

A Direct Particle Theory of Weak Interactions.

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Summary. — A theory of weak interactions is developed in terms of direct particle action. In its simplest form the theory leads to the formulation given by Feynman and Gell-Mann.

1. - Introduction.

Of the four known interactions in physics, those of classical origin—electromagnetism and gravitation—can be formulated in terms of direct interparticle action (¹⁻⁴). The direct-action formulation of electromagnetism can be quantized and the familiar results of quantum electrodynamics can be obtained (^{5,6}). This raises the question: can similar ideas be extended to the strong and weak interactions? In the present paper we give an affirmative answer so far as the weak interaction is concerned.

The strong and weak interactions differ from the classical interactions in being of short range, and it is at first sight surprising to find a short-range interaction arising from concepts that were designed to represent long-range interactions. It will be helpful to see how an apparently long-range interaction based on the function $\delta(S^2)$ can become a short-range interaction. In the

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(2) J. A. WHEELER and R. P. FEYNMAN: *Rev. Mod. Phys.*, **21**, 424 (1949).

(3) J. E. HOGARTH: *Proc. Roy. Soc.*, A **267**, 365 (1962).

(4) F. HOYLE and J. V. NARLIKAR: *Proc. Roy. Soc.*, A **277**, 1 (1963).

(5) F. HOYLE and J. V. NARLIKAR: *Ann. of Phys.*, **54**, 207 (1969).

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previous paper we have seen how the concept of mass and the phenomenon of gravitation can be based on

$$(1) \quad \iint \tilde{G}(A, B) da db = \iint \frac{\delta(S^2_{AB})}{4\pi} da db ,$$

when the cosmological space is transformed conformally to Minkowski space. Here we have the simplest coupling of the paths of two particles a, b . The next possibility is to couple the tangents $da^i/da, db^k/db$:

$$(2) \quad \iint \frac{\delta(S^2_{AB})}{4\pi} \frac{da^i}{da} \frac{db_i}{db} da db .$$

The latter coupling gives the whole of the electromagnetic theory, leading to QED when quantized. Consider now the segments 1 to 1' and 2 to 2' of particles a and b . Couple these segments by the interaction

$$(3) \quad \frac{1}{4\pi} [\delta(S^2_{21}) + \delta(S^2_{2'1'}) - \delta(S^2_{2'1}) - \delta(S^2_{21'})] ,$$

i.e. by a criss-cross mass interaction between the ends of the segments. This interaction can be written analogously to (1) and (2):

$$(4) \quad \frac{1}{4\pi} \iint \frac{\partial^2 \delta(S^2_{AB})}{\partial a \partial b} da db .$$

We shall find that (4), given a suitable matrix structure, leads to the Feynman-Gell-Mann theory of weak interactions. The second derivative on $\delta(S^2_{AB})$ leads to $\square \delta(S^2_{AB})$ when the appropriate transition element is evaluated, and this is just $4\pi\delta_4(A, B)$. We anticipate therefore that (4) will lead to a point interaction.

To avoid complications from strong interactions we confine our attention to leptonic decay. It is possible to proceed either in terms of 4-component spinors or in terms of 2-component spinors. We prefer the latter. Since 2-component spinors are not very common in the literature we devote the next Section to definitions and properties of such spinors.

2. - Two-component spinors.

We denote a covariant spinor by $u_\alpha, \alpha = 1, 2$, and its complex conjugate by $u^{\dot{\alpha}}, \dot{\alpha} = \dot{1}, \dot{2}$. The contravariant spinors $u^\alpha, u^{\dot{\alpha}}$ are related to $u_\alpha, u_{\dot{\alpha}}$ by

$$\begin{aligned} u^\alpha &= \varepsilon^{\alpha\beta} u_\beta, & u^{\dot{\alpha}} &= \varepsilon^{\dot{\alpha}\dot{\beta}} u_{\dot{\beta}}, \\ u_\beta &= \varepsilon_{\alpha\beta} u^\alpha, & u_{\dot{\beta}} &= \varepsilon_{\dot{\alpha}\dot{\beta}} u^{\dot{\alpha}}, \end{aligned}$$

