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On the Closure of the Universe : A Note on the Cosmological Constant

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ABSTRACT

If we formulate quantum cosmology respecting the causal property of the semi-classical Einsten equation, we get the result that any classical universe should admit at least one totally-geodesic spatial surface. This implies the following consequences. First, the universe should either be closed or have the negative cosmological constant. Second, if we believe in the inflationary scenario, the cosmological constant should not be positive. Combined with $\Omega_{total} = 1$, which results from inflation, and $\Omega_{baryon} \leq 0.2$, which is required from the nucleosynthesis, the above result inevitablly implies the existence of the non-baryonic dark-matter with $\Omega_{non-baryon} \geq 0.8$.

We will see below how strongly the general principles of theoretical physics restrict the possible models of our universe. We assume (a) the validity of the combination of general relativity and quantum mechanics, (b) the emergence of the semiclassical stage of the spacetime, (c) causality (the retarded nature) of the semiclassical Einstein equation, and (d) the cosmological principle.

One can construct the formulation of quantum cosmology which respects (a)-(c) by the following quantity $(h\ \phi)$,

$$(h\ \phi) = \int_{c(h,\phi)} (dg)(d\phi) \exp i(S_g + S_M) \tag{1}$$

Here, $c(h, \phi)$ denotes a class of histories of spacetime and matter along the so-called closed-time path[1], the bondary values being fixed at h_{ij} (for the spatial metric) and ϕ (for matter) (the symbol h is the abbreviation for h_{ij}).[2] Note that the requirement (c) is especially important and that it forces us to use the closed-time path formulation.[2],[3] As is explained in detail in Ref.[1]-[4], in the closed-time path formulatiuon, we take the following time-contour for the path-integral: It goes forward in time from $\tau = 0$ to $\tau = T$ ("+"-branch), changing the direction at $\tau = T$, and then goes backward in time from $\tau = T$ to $\tau = 0$ ("-"-branch). (Here, τ is a time-parameter needed for the path-integral, and T is chosen as an arbitrary, positive value.)

Now, from the assumption (b), it is reasonable to perform the stationary phase approximation for eq.(1), when we are interested in the late universe (roughly, when h corresponds to a sufficiently large spatial-geometry). Thus we get two equations, one from the "+"-branch and the other from the "-"-branch, from the 1st variation of the phase on the right-hand side of eq.(1):

$$G_{g_+} - 8\pi G/c^3\ {}_{g_-}< m|T_{g_+}|m >_{g_+} / {}_{g_-}< m|m >_{g_+} \quad , \tag{2-a}$$

$$G_{g_-} - 8\pi G/c^3\ {}_{g_-}< m|T_{g_-}|m >_{g_+} / {}_{g_-}< m|m >_{g_+} \quad , \tag{2-b}$$

where g_+ (g_-) denotes a stationary-phase history along the "+'-("-"-)branch and $|m>$ is some matter state. The suffix $g_\pm$ implies the dependence on the history

$g_\pm$. Especially, $g_\pm$ attached to inner-products, like ${}_{g_-} < m|m >_{g_+}$, indicate that $g_{/pm}$ play the role of source for matter field in the stationary-phase approximation. Now, since $G_{g_\pm}$ is real, the second term on the left-hand side should also be real, implying that $g_+ \equiv g_-$.[2],[3] Thus, eqs.$(2-a, b)$ reduce to one semiclassical Einstein equation,

$$G_{ab} = 8\pi G/c^3 < m|T_{ab}|m > . \tag{3}$$

Since we are considering the closed-time contour bent at $\tau = T$, the phase should also be stationary at $\tau = T$. This means that the same history g_+ and g_- should be joined smoothly at $\tau = T$. Hence, the "velocity" of the metric, or the extrinsic curvature K_{ij} should vanish on the $\tau = T$ spatial surface (such a surface is sometimes called a totally-geodesic surface). In this way, we have arrived at the following result: Under the assumptions (a)-(c), any classical universe should admit at least one totally-geodesic surface.

¿From this result, we can deduce several features of the global structure of the universe.

Assuming that the universe is spatially closed and that the matter satisfies a suitable energy condition, we can deduce from the above result that the spatial topology of the universe should be essentially S^3, and that there should be the initial and the final singularity. [2] Furthermore, assuming the cosmological principle (the principle (d)), and the equation of state for matter ($P/c^2 = \mu\rho$), we obtain the upperbound Λ_{crit} for the cosmological constant,[5]

$$\Lambda_{crit} = (1+3\mu)\left(\frac{2}{3\Omega}\right)^{\frac{2}{1+3\mu}}\left(\frac{3\Omega}{2} - \frac{q+1}{1+\mu}\right)^{\frac{3(1+\mu)}{1+3\mu}}(H/c)^2, \tag{4}$$

where $H := \dot{a}/a$, $\Omega := \rho/\frac{3H^2}{8\pi G}$, and $q := -\frac{\ddot{a}}{a}H^{-2}$. It should also be noted that we can derive a series of validity conditions for semiclassical gravity from eq.(1).[3]

Here, let us consider other some notable points derived from $(h\ \phi)$.

