The influence of the cosmological expansion on local systems

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Abstract

Following renewed interest, the problem of whether the cosmological expansion affects the dynamics of local systems is reconsidered. The cosmological correction to the equations of motion in the locally inertial Fermi normal frame (the relevant frame for astronomical observations) is computed. The evolution equations for the cosmological perturbation of the two–body problem are solved in this frame. The effect on the orbit is insignificant as are the effects on the galactic and galactic–cluster scales.

To appear in *Astrophys. J.*

1 Introduction

A recurring issue in cosmology concerns the nature and extent of the cosmological expansion. If expansion were to occur in proportion and in every minutia of detail to every element of the universe, then every clock and every measuring apparatus of distance would be altered in proportion. If in addition, the laws of physics were to remain unaltered in the process, the very concept of expansion would lose its meaning as it would be intrinsically unobservable. This is not what is being contemplated for our universe since we observe a systematic redshift of distant galaxies and hence we are able to deduce that there is an expansion in progress, at least on the cosmological scale. While the effect is actually registered in the small distances of a wavelength of light, this is simply an imprint of the expansion at the largest (Hubble) scale.

Recently (Anderson 1995; Bonnor 1996) there has been a revival of interest in the question as to whether the cosmological expansion also proceeds at smaller scales. There is a tendency to reject such an extrapolation by confusing it with the intrinsically unobservable "expansion" (let us refer to this as "pseudo-expansion") described above. By contrast, the metric of Friedman–Robertson–Walker (FRW) in general relativity is intrinsically dynamic with the increase (decrease) of proper distances correlated with red–shift (blue–shift). It does so on any scale provided the light travel time is much longer than the wave period. Thus, the cosmological metric alone does not dictate a scale for expansion and in principle, it could be present at the smallest practical scale as real– as opposed to pseudo–expansion, and observable in principle.

However, it is reasonable to pose the question as to whether there is a cut–off at which systems below this scale do not partake of the expansion. It would appear that one would be hard put to justify a particular scale for the onset of expansion. Thus, in this debate, we are in agreement with Anderson (1995) that it is most reasonable to assume that the expansion does indeed proceed at all scales. However, there is a certain ironical quality attached to the debate in the sense that even if the expansion does actually occur at all scales, we will show that the effects of the cosmological expansion on smaller spatial and temporal scales would be undetectable in general in the foreseeable future and hence one could just as comfortably hold the view that the expansion occurs strictly on the cosmological scale.

The question of whether the expansion of the universe affects local systems like clusters of galaxies or planetary systems was first raised many years ago and has received continued scrutiny (McVittie 1933; Järnefelt 1940, 1942; Pachner 1963; Dicke & Peebles 1964; Callan *et al.* 1965; Irvine 1965; Noerdlinger & Petrosian 1971) with the most recent consideration by Anderson (1995) who extends the question to the stellar scale and even below this. The recurrent attention paid to this issue indicates that to this point a definitive answer is still lacking. However, it is our sense that the prevalent perception is that the physics of systems which are small compared to the radius of curvature of the cosmological background is essentially unaffected by the expansion of the universe.

In the presence of spherical symmetry, the analysis of a spherical cavity embedded in an FRW universe is well known: as a consequence of Birkhoff's theorem, the metric inside the spherical cavity is the Minkowski one, and the physics is the same as in flat space (Einstein & Straus 1945; Schücking 1954; Dicke & Peebles 1964; Callan *et al.* 1965; Bonnor 1996). However, when spherical symmetry is absent, a satisfactory *quantitative* answer is missing in the literature, and certain statements about small systems being sheltered from the cosmological expansion recur (see e.g. Misner *et al.* 1973 and the discussion in Anderson 1995). Noerdlinger & Petrosian provide a quantitative treatment, but it is limited to the particular problem of the collapse of galaxy clusters. Anderson's (1995) paper employs the Einstein–Infeld–Hoffmann method to derive the cosmological corrections to the equations of motion of a system of particles subject to external forces. When the dynamics of a single particle are considered, the correction to the particle's acceleration is found to be proportional to the velocity of the particle. However, it is not clear how to relate the coordinates used by Anderson to the coordinates used by observers making astronomical observations. This is an important issue because the computation does not provide a coordinate–invariant quantity, but rather a correction to the 3–dimensional equations of motion, that are dependent upon the chosen coordinate system. Bonnor (1996) studies a distribution of pressureless charged dust in equilibrium between electrical repulsion and gravitational attraction, and concludes that it participates in the universal expansion.

