleration g. The world line of such an observer in the Minkowski coordinates (t, x, y, z) is given by the relations[4]

$$t = t_0 + g^{-1}\sinh(g\tau)\,, \quad x = x_0 + g^{-1}\cosh(g\tau)\,, \quad y = y \quad \text{and} \quad z = z\,, \tag{6}$$

where t_0 and x_0 are constants and τ is the proper time as measured by a clock in the accelerated frame. (These are the set of transformations that are typically used in literature to relate the Rindler and Minkowski coordinates. But, in the limit of $g \to 0$, these transformations do not reduce to an identity transformation because they involve a g-dependent shift. We shall discuss the $g \to 0$ limit in detail in an appendix.) From the above relations it can be easily shown that

$$(x - x_0)^2 - (t - t_0)^2 = g^{-2}\,. \tag{7}$$

This relation then implies that the world line of a uniformly accelerating observer is a hyperbola in the (t, x) plane parametrized by the two constants t_0 and x_0. The asymptotes of this hyperbola are the past and the future light cones that intersect at the point (t_0, x_0). To see how the plane wave (4) will be viewed by such an observer, we substitute the coordinate transformations (6) in the Fourier integral (5), and obtain[5]

$$\begin{aligned}\tilde{\Phi}(\Omega) &= \int_{-\infty}^{\infty} d\tau\, e^{-i\Omega\tau} \cos\left(\omega[t_0 - x_0 + g^{-1}\sinh(g\tau) - g^{-1}\cosh(g\tau)]\right) \\ &= \int_{-\infty}^{\infty} d\tau\, e^{-i\Omega\tau} \cos\left(\omega g^{-1} e^{-g\tau} - \beta\right) \\ &= \left(\frac{1}{2g}\right) e^{-i\phi}\left(e^{-(\Omega/4\Omega_0)} e^{-i\beta} + e^{(\Omega/4\Omega_0)} e^{i\beta}\right)\Gamma(i\Omega g^{-1}), \end{aligned} \tag{8}$$

where

$$\phi = \Omega g^{-1}\ln(\omega g^{-1}), \qquad \Omega_0 = g/2\pi \quad \text{and} \quad \beta = \omega(t_0 - x_0). \tag{9}$$

In the above integral we have assumed that the plane wave is traveling to the right so that $k = \omega$. The resulting power spectrum per logarithmic interval in frequency is given by $\mathcal{P}(\Omega) \equiv (\Omega|\tilde{\Phi}(\Omega)|^2)$ and can be written in a remarkable form:

$$\begin{aligned}\mathcal{P}(\Omega) \equiv \Omega|\tilde{\Phi}(\Omega)|^2 &= \left(\frac{\pi}{2g}\right)\left(\coth(\Omega/2\Omega_0) + \operatorname{csch}(\Omega/2\Omega_0)\cos(2\beta)\right) \\ &= \left(\frac{\pi}{g}\right)\left\{\frac{1}{2} + N + \sqrt{N(N+1)}\cos(2\beta)\right\}, \end{aligned} \tag{10}$$

where

$$N(\Omega) = \left(\frac{1}{\exp(\Omega/\Omega_0) - 1}\right). \tag{11}$$

We shall now consider the various features of this result.

To begin with we note that this result is a purely classical one and hence $\hbar$ does not appear anywhere. In ordinary units, $\Omega_0 = (g/2\pi c)$ has the correct dimensions (*viz.* per second) for a frequency. The quantity $N(\Omega)$ is a Planckian in terms of *frequencies* and is again independent of $\hbar$. Usually, one tries to express the Planckian distribution in terms of energies of the "quanta" labeled by frequency Ω and in such a case we need to write frequencies as, say $\Omega = (E/\hbar)$, thereby *artificially* introducing $\hbar$; but the result, stated as a power spectrum in frequency space, makes perfect conceptual sense as it stands. For example, radio astronomers measure the power spectrum in frequency space and may not think in terms of photons. Of course, to obtain a quantity with the dimension of temperature we again need to introduce a $\hbar$ into the quantity Ω_0. Since, in the situation we are considering, there is no real concept of temperature we will not introduce $\hbar$.

The analysis done above could have been carried out even in the days before quantum theory — it uses only classical relativity. Had it been done, there would have been no simple way of understanding the terms which arise in (10). But it is our knowledge of quantum theory that allows a suggestive interpretation of the three terms in the power spectrum: The first term — *viz.* the factor (1/2) — is typical of the ground state energy of a quantum oscillator. The second term N is a Planckian distribution in Ω, as already mentioned. *Note that these two terms are totally independent of the original frequency ω of the plane wave!*

The third term is still more remarkable. When we vary the constants t_0 and x_0 this term varies between $-\sqrt{N(N+1)}$ and $+\sqrt{N(N+1)}$. The magnitude of this variation (which is the root-mean-square deviation about the mean value) is exactly what one would have obtained for a strictly thermal distribution of massless bosonic quanta in quantum field theory. Thus, a classical plane wave, viewed in the accelerated frame, has a power spectrum reminiscent of Planck spectrum with associated thermal fluctuations.