where

$$\begin{aligned}\varepsilon_{12} &= \varepsilon^{12} = \varepsilon_{i\dot{2}} = \varepsilon^{i\dot{2}} = 1, \\ \varepsilon_{21} &= \varepsilon^{21} = \varepsilon_{\dot{2}i} = \varepsilon^{\dot{2}i} = -1, \\ \varepsilon_{11} &= \varepsilon_{22} = \varepsilon_{i\dot{i}} = \varepsilon_{\dot{i}i} = 0, \\ \varepsilon^{11} &= \varepsilon^{22} = \varepsilon^{i\dot{i}} = \varepsilon^{\dot{i}i} = 0.\end{aligned}$$

The mixed spinor ε^α_β is given by

$$-\varepsilon^\alpha_\beta = \varepsilon_\beta^\alpha = \delta^\alpha_\beta,$$

where δ^α_β is the Kronecker delta. The Pauli metrics $\sigma^{i\alpha\dot{\beta}}$ satisfy the relations

$$(5) \quad \sigma^{i\alpha\dot{\beta}} \sigma^k_{\alpha\dot{\beta}} = 2\eta^{ik}, \quad \sigma^{i\alpha\dot{\beta}} \sigma_{i\gamma\dot{\sigma}} = 2\varepsilon^\alpha_\gamma \varepsilon^{\dot{\beta}}_{\dot{\sigma}}.$$

They can be represented by 2×2 matrices:

$$\sigma^{1\alpha\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^{2\alpha\dot{\beta}} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \sigma^{3\alpha\dot{\beta}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma^{4\alpha\dot{\beta}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

in which α is taken to refer to rows and $\dot{\beta}$ to columns. With this convention $\sigma^{i\alpha\dot{\beta}}$ is the same as $\sigma^{i\dot{\beta}\alpha}$. The associated quantities $\sigma_i^{\alpha\dot{\beta}}$, $\sigma_{i\alpha\dot{\beta}}$ are determined by (5) by means of the above rules for raising and lowering suffixes. We also have the relation

$$(6) \quad \sigma^{i\alpha\dot{\lambda}} \sigma^k_{\dot{\lambda}\beta} = S^{ik\alpha\beta} - \eta^{ik} \varepsilon^{\alpha\beta},$$

where S is symmetric in (α, β) and antisymmetric in (i, k) . Hence if A_i is a 4-vector,

$$(7) \quad A_i A_k \sigma^{i\alpha\dot{\lambda}} \sigma^k_{\dot{\lambda}\beta} = -A^2 \varepsilon^{\alpha\beta},$$

where $A^2 = A_i A^i$. In particular, if A_i is the gradient operator ∇_i ,

$$(8) \quad \sigma^{i\alpha\dot{\lambda}} \sigma^k_{\dot{\lambda}\beta} \nabla_i \nabla_k = -\varepsilon^{\alpha\beta} \nabla^i \nabla_i = -\varepsilon^{\alpha\beta} \square,$$

where \square is the wave operator.

If we define the operator $\nabla^{\alpha\lambda}$ by

$$(9) \quad \nabla^{\alpha\lambda} \equiv \sigma^{i\alpha\lambda} \nabla_i,$$

then (8) implies

$$(10) \quad \nabla^{\alpha\lambda} \nabla_{\lambda}^{\beta} = -\varepsilon^{\alpha\beta} \square.$$

We shall apply these relations to spinor propagators.