A qualitative answer to the problem of whether local systems are affected by the expansion of the universe is easily provided if one considers the equivalence principle and its geometric formulation. Although the cosmological expansion is described by the time–dependent scale factor in the FRW metric, and we believe affects lengths at all scales, the curved spacetime manifold can be locally approximated by its (flat) tangent space at every spacetime point p. This approximation is valid only in a neighborhood $U(p)$ of the point p considered; the error involved in the approximation increases with the size of the neighborhood $U(p)$, and the approximation breaks down completely when the size of $U(p)$ becomes comparable to the radius of curvature of spacetime (the Hubble radius in the case of an FRW spacetime). From the physical point of view, the tangent space at p describes the spacetime seen by a freely falling observer in the so–called locally inertial frame (hereafter called "LIF" – see Landau & Lifshitz 1989). This frame is the one in which astronomical observations are carried out. Thus, the effect of the cosmological expansion is seen to be negligible locally and grows in significance with distance, reaching full import on the cosmological scale. This conclusion is qualitative, and is certainly well–known to most relativists but, to the best of the authors' knowledge, has yet to be well–formulated quantitatively. In earlier treatments, the coordinate systems adopted do not correspond to those used by a physical observer.

The purpose of the present paper is to provide a clear quantitative answer to the problem. The motion of a particle subject to external forces in the (approximate) LIF using Fermi normal coordinates is analyzed. It is the locally inertial frame based on a geodesic observer and it continues to be locally inertial following the observer in time. This is the frame in which astronomical observations are performed, and we compute the corrections to the dynamics due to cosmology. In this paper, we assume that homogeneous isotropic expansion is actually universal and we analyze the consequences of this assumption.

The plan of the paper is as follows: in Sec. 2 the equation of timelike geodesics in the LIF in an Einstein–de Sitter universe is investigated and the cosmological perturbations to the 3–dimensional equations of motion in the LIF are derived. In Sec. 3 the orders of magnitude of the effects in realistic astrophysical systems are estimated, and it is demonstrated that they are very small and unobservable with present and foreseeable technology. In Sec. 4 the two–body problem in the LIF is studied in detail using the correction to the equations of motion computed in Sec. 3, thus providing a solution to the evolution equations for the perturbations. It is shown that cumulative effects of the cosmological expansion on the present orbital radius of the earth and its orbital motion are essentially negligible. Section 5 contains a discussion and the conclusions.

2 Equations of motion in the LIF

In this section we find the equations of motion for a particle in the LIF using the geodesic deviation equation. We refer the reader to the Appendix for the details of the calculation.

The metric in FRW coordinates for an Einstein – de Sitter universe is given by

$$ds^2 = -dt^2 + a(t)^2 \left[dx^2 + dy^2 + dz^2\right] \ , \tag{2.1}$$

where $a(t)$ is the scale factor. Consider an observer whose world line is the geodesic $t = \tau$ and $\vec{r} = 0$. In Fermi normal coordinates, the metric on the geodesic is $g_{\mu\nu} = \eta_{\mu\nu}$ and a parallely propagated orthonormal tetrad is given by