To avoid possible misunderstanding, we stress here the following fact: The system we are considering has no fluctuations or temperature in the sense of statistical physics. Being a purely classical system, it does not have any quantum fluctuations either. But the terms which we get in the accelerated frame have the most natural interpretation in terms of notions like thermal spectrum and its fluctuations.

The quantity β is related to t_0 and x_0 by Eq. (9). If the original plane wave had an extra phase δ, then the argument of the cosine term will pick up 2δ additionally. For a specific choice of the constants δ, t_0 and x_0, it possible to kill the fluctuations in the power spectrum. It is also easy to verify that one *cannot* choose the constants to cancel the first two terms as well. But, in general, all the three terms are present in the power spectrum. We believe this result is unlikely to be a mere curiosity and deserves attention. We shall now comment on the related aspects of this result.

It may be noted that the existence of the three terms is a direct consequence of our choosing a *real* plane wave which, in classical field theory, *is* mandatory. If the same analysis is repeated for a complex mode for the scalar field, say $\Phi(t, x) = \exp -i(\omega t - kx)$, then the resultant power spectrum per logarithmic frequency interval is given by

$$\mathcal{P}(\Omega) = \left(\frac{2\pi}{g}\right) N\,, \tag{12}$$

where N is given by (11). We do not get the zero-point term or the fluctuations. Of course, in classical field theory, one must use *real* modes and that is exactly what we have done here.

Finally, let us consider the limit of $\omega \to 0$. In this limit, the field in the inertial frame reduces to an unimportant constant — which could be thought of as closest to the concept of a "vacuum" in the classical theory. The Fourier integral as well as the phase ϕ in Eq. (9) diverges when $\omega \to 0$; but the power spectrum — which is the modulus squared of the amplitude — is well defined:

$$\mathcal{P}(\Omega)|_{\omega\to 0} = \left(\frac{\pi}{g}\right)\left\{\frac{1}{2} + N + \sqrt{N(N+1)}\right\}. \tag{13}$$

However, as long as ω is treated as a "regulator" one can say that the accelerated observer will see these terms even in the limit of $\omega \to 0$. This is very reminiscent of the inertial vacuum appearing as a Planckian spectrum to the accelerated observer in a manner which is completely independent of the original wave mode. Mathematically, this result arises because our limiting procedure does not commute with that of Fourier transforming the mode. If we consider the $\omega \to 0$ limit first and then evaluate the Fourier transform, we will, of course, get the square of the Dirac delta function as the power spectrum. But, when we compute the power spectrum first and *then* take the limit of $\omega \to 0$ we get a different, and finite, result. Once again, the situation is reminiscent of regularization procedures (like the "$i\epsilon$ prescription") in quantum theory in which the order of operations matter. In a way, this limiting value turns out to be a more generic feature. (In the above discussion we have assumed that the wave and the observer are moving along same direction, *viz.* the x-axis. But the result for the $\omega \to 0$ should hold irrespective of this condition; see Sec. 3.)

2.2. *Power spectrum of a plane electromagnetic wave*

The analysis we have carried out for a real, monochromatic scalar field mode can analogously be carried out for a plane electromagnetic wave. Given a vector potential A^μ the electromagnetic field tensor is defined as[6]

$$F_{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu) . \tag{14}$$

The components of the field tensor are then given by

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} , \tag{15}$$

where $\mathbf{E} = (E_x, E_y, E_z)$ and $\mathbf{B} = (B_x, B_y, B_z)$ are the electric and magnetic field vectors respectively.

A real and monochromatic, plane electromagnetic wave traveling along the x-axis can be described by the following vector potential:

$$A^\mu = (0, 0, 1, 1) \cos(\omega t - kx) , \tag{16}$$

where $\omega = |k|$. The electromagnetic field tensor corresponding to such a vector potential is then given by

$$F_{\mu\nu} = \begin{pmatrix} 0 & 0 & \omega & \omega \\ 0 & 0 & -k & -k \\ -\omega & k & 0 & 0 \\ -\omega & k & 0 & 0 \end{pmatrix} \times \cos(\omega t - kx) . \tag{17}$$

In the frame of a uniformly accelerating observer whose world line is given by the transformations (6), the electromagnetic field tensor transforms to

$$F'_{\mu\nu} = \begin{pmatrix} 0 & 0 & \omega e^{-g\tau} & \omega e^{-g\tau} \\ 0 & 0 & ke^{-g\tau} & -ke^{-g\tau} \\ -\omega e^{-g\tau} & -ke^{-g\tau} & 0 & 0 \\ -\omega e^{-g\tau} & ke^{-g\tau} & 0 & 0 \end{pmatrix} \times \cos\big(\omega t(\tau) - kx(\tau)\big)\,. \quad (18)$$

Notice that the acceleration of the observer is along the same axis as that of the direction of propagation of the wave. Let us now assume that the electromagnetic wave is traveling to the right, i.e. $k = \omega$. Then, from the above equation it can be easily seen that the all the transformed components of the field tensor are of the following form:

$$F'(\tau) = \pm\omega e^{-g\tau}\cos\big[\omega\big(t(\tau) - x(\tau)\big)\big]\,. \quad (19)$$

