

# When a mass term does not represent a mass

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## Abstract

The definition of mass of a scalar field in a curved space has often been generalized by grouping coupling terms between the field and the Ricci curvature with non-curvature-related mass terms. In a broader point of view, one sees that a common misunderstanding resulting from such an identification leads one, in the case of the spin 2 field, to regard the cosmological constant as a non-vanishing mass of cosmological origin for the graviton. Similarly, there are inconsistencies for the spin 1 field. Instead, the intrinsic mass of a field should be regarded as being independent of the background curvature.

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# 1 Introduction

The generalization of the Klein–Gordon equation for a scalar field  $\phi$  on a curved spacetime [1]

$$\square\phi - m^2\phi - \xi R\phi = 0 \tag{1.1}$$

presents the possibility of a non-minimal coupling between the scalar field and the Ricci curvature  $R$  of spacetime. Analogously, direct couplings with the Ricci and Riemann tensors appear, respectively, in the wave equations for the electromagnetic field and for gravitational waves propagating on a curved background spacetime. Various authors have grouped the scalar field–curvature coupling term  $\xi R\phi$  with the mass term  $m^2\phi$  in Eq. (1.1), particularly when the Ricci curvature of the background spacetime is constant. In this case, an effective mass  $\mu$  given by

$$\mu^2 = m^2 + \xi R \tag{1.2}$$

is sometimes introduced. Equation (1.2) has led to problems of interpretation.

The interpretation of the terms coupling the field with the curvature as mass terms has also been extended to wave equations for fields of higher spin. In fact, Eq. (1.1) is the prototype of the wave equation on a curved spacetime, and is a guide for the study of more complicated wave equations. For the case of spin 2 fields (gravitational waves), the identification of field–curvature coupling terms with mass terms frequently led to attribute to the graviton a mass of cosmological origin related to the cosmological constant [2]–[7]. Despite some attempts [8, 5], the past and recent misunderstandings on this issue still await clarification in the literature. The interpretation of terms like  $\xi R\phi$  in Eq. (1.1) as mass terms leads to properties of the “mass”  $\mu$  which are physically unacceptable. The proper interpretation is to regard the mass of the field as being independent of the background curvature.

## 2 The Klein–Gordon field

Intuitively, one would be inclined to consider the mass of a particle as an intrinsic characteristic which does not depend on whether the particle is in flat space or in curved space. The definition (1.2) of mass of a scalar field in curved spacetime does not have this property.

Consider the de Sitter space with arbitrarily large (constant) Ricci curvature (i.e. arbitrarily large cosmological constant  $\Lambda$ ), and consider Eq. (1.1) with  $m = 0$  and  $\xi = 1/6$  in this space:

$$\square\phi - \frac{1}{6}R\phi = 0. \quad (2.3)$$

Equation (1.2) gives a value of the effective mass  $\mu$  which is arbitrarily large. However, the solutions of the wave equation (2.3) propagate on the light cones of de Sitter space. This result is well known in the investigations of Huygens' principle in curved spacetimes and of "tails" of radiation [9]–[12]. The Green function  $G(x', x)$  of Eq. (1.1) is, for a general given spacetime, the sum of two contributions; the first contribution has support on the light cone, and describes *lightlike* propagation of the waves. The second contribution has support *inside* the light cone, and describes *timelike* propagation of waves. In general, the second contribution is different from zero even when  $m = 0$  in Eq. (1.1), and is associated with *tails* (violations of Huygens' principle [9]–[12] – see [13] for a pedagogical introduction).

Equation (2.3) is conformally invariant and the de Sitter universe is conformally flat: therefore the second contribution to the Green function  $G(x', x)$  of Eq. (2.3) vanishes (no tails) and the waves propagate strictly on the light cones [14]–[17]. In other words, the tail-free propagation property experienced by scalar waves in flat space is transferred to the conformally flat de Sitter space. Therefore, we have an arbitrarily large effective mass  $\mu = (R/6)^{1/2}$ , but the solutions of Eq. (2.3) propagate on light cones. It appears reasonable to require that, whatever definition of mass is adopted in curved spacetimes, particles with nonzero mass necessarily propagate strictly *inside* the light cone. The definition given by Eq. (1.2) clearly does not satisfy this requirement.

If  $\xi \neq 1/6$ , Eq. (1.1) (with  $m = 0$ ) is not conformally invariant. It may appear that, in this case, the conclusions reached using our example with  $\xi = 1/6$  are not valid and, at most, the previous example suggests some

caution in the use of Eq. (1.2). However, the Einstein equivalence principle [18] applied to the scalar field  $\phi$  restricts the value of  $\xi$  to be  $1/6$ , a result that is now well established [19, 20, 21].

The previous example shows clearly the unphysical properties of the mass  $\mu$  defined by Eq. (1.2). Therefore, the term “mass” of the scalar field  $\phi$  in Eq. (1.1) should be used exclusively for the coefficient  $m$ , rejecting the alternative definition (1.2). In other words, the intrinsic mass is independent of the background curvature. Further support for this identification is given in the next section.