$$\hat{e}_t^F = (1, 0, 0, 0) \qquad \hat{e}_x^F = (0, 1, 0, 0) \tag{2.2}$$

$$\hat{e}_y^F = (0,0,1,0) \qquad \hat{e}_z^F = (0,0,0,1)\,. \tag{2.3}$$

The FRW basis vectors $\vec{e}_\nu^{\,FRW}$ are related to the $\hat{e}_\mu^F$ via

$$\hat{e}_\mu^F = \Lambda_\mu{}^\nu \vec{e}_\nu^{\,FRW} \tag{2.4}$$

where Λ_μ^ν =diagonal(1,a^{-1}, a^{-1}, a^{-1}). The Riemann tensor in Fermi normal coordinates $\{x_F^\mu\}$, along the geodesic, is given by (Greek indices range from 0 to 3, Latin indices from 1 to 3)

$$R^F_{\alpha\beta\mu\nu} = \Lambda_\alpha{}^\sigma \Lambda_\beta{}^\lambda \Lambda_\mu{}^\omega \Lambda_\nu{}^\kappa R^{FRW}_{\sigma\lambda\omega\kappa} \tag{2.5}$$

(where the superscript F denotes quantities in Fermi normal coordinates) and the geodesic deviation equation

$$\frac{d^2x^k}{d\tau^2} + \Gamma^k_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} + R^k{}_{0l0}x^l = 0 \tag{2.6}$$

becomes

$$\frac{d^2x_F^k}{d\tau^2} + R^k{}_{0l0}x_F^l = 0 \tag{2.7}$$

in Fermi normal coordinates, since $\Gamma^{(F)\mu}_{\alpha\beta} = 0$. Thus, to lowest order in x^μ and $dx^\mu/d\tau$ (the order to which the geodesic deviation equation is valid), the equations of motion in Fermi normal coordinates are (see the Appendix)

$$\frac{d^2x_F^k}{d\tau^2} - \left(\frac{\ddot{a}}{a}\right)x_F^k = 0\,, \tag{2.8}$$

where a overdot denotes differentiation with respect to the comoving time t.

3 Order of magnitude estimates

In this section, the order of magnitude of the effect created by the cosmic expansion on the dynamics of local systems is estimated. Astronomical systems for which the velocities involved are non–relativistic are considered. The present value of the age of the universe is taken to be 6.3×10^{17} seconds.

Acceleration on the scale of the Solar System

It is sufficient for our purposes to use the present value of the average size of the earth-sun system, i.e. the astronomical unit r_0 of 1.5×10^{11} m and the present orbital frequency ω_0 of 2×10^{-7} sec^{-1}. From eq. (4.10) of the following section, the correction to the acceleration for this distance and frequency due to the cosmological expansion at the present matter–dominated ($a(t) \propto t^{2/3}$) epoch is

$$\delta\ddot{r} = -\frac{4r_0}{3t^4\omega_0^2} = -3.17 \times 10^{-47}\text{m/sec}^2 \,. \tag{3.1}$$

This is to be compared to the predominant gravitational acceleration of the earth towards the sun

$$g = \frac{GM_\odot}{r_0^2} = 6 \times 10^{-3}\text{m/sec}^2 \tag{3.2}$$

which completely overwhelms the effect of the cosmological expansion by 44 orders of magnitude. This is in qualitative agreement with a cruder order of magnitude estimate in Lightman *et al.* (1975).

Acceleration on the galactic scale

As an example, consider our galaxy, a spiral in which the sun is located at $r_0 = 8.5 \pm 1$ Kpc from the center and has orbital velocity $v_0 = 220 \pm 15$ Km s^{-1} (Binney & Tremaine 1987).

Thus, the gravitational acceleration of the sun towards the center of the galaxy $g = v_0^2/r_0 \simeq 1.9 \times 10^{-10}$ m/sec^2. Since the orbital period is $\simeq 2 \times 10^8$ yrs, the angular velocity ω_0 is of the order 10^{-15} sec^{-1}. Thus, the correction of the acceleration of the sun due to the cosmological expansion

$$\ddot{r} = -\frac{4r_0}{3t^4\omega_0^2} \tag{3.3}$$

is of the order 10^{-21} m/sec^2 at the present epoch which is 11 orders of magnitude smaller than the galactic g.