Fourier transforming $F'(\tau)$ with respect to the proper time of the uniformly accelerating observer, we obtain

$$\tilde{F}'(\Omega) = \pm\left(\frac{\Omega}{2g}\right)e^{-i\phi}\big(e^{-(\Omega/4\Omega_0)}e^{-i\beta} - e^{(\Omega/4\Omega_0)}e^{i\beta}\big)\Gamma\big(i\Omega g^{-1}\big)\,, \quad (20)$$

where ϕ, Ω_0 and β are given by Eq. (9). The resulting power spectrum per unit logarithmic interval in frequency, viz. $\mathcal{P}(\Omega) \equiv \big(\Omega|\tilde{F}'(\Omega)|^2\big)$, is then given by

$$\mathcal{P}(\Omega) = \left(\frac{\pi}{g}\right)\Omega^2\left\{\frac{1}{2} + N - \sqrt{N(N+1)}\cos(2\beta)\right\}, \quad (21)$$

where N is given by Eq. (11). In the limit of $\omega \to 0$ this power spectrum reduces to

$$\mathcal{P}(\Omega)|_{\omega\to 0} = \left(\frac{\pi}{g}\right)\Omega^2\left\{\frac{1}{2} + N - \sqrt{N(N+1)}\right\}. \quad (22)$$

Thus, even in the case of the electromagnetic field, the power spectrum is well defined in the limit of $\omega \to 0$.

The power spectrum per unit logarithmic frequency interval obtained above has a factor Ω^2 multiplying the term in braces which was absent in Eq. (10). This extra factor has a simple explanation. The power spectrum for the scalar field of the form given in Eq. (2) in the Minkowski frame is just a constant independent of ω. The analogous spectrum per logarithmic frequency interval for the scalar field in the Rindler frame given in Eq. (10) has no term dependent on Ω multiplying the term in braces. In contrast, the power spectrum for the electromagnetic field is $\propto \omega^2$ in the Minkowski frame. Extrapolating the result for the scalar field, one sees that the power spectrum of the electromagnetic wave (per unit logarithmic frequency interval) in the Rindler frame should be $\propto \Omega^2$. And, this is exactly what we have obtained in Eq. (21).

3. Generalization to Other Field Configurations

In the last section, we have carried out our analysis for real Minkowski waves that were traveling to the right. It is straight forward to verify that the same power spectrum can be obtained for left moving waves, i.e. when $k = -\omega$.

A more general case is as follows. Consider a function of $\Phi(t-x)$ that satisfies the Klein–Gordon equation and is either odd or even in $(t-x)$. Such a function $\Phi(t-x)$, which will represent a wave packet that is traveling along the x axis, can be Fourier decomposed into the following form:

$$\Phi(t-x) = \int_{-\infty}^{\infty} d\alpha\, f(\alpha) \exp i\alpha(t-x)\,. \tag{23}$$

The function $f(\alpha)$ will prove to be odd or even depending on whether $\Phi(t-x)$ is odd or even. Substituting the transformation equations (6) in (23) and Fourier transforming as before with respect to the proper time of the Rindler observer, we obtain

$$\tilde{\Phi}(\Omega) = g^{-1}\Gamma(i\Omega g^{-1})\big(e^{(\Omega/4\Omega_0)}F_1(\Omega) \pm e^{-(\Omega/4\Omega_0)}F_2(\Omega)\big)\,, \tag{24}$$

where the plus sign is to be chosen if $\Phi(t-x)$ is an even function and the minus sign if $\Phi(x-t)$ is an odd function [Ω_0 is given by (9)]. The distributions $F_1(\Omega)$ and $F_2(\Omega)$ are described by the integrals

$$F_1(\Omega) = \int_0^{\infty} d\alpha\, f(\alpha) e^{i\alpha(t_0-x_0)} \exp -\big(i\Omega g^{-1}\ln(g^{-1}\alpha)\big) \tag{25}$$

and

$$F_2(\Omega) = \int_0^{\infty} d\alpha\, f(\alpha) e^{-i\alpha(t_0-x_0)} \exp -\big(i\Omega g^{-1}\ln(g^{-1}\alpha)\big)\,. \tag{26}$$

Now, we obtain

$$\begin{aligned}\mathcal{P}(\Omega) &\equiv \Omega|\tilde{\Phi}(\Omega)|^2\\ &= \left(\frac{\pi}{g\sinh(\Omega/2\Omega_0)}\right)\big\{e^{(\Omega/2\Omega_0)}|F_1(\Omega)|^2\\ &\quad + e^{-(\Omega/2\Omega_0)}|F_2(\Omega)|^2 \pm \big(F_1^*(\Omega)F_2(\Omega) + F_1(\Omega)F_2^*(\Omega)\big)\big\}\,.\end{aligned} \tag{27}$$

This spectrum, of course, does not have a thermal nature since it depends explicitly on the form of $f(\alpha)$.