Let q^i denote the displacement vector from the space-time point 1 to the space-time point 2, and let \square denote the wave operator with respect to the co-ordinates of 2. Define the propagator $K(2,1)$ by

$$(11) \quad K(2,1)^{\beta\lambda} = \frac{1}{4\pi} \sigma^{i\beta\lambda} \delta(q^2)_{;i} = \frac{1}{4\pi} \nabla^{\beta\lambda} \delta(q^2).$$

Using the relation

$$(12) \quad \square \delta(q^2) = 4\pi \delta_4(2,1)$$

we get

$$(13) \quad \nabla_{\alpha\lambda} K^{\beta\lambda} = \frac{1}{4\pi} \nabla_{\alpha\lambda} \nabla^{\beta\lambda} \delta(q^2) = \delta_4(2,1) \delta^{\beta}_{\alpha}.$$

The derivatives here are all with respect to point 2. Thus K satisfies the inhomogeneous Dirac equation for the neutrino. Using the definition

$$(14) \quad q^{\beta\lambda} = q_i \sigma^{i\beta\lambda},$$

we can rewrite (11) as

$$(15) \quad K^{\beta\lambda} = 2\delta'(q^2) q^{\beta\lambda}.$$

Suppose Σ is a closed surface surrounding a volume V . Let n_i denote the unit outward normal to $d\Sigma$, a typical surface element of Σ . If Φ is a well-behaved function of position, which may be a spinor or a tensor, we have

$$(16) \quad \int n_{\alpha\lambda} \Phi d\Sigma = \int \nabla_{\alpha\lambda} \Phi dV,$$

where $n_{\alpha\lambda} = \sigma^i_{\alpha\lambda} n_i$. Let

$$(17) \quad \Phi = K^{\beta\lambda} v^{\alpha},$$

where v^{α} satisfies the neutrino equation

$$(18) \quad \nabla_{\alpha\lambda} v^{\alpha} = 0.$$

Then (13) and (18) give

$$(19) \quad \int n_{\alpha\lambda} K^{\beta\lambda} v^\alpha d\Sigma = \int (-\nabla_{\alpha\lambda} K^{\beta\lambda}) v^\alpha dV = -v^\beta.$$

That is, if $v^\beta(1)$ is given everywhere on Σ , we can determine $v^\beta(2)$ at an interior point of Σ by

$$(20) \quad v^\beta(2) = -\int_{\Sigma} K^{\beta\lambda}(2,1) n_{\alpha\lambda} v^\alpha(1) d\Sigma_1.$$

The minus sign has appeared because in the transformation of (20) to a volume integral the derivative on $K^{\beta\lambda}$ is with respect to point 1 ranging now through the interior of Σ . In (13), on the other hand, the derivative is with respect to point 2.

Suppose $\{v_n^\alpha\}$ is a complete set of solutions of (18), normalized so that

$$(21) \quad \int v_m^\lambda(1) \sigma_{\alpha\lambda}^A v_n^\alpha(1) d^3\underline{x}_1 = \delta_{mn},$$

the integration being over a time section $t = t_1 = \text{constant}$. Then for $t_2 > t_1$ we can write

$$(22) \quad K^{\beta\lambda}(2,1) = \sum_n v_n^\beta(2) v_n^\lambda(1),$$

while for $t_2 < t_1$

$$(23) \quad K^{\beta\lambda}(2,1) = 0.$$

The proof is straightforward.

We now consider a power-series expansion of $K^{\beta\lambda}(2,1)$ for $t_2 - t_1 > 0$ small. First we note that, if we consider Σ to be the hyperplane $t = t_1$, (20) gives

$$(24) \quad \lim_{t_2 \rightarrow t_1} K^{\beta\lambda}(2;1) \sigma_{\alpha\lambda}^A = \delta_3(\underline{x}_2 - \underline{x}_1) \delta_\alpha^\beta.$$

Also, multiplication of (13) by $\sigma^{A\alpha\dot{\nu}}$ gives

$$(25) \quad \frac{\partial K^{\beta\dot{\nu}}}{\partial t} = \delta_4(2,1) \sigma^{A\beta\dot{\nu}} - \sigma^{A\alpha\dot{\nu}} \sigma_{\alpha\dot{\mu}}^\lambda \frac{\partial K^{\beta\dot{\mu}}}{\partial x^\lambda}.$$