[A] **The closure of the universe.**

The fundamental formulation itself does not require us to assume the closure of the universe from the beginning, although it is convenient to assume it in order to avoid some technical complications. One among them maybe the fact that the action of the open universe becomes divergent in general. We can at least construct quantum field theory on the Minkovski space, which is non-compact. Here. let us assume that technical difficulties induced from the non-compactness of the spacetime can be overcome in some manner. Then, whether the universe is spatially closed or not turns to be a question which should be answered rather than an assumption. To make a problem clear, we assume here (d) the cosmological principle. We also assume the equation of state for matter (eq.$(5-c)$, below). Then the semiclassical universe can be described by

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} = \frac{\Lambda c^2}{3} + \frac{8\pi G}{3}\rho \ , \qquad (5-a)$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P/c^2) + \frac{\Lambda c^2}{3} \ , \qquad (5-b)$$

$$P/c^2 = \mu\rho \quad (\mu : \text{constant}) \ . \qquad (5-c)$$

Since the universe should possess a totally geodesic surface ($\dot{a} = 0$ in this case), the universe with $\Lambda \geq 0$ and $k = 0$ or -1 is excluded. The allowed universe is the one with $k = 1$ or $\Lambda < 0$.

Although it is not certain about the sign and the value of Λ, $\Lambda \geq 0$ is preferable considering the present age of the universe.[6] We can conclude that the universe should be spatially closed if the assumptions (a)-(d) and $\Lambda \geq 0$ are valid. If the inflationary scenario is correct, the universe becomes nearly flat so that it maybe difficult to determine whether $k = 1$ or not by usual observations. However, this result has a theoretical significance.

We can obtain further constraints on the global structure of the universe if we assume the inflationary scenario along with (a)-(c) (in this case, (d) can be regarded as a consequence of inflation).

[B] **Constraints on the age of the universe and the Hubble constant.**

After inflation, Ω_{total} is supposed to approach to 1. When Ω_{total} were exactly 1, i.e., when the universe is flat, the age of the universe is given by $\frac{2}{3}{H_0}^{-1}$, assuming that the matter can be treated as dust. According to the present theory, the universe shoud be a recollapsing universe, i.e. $\Omega = 1{+}0$. Thus, we get an inequality, $t_{univ} < \frac{2}{3}{H_0}^{-1}$. If we introduce the non-dimensional parameter τ and h by $t_{univ} = \tau \cdot 10^{10}$ years, $H_0 = h \cdot 100$ km/sec $\cdot$ Mpc , this implies

$$\tau h < \frac{2}{3} \quad . \tag{6}$$

Let us make a quite reasonable assumption,

$$h \geq 1/2, \quad \tau \geq 1 \quad . \tag{7}$$

Then, we get from (6) and (7),

$$1/2 \leq h < 2/3, \quad 1 \leq \tau < 4/3 \quad . \tag{8}$$

These are quite stringent constraints required to reconcile the inflationary scenario with (a)-(c).

[C] **The necessity of the non-baryonic dark-matter.**

Another result of the assumptuion of the inflationary scenario is the inevitable existence of the non-baryonic dark-matter. After inflation, we get $\Omega_{total} = 1 + 0$. It is quite reasonable to assume $\Omega_{baryon} \leq 0.2$ from the nucleosynthesis theory. However, as discussed in [A], Ω_Λ should be $\Omega_\Lambda \leq 0$. Thus models with Ω_Λ for filling up the gap between Ω_{baryon} and $\Omega_{total} = 1$ are not allowed from our point of view. This implies that there should be non-baryonic dark-matter, e.g. massive neutrinos or axions.

We have observed above that (a)-(d), and the inflationary scenario for [B] and [C], restrict possible global properties of the universe to a great extent. Although a formalism which fulfills (a)-(c) may not be unique, the present one seems to be one natural choice. If so, [A]-[C] are direct consequences. If some of the above results were not realized in the real world, it would imply that there are at least one fundamental problem in (a)-(d) or in the inflationary scenario. This would provide one motivation for a further pursuit of observations and observational cosmology.

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