But a simplification occurs if we treat $f(\alpha)$ as a stochastic variable so that when averaged over an ensemble of realizations, it satisfies the relation

$$\langle f(\alpha)\, f^*(\alpha')\rangle = P(\alpha)\,\delta(\alpha-\alpha')\,, \tag{28}$$

with some power spectrum $P(\alpha)$, such that $\int_{-\infty}^{\infty} d\alpha\, P(\alpha) = 2C$. In such a case, when $|F_1(\Omega)|^2$ and $|F_2(\Omega)|^2$ are averaged over the stochastic variable $f(\alpha)$, both reduce to a constant independent of Ω, i.e.

$$\langle|F_1(\Omega)|^2\rangle = \langle|F_2(\Omega)|^2\rangle = \int_0^{\infty} d\alpha\, P(\alpha) = C\,. \tag{29}$$

The power spectrum (27), when it is averaged over the stochastic variable $f(\alpha)$ is then given by

$$\langle \mathcal{P}(\Omega) \rangle = \left(\frac{4\pi C}{g} \right) \left\{ \frac{1}{2} + N \pm \sqrt{N(N+1)} \cos(2\beta') \right\}, \tag{30}$$

where β' is a function of $(t_0 - x_0)$ and is defined by the relation

$$\begin{aligned} \cos(2\beta') &= \left(\frac{1}{2C} \right) \langle F_1^*(\Omega) F_2(\Omega) + F_1(\Omega) F_2^*(\Omega) \rangle \\ &= \left(\frac{1}{C} \right) \int_0^\infty d\alpha \; P(\alpha) \cos[2\alpha(t_0 - x_0)] \, . \end{aligned} \tag{31}$$

So a stochastic wave field in the Minkowski frame will also reproduce all the three terms in the power spectrum obtained earlier.

The wave field described above did not have explicit random phases. It is possible to define a different random field in the following way. Consider a random superposition of real modes for the scalar field:

$$\Phi(t, x) = \int_{-\infty}^{\infty} d\omega \; A(\omega) \cos[\omega(t - x) + \theta(\omega)] \, , \tag{32}$$

where $A(\omega)$ is a stochastic variable satisfying the relation

$$\langle A(\omega) A(\omega') \rangle = \bar{P}(\omega) \delta(\omega - \omega') \, , \tag{33}$$

and $\bar{P}(\omega)$ is an arbitrary function of ω such that $\bar{C} = \int_{-\infty}^{\infty} d\omega \bar{P}(\omega)$ is a finite constant. Further, we shall assume that $\theta(\omega)$ is a random phase factor distributed uniformly in the range $(0, 2\pi)$. We can now set $t_0 = x_0 = 0$ in (6) without any loss of generality. Substituting the coordinate transformations (6) in the scalar field configuration given by (32) and Fourier transforming the same with respect to the proper time of the uniformly accelerated observer, we obtain

$$\begin{aligned} \tilde{\Phi}(\Omega) &= \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\omega \; A(\omega) \cos\left(\omega[t(\tau) - x(\tau)] + \theta(\omega) \right) e^{-i\Omega\tau} \\ &= \int_{-\infty}^{\infty} d\omega \; A(\omega) \int_{-\infty}^{\infty} d\tau \cos\left(\omega g^{-1} e^{-g\tau} - \theta(\omega) \right) e^{-i\Omega\tau} \\ &= \left(\frac{1}{2g} \right) \Gamma(i\Omega g^{-1}) \int_{-\infty}^{\infty} d\omega \; A(\omega) e^{-i\phi} \left(e^{-(\Omega/4\Omega_0)} e^{-i\theta(\omega)} + e^{(\Omega/4\Omega_0)} e^{i\theta(\omega)} \right), \end{aligned} \tag{34}$$

where ϕ and Ω_0 are given by (9). The power spectrum per logarithmic frequency interval, viz. the quantity $(\Omega |\tilde{\Phi}(\Omega)|^2)$, when it is averaged over the stochastic variables $A(\omega)$ and $\theta(\omega)$ reduces to

$$\langle \mathcal{P}(\Omega) \rangle = \left(\frac{\pi \bar{C}}{g} \right) \left\{ \frac{1}{2} + N \right\}. \tag{35}$$

In this case, the random phases have averaged out the fluctuation term, viz. the factor $\sqrt{N(N+1)}$ that had appeared in the power spectrum (10). A somewhat similar result was obtained earlier by Boyer.[7] He modeled the zero-point fluctuations as due to random superposition of Minkowski plane wave modes, and used it as a basis for investigating the "spectrum" observed by a uniformly accelerating observer. He showed that the correlation function of an accelerating observer "in a random classical scalar zero-point radiation" exactly matches the correlation function of an inertial observer in a thermal background. Our analysis here shows that the effect reported by Boyer arises when a random superposition of Minkowski real modes are simply Fourier analyzed in the frame of a uniformly accelerating observer [cf. Eq. (35) above]. *But notice that, such an approach has killed a very interesting* $\sqrt{N(N+1)}$ *term which was originally present.*