Using (24) as the initial condition we integrate (25) to get for $t_2 > t_1$

$$(26) \quad K^{\beta\dot{\nu}}(2,1) = \delta_3(\underline{x}_2 - \underline{x}_1) \sigma^{A\beta\dot{\nu}} - \sigma^{A\alpha\dot{\nu}} \sigma_{\alpha\dot{\mu}}^\lambda \int_{t_1}^{t_2} \frac{\partial K^{\beta\dot{\mu}}}{\partial x^\lambda} dt.$$

To first order in ε we have

$$(27) \quad K^{\beta\dot{\nu}}(t_1 + \varepsilon, \underline{x}_2; t_1, \underline{x}_1) = \delta_3(\underline{x}_2 - \underline{x}_1) \sigma^{\dot{\nu}} - \varepsilon \sigma^{\dot{\nu}} \sigma_{\alpha\dot{\mu}}^{\lambda} \nabla_{\lambda} \delta_3(\underline{x}_2 - \underline{x}_1) \sigma^{\dot{\mu}}$$

This method can be extended to obtain higher-order terms.

It will be useful in relation to the theory of the following Section to work out path integral transition elements, giving some explicit examples. The transition amplitude of the space-time function F is defined for the values of F on the plane $t = \text{constant}$ by

$$(28) \quad \langle w|F|v \rangle_t = \int w^{\dot{\beta}}(t, \underline{x}_1) \sigma^{\dot{\alpha}}_{\alpha\dot{\beta}} v^{\alpha}(t, \underline{x}_1) F(t, \underline{x}_1) d^3 \underline{x}_1,$$

where v, w are the initial and final wave functions. The trivial case $F = 1$ gives the transition probability amplitude

$$(29) \quad \int w^{\dot{\beta}} \sigma^{\dot{\alpha}}_{\alpha\dot{\beta}} v^{\alpha} d^3 \underline{x}_1.$$

The situation is more complicated when we ask for a derivative of F taken with respect to particle paths. The transition amplitude for dF/dt taken along paths at $t = \text{constant}$ is (cf. (7), p. 164)

$$(30) \quad \langle w | \frac{dF}{dt} | v \rangle = \lim_{\varepsilon \rightarrow 0} \int \int w^{\dot{\beta}}(t + \varepsilon, \underline{x}_2) \sigma^{\dot{\alpha}}_{\alpha\dot{\beta}} K^{\alpha\dot{\lambda}}(t + \varepsilon, \underline{x}_2; t, \underline{x}_1) \cdot \sigma^{\dot{\nu}}_{\gamma\dot{\lambda}} v^{\gamma}(t, \underline{x}_1) \left[\frac{\partial F}{\partial t} + \frac{\underline{x}_2 - \underline{x}_1}{\varepsilon} \cdot \nabla F \right] d^3 \underline{x}_1 d^3 \underline{x}_2.$$

More strictly, we should have written $v^{\gamma}(t, \underline{x}_1)$ but the difference goes to zero with ε . The derivatives $\partial F/\partial t, \partial F/\partial x^{\nu} \equiv \nabla F$ are evaluated at t, \underline{x}_1 . Consider $\partial F/\partial t$ first. If we use the first term in the expansion, (27) gives

$$(31) \quad \lim_{\varepsilon \rightarrow 0} \int \int w^{\dot{\beta}}(t + \varepsilon, \underline{x}_2) \sigma^{\dot{\alpha}}_{\alpha\dot{\beta}} \sigma^{\alpha\dot{\lambda}} \delta_3(\underline{x}_2 - \underline{x}_1) \sigma^{\dot{\nu}}_{\gamma\dot{\lambda}} \frac{\partial F}{\partial t} v^{\gamma}(t, \underline{x}_1) d^3 \underline{x}_1 d^3 \underline{x}_2 = \int w^{\dot{\beta}} \frac{\partial F}{\partial t} v^{\gamma} \sigma^{\dot{\alpha}}_{\alpha\dot{\beta}} \sigma^{\alpha\dot{\lambda}} \sigma^{\dot{\nu}}_{\gamma\dot{\lambda}} d^3 \underline{x}_1.$$