### 3 The electromagnetic field

We now consider the Maxwell field tensor  $F_{\mu\nu}$  which, in the absence of sources, satisfies the curved space Maxwell equations:

$$\nabla^\nu F_{\mu\nu} = 0 , \quad (3.1)$$

$$\nabla_{[\mu} F_{\nu\rho]} = 0 . \quad (3.2)$$

Alternatively, the photon field can be described using the four-vector potential  $A^\mu$ . In the Lorentz gauge  $\nabla^\alpha A_\alpha = 0$ ,  $A^\mu$  satisfies the wave equation

$$\square A_\mu - R_{\mu\nu} A^\nu = 0 . \quad (3.3)$$

We consider again the de Sitter universe, in which the Ricci tensor is given by  $R_{\rho\sigma} = \Lambda g_{\rho\sigma}$ . In this space, Eq. (3.3) reduces to

$$\square A_\mu - \Lambda A_\mu = 0 . \quad (3.4)$$

According to the identification underlying Eq. (1.2), one would conclude from Eq. (3.4) that the cosmological constant provides a mass for the photon field. One can actually derive a wave equation for the Maxwell tensor by applying the operator  $\nabla^\mu$  to Eq. (3.2) and using Eq. (3.1):

$$\square F_{\mu\nu} + 2R_{\alpha\mu\nu\beta} F^{\beta\alpha} + 2R_{\beta[\mu} F_{\nu]}{}^\beta = 0 . \quad (3.5)$$

In the de Sitter space one has

$$R_{\mu\nu\rho\sigma} = \frac{2\Lambda}{3} g_{\mu[\rho} g_{\sigma]\nu} , \quad (3.6)$$

which reduces Eq. (3.5) to

$$\square F_{\mu\nu} - \frac{4\Lambda}{3} F^{\mu\nu} = 0 , \quad (3.7)$$

in which a “mass term” appears, leading to a “mass”  $(4\Lambda/3)^{1/2}$  for the photon field. This conclusion is incorrect, which can be seen as follows: The Maxwell equations (3.1), (3.2) in four dimensions are conformally invariant and the de Sitter space is conformally flat. Thus, the electromagnetic field propagates on the light cone [14, 16, 17]. Since the propagation of electromagnetic waves in this case is restricted to the light cones, we know that the Maxwell field is massless according to any useful definition of mass for a field in a curved space.

The identification of the linear term in the field as a mass term, as expressed in Eq. (1.2) leads to another inconsistency. According to this identification, in the case of a spin 1 field in de Sitter spacetime, one would read a mass  $\sqrt{\Lambda}$  from Eq. (3.4) for the vector potential  $A^\mu$ . However, the wave equation (3.7) for the field  $F_{\mu\nu}$  would provide a mass  $\sqrt{4\Lambda/3}$ , so that the “mass” depends on whether one chooses to consider the potential or the field. This situation is markedly different from the case of the massive spin 1 field in Minkowski space. In flat space, the equations for the Proca 4-potential are

$$\partial^\mu A_\mu = 0 , \quad (3.8)$$

$$\square A^\mu - m^2 A^\mu = 0 , \quad (3.9)$$

where  $m$  is the mass of the Proca potential. The Proca field  $F_{\mu\nu}$  satisfies

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu , \quad (3.10)$$

$$\partial_\mu F^{\mu\nu} = -m^2 A^\nu . \quad (3.11)$$

From Eqs. (3.10) and (3.11) and from the identity  $\partial_{[\rho} F_{\mu\nu]} = 0$ , it is easy to derive a wave equation for  $F_{\mu\nu}$ :

$$\square F_{\mu\nu} - m^2 F_{\mu\nu} = 0 . \quad (3.12)$$

Equations (3.9) and (3.12) provide the same value for the mass of the Proca field, and there is no trace of the ambiguity encountered in Eqs. (3.4) and (3.7).

The presence of terms coupling the field tensor with the Riemann and Ricci tensors follows from the fact that, in the derivation of the wave equation for the field tensor  $F_{\mu\nu}$  in a curved space, one can take the combination of covariant derivatives  $\nabla_\mu \square A_\nu - \nabla_\nu \square A_\mu$  to generate  $\square F_{\mu\nu}$ . However, due to the non-commutativity of covariant derivatives, contractions of the field  $F_{\mu\nu}$  with the Riemann tensor inevitably appear in the wave equation for  $F_{\mu\nu}$  in addition to the term  $\square F_{\mu\nu}$ . By contrast, in a flat space,  $\nabla_\mu \square A_\nu - \nabla_\nu \square A_\mu$  is precisely  $\square F_{\mu\nu}$  and hence the mass which is identified from the equation for the 4-vector potential is the same as the mass that is identified from the equation for  $F_{\mu\nu}$ .