Finally, we discuss a case in which the observer is moving in a direction perpendicular to the wave vector. Consider an observer who is uniformly accelerating along the y axis, i.e. in a direction perpendicular to which the plane wave is traveling (which we always take to be the x-axis). If the proper acceleration of the observer is g, then the coordinate transformations to the uniformly accelerated frame are given by

$$t = t_0 + g^{-1}\sinh(g\tau)\,, \quad x = x\,, \quad y = y_0 + g^{-1}\cosh(g\tau) \quad \text{and} \quad z = z\,. \tag{36}$$

Substituting these transformations in the Fourier transform (5), we obtain

$$\begin{aligned}\tilde{\Phi}(\Omega) &= \int_{-\infty}^{\infty} d\tau \cos\left(\omega\left(t_0 + g^{-1}\sinh(g\tau)\right) - kx\right)e^{-i\Omega\tau} \\ &= g^{-1}K_{(i\Omega/g)}\left(\omega g^{-1}\right)\left(e^{-(\Omega/4\Omega_0)}e^{-i(\omega t_0 - kx)} + e^{(\Omega/4\Omega_0)}e^{i(\omega t_0 - kx)}\right),\end{aligned} \tag{37}$$

where $K_{(i\Omega/g)}$ is the Bessel function of imaginary order. The resulting power spectrum

$$\begin{aligned}\mathcal{P}(\Omega) &\equiv \Omega\left|\tilde{\Phi}(\Omega)\right|^2 \\ &= 2\Omega g^{-2}\left|K_{(i\Omega/g)}\left(\omega g^{-1}\right)\right|^2\left\{\cosh(\Omega/2\Omega_0) + \cos[2(\omega t_0 - kx)]\right\} \\ &= 4\Omega g^{-2}\sinh(\Omega/2\Omega_0)\left|K_{(i\Omega/g)}\left(\omega g^{-1}\right)\right|^2 \\ &\quad \times\left\{\frac{1}{2} + N + \sqrt{N(N+1)}\cos[2(\omega t_0 - kx)]\right\}\end{aligned} \tag{38}$$

does not have a thermal nature because of the coefficients multiplying the expression in the curly brackets. Therefore, "thermal" ambience arises only for observers whose acceleration is along the same axis as the direction of propagation of the wave.

It is however interesting to ask: What happens to the power spectrum (38) in the limit of $\omega \to 0$? In this limit, the original wave field is a constant and hence any direction of motion for the observer should be equivalent. Hence we expect to see the "thermal" ambience in this limit even for this observer. This is indeed the case: In the limit of $\omega \to 0$,

$$K_{(i\Omega g^{-1})}(\omega g^{-1}) \approx 2^{(i\Omega g^{-1}-1)}(\omega g^{-1})^{-(i\Omega g^{-1})}\Gamma(i\Omega g^{-1}) \\ - 2^{(-i\Omega g^{-1}-1)}(\omega g^{-1})^{(i\Omega g^{-1})}\Gamma(-i\Omega g^{-1}) . \qquad (39)$$

Substituting the above approximation for $K_{(i\Omega g^{-1})}(\omega g^{-1})$ in (38) one recovers the result given in Eq. (10) with β set to zero (but with a factor 2 multiplying the expression). This result also holds for a wave propagating in an arbitrary direction, as is to be expected.

4. Conclusions

It will be interesting to investigate whether the power spectrum we have evaluated in the last two sections can be measured physically. We shall present here a model of a detector that is capable of measuring the Fourier spectrum of the classical field with respect to its proper time.

By a detector we have in mind a pointlike object which nevertheless has internal degrees of freedom. We shall also assume that the world line of the detector is given to us *a priori* and does not form a part of the dynamics. One such detector would be a simple harmonic oscillator that is coupled directly to the components of the classical field through a linear coupling. If the internal degree of freedom of the oscillator is q, then the interaction Lagrangian between the field and the detector would be of the form qF, where F is one of the components of the classical field. Varying the total action of the detector and the field with respect to the degree of freedom q, we find that the equation of motion satisfied by the harmonic oscillator q is given by

$$\frac{d^2q}{d\tau^2} + \alpha^2 q = F'(\tau) , \qquad (40)$$

where α is the frequency of the oscillator and $F'(\tau)$ is the component of the classical field in the frame of the oscillator. The total energy gained by any forced harmonic oscillator is proportional to the modulus square of the Fourier transform of the driving force. Therefore, the total energy ε absorbed by the harmonic oscillator that is coupled to the field F is then given by

$$\varepsilon(\alpha) = \left| \int_{-\infty}^{\infty} d\tau \, F'(\tau) e^{-i\alpha\tau} \right|^2 . \qquad (41)$$

Consider, for instance, a simple harmonic oscillator, say, a bound electric charge, that is coupled to the y-component of the electric field. Let us assume that the electric field is the plane electromagnetic wave we have considered in Subsec. 2.2. Let us also assume that the harmonic oscillator is accelerating uniformly described by worldline (6). We saw in Sec. 2.2 that in the frame of the uniformly accelerating observer the y-component of the electromagnetic wave is given by

$$E'_y(\tau) = \omega e^{-g\tau} E_y(\tau) = \omega e^{-g\tau} \cos\left(\omega t(\tau) - kx(\tau)\right) . \qquad (42)$$