The subsequent terms in the expansion (27) make no contribution in the limit $\varepsilon \rightarrow 0$. By means of the identity

$$(32) \quad \sigma^{\dot{\alpha}}_{\alpha\dot{\beta}} \sigma^{\alpha\dot{\lambda}} \sigma^{\dot{\nu}}_{\gamma\dot{\lambda}} = \sigma^{\dot{\nu}}_{\gamma\dot{\beta}},$$

(7) R. P. FEYNMAN and A. R. HIBBS: *Quantum Mechanics and Path Integrals* (New York, 1965).

the integral (31) becomes

$$(33) \quad \int w^{\dot{\beta}} \sigma^{\epsilon}{}_{\gamma \dot{\beta}} v^{\gamma} \frac{\partial F}{\partial t} d^3 \underline{x}_1.$$

Consider next the ∇F term. Because of the $\underline{x}_2 - \underline{x}_1$ factor, the first term in the expansion (27) does not contribute. The second term gives

$$(34) \quad \int w^{\dot{\beta}} \sigma^{\epsilon}{}_{\alpha \dot{\beta}} \sigma^{\epsilon}{}_{\gamma \dot{\mu}} \sigma^{\lambda}{}_{\gamma \dot{\eta}} \sigma^{\epsilon}{}_{\xi \dot{\mu}} \sigma^{\epsilon}{}_{\eta \dot{\lambda}} v^{\epsilon} \frac{\partial F}{\partial x^{\lambda}} d^3 \underline{x}_1 = \int w^{\dot{\beta}} \sigma^{\lambda}{}_{\gamma \dot{\beta}} v^{\gamma} \frac{\partial F}{\partial x^{\lambda}} d^3 \underline{x}_1$$

after using an identity similar to (32). Combining (33) and (34) we get

$$(35) \quad \langle w | \frac{dF}{dt} | v \rangle = \int w^{\dot{\beta}} \sigma^{\epsilon}{}_{\gamma \dot{\beta}} \nabla_{\epsilon} F v^{\gamma} d^3 \underline{x}_1.$$

If in particular $F = x^j$, we have

$$(36) \quad \langle w | \frac{dx^j}{dt} | v \rangle = \int w^{\dot{\beta}} \sigma^j{}_{\gamma \dot{\beta}} v^{\gamma} d^3 \underline{x}_1$$

as the transition element of velocity.

These considerations apply to F as a scalar function. The above work can be extended trivially by writing $F^{\alpha}{}_{\lambda} = F \delta^{\alpha}{}_{\lambda}$:

$$(37) \quad \langle w | F^{\alpha}{}_{\lambda} | v \rangle = \int w^{\dot{\beta}} \sigma^{\epsilon}{}_{\alpha \dot{\beta}} F \delta^{\epsilon}{}_{\lambda} v^{\epsilon} d^3 \underline{x}_1.$$

Similarly, for a general spinor $F^{\alpha}{}_{\lambda}$ the transition element is given by the definition

$$(38) \quad \langle w | F^{\alpha}{}_{\lambda} | v \rangle = \int w^{\dot{\beta}} \sigma^{\epsilon}{}_{\alpha \dot{\beta}} F^{\epsilon}{}_{\lambda} v^{\epsilon} d^3 \underline{x}_1.$$

A similar analysis to that given above then leads to

$$(39) \quad \langle w | \frac{dF^{\alpha}{}_{\lambda}}{dt} | v \rangle = \int w^{\dot{\beta}} \sigma^{\epsilon}{}_{\gamma \dot{\beta}} \nabla_{\epsilon} F^{\gamma}{}_{\lambda} v^{\epsilon} d^3 \underline{x}_1.$$

We shall use (39) and similar expressions in formulating the theory of weak interactions.

3. - A model for weak interactions.