## 4 Gravitational waves

A recurring point in the literature involves the effect of the cosmological constant in determining the character of gravitational waves which propagate in curved spacetimes. It has been explicitly stated or implicitly suggested that the cosmological constant endows the graviton with a rest mass [2]–[7]. This conclusion is incorrect and, despite some attempts [8], the issue has not been clarified in the literature.

The old argument supporting the idea of the cosmological constant as endowing the graviton with a mass relies on the fact that the weak field limit of the Einstein equations with a cosmological constant produces a Yukawa (instead of a Coulombic) potential [2]–[4]. This feature has been known for many years and its astrophysical consequences have been explored [22]. A Yukawa potential is also obtained by postulating a finite range of gravitation arising from a nonvanishing graviton mass [4].

Previous claims [8, 5] against the identification of the cosmological constant as the mass of the graviton [23] proceed as follows. An exact solution

$g_{\mu\nu}^{(0)}$  of the Einstein equations with cosmological constant and no matter,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 0 \quad (4.1)$$

was considered. This solution was perturbed as

$$g_{\mu\nu} = g_{\mu\nu}^{(0)} + h_{\mu\nu} , \quad (4.2)$$

where  $h_{\mu\nu}$  ( $|h_{\mu\nu}| \ll |g_{\mu\nu}^{(0)}|$ ) describe gravitational waves. The linearization of the Einstein equations (4.1) with the gauge choice

$$\nabla^\nu \left( h_{\mu\nu} - \frac{1}{2} g_{\mu\nu}^{(0)} h \right) = 0 \quad (4.3)$$

(where  $h \equiv g^{(0)\mu\nu} h_{\mu\nu}$ ) gives

$$\square h_{\mu\nu} - 2R_{\alpha\mu\nu\beta} h^{\alpha\beta} + R_{\mu\rho} h_{\nu}{}^{\rho} + R_{\nu\rho} h_{\mu}{}^{\rho} - 2\Lambda h_{\mu\nu} = 0 . \quad (4.4)$$

The last term in the left hand side of Eq. (4.4) is susceptible to being interpreted as a mass term. However, in Ref. 8 it is noted that  $R_{\mu\rho} h_{\nu}{}^{\rho} + R_{\nu\rho} h_{\mu}{}^{\rho} = 2\Lambda h_{\mu\nu}$ . Using this result, Eq. (4.4) is reduced to

$$\square h_{\mu\nu} - 2R_{\alpha\mu\nu\beta} h^{\alpha\beta} = 0 . \quad (4.5)$$

According to Ref. 8, the substitution has removed all trace of  $\Lambda$  and the propagation equation is reduced to the same form as it would have in a spacetime free of any cosmological constant. The claim that this proves that the propagation describes a massless field on this basis is unfounded: if one proceeds to substitute the form (3.6) of the Riemann tensor, one obtains

$$\square h_{\mu\nu} - \frac{2\Lambda}{3} h_{\mu\nu} = 0 , \quad (4.6)$$

in which a “mass term” re-appears in the wave equation for gravitational waves, and one is presented with a quandary. Equation (4.6) agrees with Eq. (2.21) of Ref. 21 and is obtained by imposing (4.3) and the additional constraint  $h = 0$  (the propagation equations for  $\nabla^\nu h_{\mu\nu}$ ,  $h$  and a proof that the constraints  $\nabla^\nu h_{\mu\nu} = 0$ ,  $h = 0$  can be imposed in a globally hyperbolic spacetime can be found, e.g., in Ref. [24]).

Armed with our argument from the previous section, we recognize that the identification of the second term in the left hand side of Eq. (4.6) as a mass term is inappropriate. Thus, it is seen that the correct conclusion was reached in Ref. 8, but the proof was incomplete. While the presence of  $\Lambda$  affects the propagation of scalar, vector and tensor fields in curved spacetime, it does not endow the fields with intrinsic mass.

The connection between the cosmological constant and the mass of the graviton is excluded. Although the introduction of the quantity  $\mu$  in Eq. (1.2) may be useful from the mathematical point of view in some cases, it should not be identified with the physical mass of the field, since such an identification would lead to unacceptable physical properties for the field.

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## References

- [1] The metric signature is  $- + + +$ . Greek indices assume the values 0, 1, 2, 3, Latin indices assume the values 1, 2, 3. The Riemann tensor is given in terms of the Christoffel symbols by  $R_{\mu\nu\rho}{}^{\sigma} = \Gamma_{\mu\rho,\nu}^{\sigma} - \Gamma_{\nu\rho,\mu}^{\sigma} + \Gamma_{\mu\rho}^{\alpha}\Gamma_{\alpha\nu}^{\sigma} - \Gamma_{\nu\rho}^{\alpha}\Gamma_{\alpha\mu}^{\sigma}$ . The Ricci tensor is  $R_{\mu\rho} \equiv R_{\mu\nu\rho}{}^{\nu}$ . Round [square] brackets denote [anti]symmetrization.  $\nabla_{\mu}$  is the covariant derivative operator, and  $\square \equiv g^{\mu\nu}\nabla_{\mu}\nabla_{\nu}$ .
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