The energy gained by such an oscillator due to its interaction with the plane electromagnetic wave is then given by

$$\begin{aligned}\varepsilon(\alpha) &= \left|\int_{-\infty}^{\infty} d\tau\ E'_y(\tau)e^{-i\alpha\tau}\right|^2 \\ &= \left(\frac{\pi\alpha}{g}\right)\left\{\frac{1}{2} + N + \sqrt{N(N+1)}\cos(2\beta)\right\}, \end{aligned} \tag{43}$$

where N is given by the equation

$$N(\alpha) = \left(\frac{1}{\exp(\alpha/\Omega_0) - 1}\right) \tag{44}$$

and Ω_0 and β are as in Eq. (9). Thus, the power spectrum we have evaluated in Secs. 2 and 3 can, in principle, be measured physically.

In conclusion, we would like to stress those aspects of our results which are unexpected and contrast them with those which could have been anticipated with some hindsight.

To begin with, the following fact is well known: In quantum field theory, the amplitude for transition of an Unruh–DeWitt detector, up to the first order in perturbation theory, is described by an integral that is similar in form to Eq. (3).[2,3] When the scalar field is decomposed in terms of the Minkowski modes, the transition probability, per unit proper time, of a uniformly accelerating Unruh–DeWitt detector turns out to be a thermal spectrum (see, for instance, Ref. 8). It might, therefore, seem that when a traveling wave is Fourier transformed with respect to the proper time of a uniformly accelerated observer, the resulting power spectrum will have a thermal nature.

However, there are some subtlities involved. To begin with, the modes of the quantum field are complex while here we are dealing with real plane wave modes. This makes the vital difference. As we have mentioned before, while a complex mode like $\exp -i(\omega t - kx)$ will give a Planckian distribution it will *not* yield the two other terms we have obtained in our analysis. In this sense, the real wave is quite different from the complex one. We stress the fact that, when a real Minkowski mode is Fourier transformed with respect to the proper time of a uniformly accelerating observer, the resulting power spectrum not only contains a Planckian distribution but also contains the root-mean-square fluctuations about the Planckian. As mentioned earlier, it is the appearance of these fluctuations that motivates us to attribute a "thermal" nature to the power spectrum. *We know of no simple way to guess at this answer.*

Secondly, note the effect survives in the power spectrum even in the limit of $\omega \to 0$. This is the closest to what one can call a "classical" vacuum — and our result shows that such a mode, with infinitesimal frequency, leads to a "thermal" ambience in the accelerated frame which is *totally independent of the properties of* *the original wave.* This result suggests that there is a deep connection between plane waves, accelerated frames and thermal fluctuations even at the classical level. This connection could be worth exploring.

A somewhat similar analysis, viz. Fourier analyzing the Minkowski modes in the frame of an uniformly accelerated observer, was carried out earlier by Gerlach.[9] He had constructed a linear superposition of Minkowski modes in $(3 + 1)$ dimensions such that the modulus square of the amplitude of these modes (which represents the total classical energy of these modes) to be equivalent to that of the ground state energy of a quantum oscillator. Fourier analyzing such a field configuration with respect to the proper time of a uniformly accelerating observer, Gerlach had obtained a power spectrum (in a particular "semiclassical" limit) similar in form to Eq. (10). He had presented his result as a "heuristic derivation of the thermal spectrum" that arises in quantum field theory due to the inequivalent quantization in Minkowski and Rindler coordinates.

Our results and emphasis are different in several ways. To begin with, the effect we are reporting here is a feature of classical field theory and no quantum processes are involved. It is physically motivated in a clear and simple manner and we do not have to resort to any superposition of modes. Secondly, our results are *exact* for a real, monochromatic plane wave while Gerlach needed to resort to an approximation because of the particular superposition of modes he had chosen. Thirdly, we would like to draw attention to the zero-frequency limit of the wave, when it takes a life of its own in the accelerated frame. This result, as far as we know, has not been noted in the literature before. Finally, Gerlach had offered no explanation for the appearance of the factor $\cos(2\beta)$ as the coefficient of the fluctuation term. Our analysis clearly shows that it arises due to the shift in the origin of the Minkowski coordinates.

It would be interesting to investigate whether the result obtained here is merely a mathematical curiosity or whether it has deeper physical significance. Work in this aspect is in progress.

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Appendix. A Note on the Limiting Behavior

In this appendix, we shall illustrate as to how the Fourier amplitude in the accelerated frame reduces to that of the Minkowski amplitude as $g \to 0$.