In this Section we show that an interaction based on (4) leads to the Feynman-Gell-Mann theory (8). The transition amplitude appropriate to the latter can be expressed in the form

$$(40) \quad A \int \int w_A^{\dot{\beta}} v_A^{\lambda} w_B^{\dot{\beta}} v_B^{\lambda} \delta_4(A, B) d^3x_A d^3x_B dt_A dt_B,$$

where v_A, v_B are the incoming (two-component) wave functions and w_A, w_B are the outgoing wave functions. A is a coupling constant. The $\delta_4(A, B)$ expresses the local character of the interaction.

We give a matrix structure to (4) in a very simple way by adding elementary spinors

$$(41) \quad \varepsilon^{\alpha_A \xi_B} \varepsilon_{\lambda_A \eta_B},$$

where the suffixes α, λ act at A and ξ, η at B . Including the coupling constant we have

$$(42) \quad \frac{A}{4\pi} \varepsilon^{\alpha_A \xi_B} \varepsilon_{\lambda_A \eta_B} \int \int \frac{d^2\delta(S^2_{AB})}{da db} da db.$$

The transition amplitude for a perturbation produced by (42) is given by a transition element of the kind (39). All we need do is use (39) both at A and B , and integrate finally with respect to t_A, t_B . The result is

$$(43) \quad \int \int w_A^{\dot{\beta}} \sigma^i_{\alpha\dot{\beta}} v_A^{\lambda} \varepsilon^{\alpha\xi} \varepsilon_{\lambda\eta} w_B^{\dot{\beta}} \sigma^k_{\xi\dot{\beta}} v_B^{\eta} [\nabla_i \nabla_k \delta(S^2_{AB})] d^3x_A d^3x_B dt_A dt_B.$$

This can be simplified since

$$(44) \quad \begin{aligned} \sigma^i_{\alpha\dot{\beta}} \sigma^k_{\xi\dot{\beta}} \varepsilon^{\alpha\xi} \varepsilon_{\lambda\eta} \nabla_i \nabla_k \delta(S^2_{AB}) &= \sigma^i_{\alpha\dot{\beta}} \sigma^{k\alpha}_{\dot{\beta}} \varepsilon_{\lambda\eta} \nabla_i \nabla_k \delta(S^2_{AB}) = \\ &= -\eta^{ik} \varepsilon_{\beta\dot{\theta}} \varepsilon_{\lambda\eta} \nabla_i \nabla_k \delta(S^2_{AB}) = \varepsilon_{\beta\dot{\theta}} \varepsilon_{\lambda\eta} \square_A \delta(S^2_{AB}) = 4\pi \varepsilon_{\beta\dot{\theta}} \varepsilon_{\lambda\eta} \delta_4(A, B). \end{aligned}$$

Besides leading immediately to

$$(45) \quad A \int \int w_A^{\dot{\beta}} v_A^{\lambda} w_B^{\dot{\beta}} v_B^{\lambda} \delta_4(A, B) d^3x_A d^3x_B dt_A dt_B$$

for the transition amplitude, (44) shows that (43) is actually symmetric between dotted and undotted suffixes, although at first sight it does not appear to be.

(8) R. P. FEYNMAN and M. GELL-MANN: *Phys. Rev.*, **109**, 193 (1958).

Thus we arrive at a local looking interaction even though our starting point was nonlocal. Analogy with QED suggests that a response of the universe changes $\delta(S^2_{AB})$ to $\delta_+(S^2_{AB})$. While this would not affect (44) it could lead to higher-order effects not present in a local theory.

● RIASSUNTO (*)

Si sviluppa una teoria delle interazioni deboli in termini della azione diretta delle particelle. Nella sua forma più semplice la teoria porta alla formulazione esposta da Feynman e Gell-Mann.

(*) *Traduzione a cura della Redazione.*

Теория слабых взаимодействий в терминах прямого действия частиц.

Резюме (*). — Развивается теория слабых взаимодействий в терминах прямого действия частиц. В простейшей форме эта теория приводит к формулировке, данной Фейнманом и Гелл-Маном.

(*) *Переведено редакцией.*