The set of transformations given by Eq. (6) are those that are usually used in literature to relate the Rindler and the Minkowski coordinates. As we have mentioned earlier, because of a g-dependent shift these transformations do not have a sensible limit as $g \to 0$. Therefore, in the $g \to 0$ limit, it is better to use the following set of transformations:

$$\begin{aligned} t &= t_0 + g^{-1}\sinh(g\tau)\,, \\ x &= x_0 + g^{-1}(\cosh(g\tau) - 1)\,, \quad y = y \quad \text{and} \quad z = z\,, \end{aligned} \tag{A.1}$$

which differ from the original set by an $(1/g)$ shift. With this new set of transformations, it can be easily shown that our results in the Rindler frame will reduce to the results in the Minkowski coordinates as $g \to 0$. We shall outline the analysis below.

These new transformations, when used in place of the old ones, merely change $\beta \to \bar{\beta} = (\beta + \omega g^{-1})$. We shall first calculate the Fourier amplitude $\tilde{\Phi}(\Omega)$ (described by the integral (5)) with g set to zero in these new set of transformations. We shall then evaluate the $g \to 0$ limit of $\tilde{\Phi}(\Omega)$ in the accelerated frame. We shall show that both give the same result.

In the limit of $g \to 0$, the transformations (A.1) reduce to

$$t = t_0 + \tau\,, \quad x = x_0\,, \quad y = y \quad \text{and} \quad z = z\,. \tag{A.2}$$

The Fourier amplitude $\tilde{\Phi}(\Omega)$ for this set of transformations is then given by

$$\begin{aligned} \tilde{\Phi}(\Omega)|_{g=0} &= \int_{-\infty}^{\infty} d\tau\, e^{-i\Omega\tau} \cos\left[\omega\big(t(\tau) - x(\tau)\big)\right] \\ &= \int_{-\infty}^{\infty} d\tau\, e^{-i\Omega\tau} \cos[\omega(t_0 - x_0) + \omega\tau] \\ &= \pi\big\{e^{i\omega(t_0 - x_0)}\delta_D(\Omega - \omega) + e^{-i\omega(t_0 - x_0)}\delta_D(\Omega + \omega)\big\}\,, \end{aligned} \tag{A.3}$$

where δ_D is the Dirac delta function. Now, for the new set of transformations (A.1), we find that $\tilde{\Phi}(\Omega)$ in the accelerated frame is given by

$$\tilde{\Phi}(\Omega) = \left(\frac{1}{2g}\right) e^{-i\phi}\big(e^{-(\Omega/4\Omega_0)}e^{-i\bar{\beta}} + e^{(\Omega/4\Omega_0)}e^{i\bar{\beta}}\big)\Gamma(i\Omega g^{-1})\,, \tag{A.4}$$

where

$$\phi = \Omega g^{-1}\ln(\omega g^{-1})\,, \quad \Omega_0 = g/2\pi \quad \text{and} \quad \bar{\beta} = \omega(t_0 - x_0 + g^{-1})\,. \tag{A.5}$$

In what follows, we shall illustrate as to how Eq. (A.4) does reduce to Eq. (A.3) in the limit of $g \to 0$. We shall first argue as to how $\lim_{g\to 0} \tilde{\Phi}(\Omega)$ reduces to zero when $\Omega \neq \pm\omega$. We shall also argue that $\lim_{g\to 0} \tilde{\Phi}(\Omega) = \infty$ when $\Omega = \pm\omega$. Then, we shall show that the integral of $\tilde{\Phi}(\Omega)$ over Ω is the same as the integral of $\tilde{\Phi}(\Omega)|_{g=0}$ over Ω.

The asymptotic expansion of $\Gamma(z)$ for large $|z|$ is given by[5]

$$\Gamma(z) \approx \sqrt{2\pi} z^{(z-1/2)} e^{-z}\,. \tag{A.6}$$

Substituting (A.5) and (A.6) in Eq. (A.4), we obtain that

$$\tilde{\Phi}(\Omega)|_{g\to 0} \approx \left(\frac{\pi}{2i\Omega g}\right)^{1/2} \left(\frac{\Omega}{\omega}\right)^{i\Omega g^{-1}} e^{-(\pi\Omega/2g)}\, e^{-i\Omega g^{-1}}$$
$$\times \left(e^{-(\pi\Omega/2g)} e^{-i\omega g^{-1}} e^{-i\beta} + e^{(\pi\Omega/2g)} e^{i\omega g^{-1}} e^{i\beta}\right)$$
$$= \left(\frac{\pi}{2i\Omega g}\right)^{1/2} \left\{ \left(\frac{-\Omega}{\omega}\right)^{i\Omega g^{-1}} e^{-i(\Omega+\omega)g^{-1}} e^{-i\beta}\right.$$
$$\left. + \left(\frac{\Omega}{\omega}\right)^{i\Omega g^{-1}} e^{-i(\Omega-\omega)g^{-1}} e^{i\beta}\right\}, \tag{A.7}$$

and, in obtaining the last equality, we have assumed that $e^{i\pi} \equiv (-1)$. Also, in the limit of $g \to 0$, we should assume that exponential factors $\exp -[i(\Omega \pm \omega)g^{-1}]$ appearing in the above expression contain an $i\epsilon$ term (where $\epsilon \to 0^+$) to ensure proper convergence. Or in other words, we should assume that these exponential factors are actually given by $\exp -[i(\Omega \pm \omega - i\epsilon)g^{-1}]$. Keeping this in mind, it is easy to see that, when $g \to 0$

$$\tilde{\Phi}(\Omega)|_{g\to 0} \to \begin{cases} 0 & \text{when} \quad \Omega \neq -\omega \quad \text{and} \quad \Omega \neq \omega\,, \\ \infty & \text{when} \quad \Omega = -\omega \quad \text{or} \quad \Omega = \omega\,. \end{cases} \tag{A.8}$$

To prove the second part, we shall first integrate $\tilde{\Phi}(\Omega)|_{g=0}$ [given by Eq. (A.3)] with respect to Ω. We obtain

$$I = \int_{-\infty}^{\infty} d\Omega\, \tilde{\Phi}(\Omega)|_{g=0} = \pi\left(e^{i\omega(t_0 - x_0)} + e^{-i\omega(t_0 - x_0)}\right). \tag{A.9}$$

Let us now integrate Eq. (A.4) (in the limit of $g \to 0$) with respect to Ω. Consider the first term that appears within the brackets in Eq. (A.4). Substituting the following integral representation of $\Gamma(z)$,[5]

$$\Gamma(z) = i^{-z} \int_0^{\infty} dt\, t^{z-1} e^{it} \tag{A.10}$$

in the first term in (A.4) and then integrating over Ω, we obtain

$$I_1 = \lim_{g\to 0} \left(\frac{1}{2g}\right) e^{-i\bar{\beta}} \int_{-\infty}^{\infty} d\Omega\, e^{-i\phi} e^{-(\Omega/4\Omega_0)} \Gamma(i\Omega g^{-1})$$
$$= \lim_{g\to 0} \left(\frac{1}{2g}\right) e^{-i\bar{\beta}} \int_{-\infty}^{\infty} d\Omega (\omega g^{-1})^{-i\Omega g^{-1}} e^{-(\pi\Omega/2g)} i^{-i\Omega g^{-1}} \int_0^{\infty} dt\, t^{(i\Omega g^{-1}-1)} e^{it}\,. \tag{A.11}$$

Changing the variable $t \to t' = (gt/\omega)$ and also interchanging the order of integration over Ω and t, we get

$$\begin{aligned} I_1 &= \lim_{g\to 0} \left(\frac{1}{2g}\right) e^{-i\bar{\beta}} \int_0^\infty \frac{dt'}{t'} e^{i\omega t' g^{-1}} \int_{-\infty}^{\infty} d\Omega (t')^{i\Omega g^{-1}} \\ &= \lim_{g\to 0} \left(\frac{1}{2g}\right) e^{-i\bar{\beta}} \int_0^\infty \frac{dt'}{t'} e^{i\omega t' g^{-1}} 2\pi\, \delta_D(g^{-1}\ln t') \\ &= \lim_{g\to 0} \pi e^{-i\bar{\beta}} e^{i\omega g^{-1}} = \pi e^{-i\omega(t_0 - x_0)} \,. \end{aligned} \qquad \text{(A.12)}$$

The second term in Eq. (A.4) can similarly be integrated using the following representation of the Gamma function [5]

$$\Gamma(z) = i^z \int_0^\infty dt\, t^{z-1} e^{-it} \,. \qquad \text{(A.13)}$$

Substituting this in Eq. (A.4) and then integrating over Ω, we find that

$$\begin{aligned} I_2 &= \lim_{g\to 0} \left(\frac{1}{2g}\right) e^{i\bar{\beta}} \int_{-\infty}^{\infty} d\Omega\; e^{-i\phi} e^{(\Omega/4\Omega_0)} \Gamma(i\Omega g^{-1}) \\ &= \lim_{g\to 0} \left(\frac{1}{2g}\right) e^{i\bar{\beta}} \int_{-\infty}^{\infty} d\Omega (\omega g^{-1})^{-i\Omega g^{-1}} e^{(\pi\Omega/2g)} i^{i\Omega g^{-1}} \\ &\quad \times \int_0^\infty dt\; t^{(i\Omega g^{-1} - 1)} e^{-it} \,. \end{aligned} \qquad \text{(A.14)}$$

I_2 can be evaluated exactly along the same lines as I_1 with the result

$$I_2 = \lim_{g\to 0} \pi e^{i\bar{\beta}} e^{-i\omega g^{-1}} = \pi e^{i\omega(t_0 - x_0)} \,. \qquad \text{(A.15)}$$

Therefore

$$\int_{-\infty}^{\infty} d\Omega\; \tilde{\Phi}(\Omega)|_{g\to 0} = I_1 + I_2 = \pi\left(e^{-i\omega(t_0 - x_0)} + e^{i\omega(t_0 - x_0)}\right), \qquad \text{(A.16)}$$

which is the same as Eq. (A.9). Thus, our analysis in this appendix clearly shows that, in the limit of $g \to 0$, $\tilde{\Phi}(\Omega)$ in Eq. (A.4) reduces to $\tilde{\Phi}(\Omega)|_{g=0}$ given by (A.3).

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