Acceleration on the galactic cluster scale

Assuming the core radius of a galaxy cluster ($r_0 \sim 250$ Kpc) and the line of sight velocity dispersion $\sigma \simeq 800$ Km s^{-1} (Binney & Tremaine 1987) and assuming that the galaxies at the edge of the core of the cluster are in orbit around the centre of the core with velocity σ, they are subject to the gravitational acceleration $g = (v_0)^2/r_0 \simeq 8 \times 10^{-11}$ m s^{-2}. This is to be compared with the correction due to the cosmological expansion

$$\ddot{r} = -\frac{4r_0^3}{3t^4v^2} = -5.6 \times 10^{-18}\text{m/sec}^2 \,. \tag{3.4}$$

While this is 7 orders of magnitude smaller than the galactic cluster g and thus of considerably greater relative significance than was found for the galactic and the solar system scales, it is still nevertheless essentially ignorable.

4 Cosmological corrections to the two–body problem in the LIF

The effects of the expansion of the universe on the dynamics of local systems are exemplified by the corrections induced in the two–body problem. The two–body problem in a cosmological background has been analyzed in previous papers (McVittie 1933; Dicke & Peebles 1964; Noerdlinger & Petrosian 1971; Anderson 1995) with differing results. McVittie (1933) reached the conclusion that the orbital radius stays constant for an observer using coordinates fixed in the solar system[1]. Dicke & Peebles (1964) used a conformal technique to show that the coordinate radius of the orbit decreases as the inverse of the scale factor and that the proper radius stays constant. Noerdlinger & Petrosian (1971) considered the two–body problem inside a cluster and found that, in a dust–dominated FRW universe, the time derivative of the average orbital radius obeys the equation

$$\dot{\langle r \rangle} = \frac{3\epsilon}{1+4\epsilon} H \langle r \rangle \ , \tag{4.1}$$

where ϵ is the ratio of average energy densities in the cluster and in the rest of the universe (compare eq. (8) of Noerdlinger & Petrosian 1971). According to this result, the orbital radius increases and the effect is proportional to $|H_0 x^i|$. The more recent result of Anderson (1995) agrees with that of Dicke & Peebles (1964), but in different coordinates. A comparison of all these results is rendered difficult by the different coordinate systems adopted in the different studies. Moreover, no treatment of the problem was given in the LIF, which is the frame of reference relevant for astronomical observations performed by a freely falling observer. In fact, the 3–dimensional equations of motion of a particle are not coordinate–invariant and, like the equations of motion themselves, the correction due to the cosmic expansion is dependent upon the frame employed. In this section, we apply the results obtained in Sec. 2 to compute the perturbations of the two–body problem in the LIF in an expanding, matter–dominated Einstein–de Sitter universe. For simplicity, we restrict ourselves to the case of circular orbits, in which the equation of motion for the two–body problem takes the form

$$\frac{d^2\vec{r}}{dt^2} - \frac{\ddot{a}}{a}\vec{r} = -\frac{GM}{r^2}\,\underline{e_r} \tag{4.2}$$

where M is the mass of the central object. In this section we only use quantities defined in the LIF, and we drop the subscripts. Cylindrical coordinates (r, θ, z) are used, with

[1]See Ferraris *et al.* (1996) for a modern criticism of McVittie's coordinates in astrophysical applications.

associated unit vectors $\underline{e_r}$, $\underline{e_\theta}$ and $\underline{e_z}$. Since the perturbation of the central force is also central, the motion is again confined to the unperturbed orbital plane. We consider the perturbation of the orbital coordinates r, θ given by

$$r(t) = r_0 + \delta r(t) \ , \tag{4.3}$$

$$\theta(t) = \omega_0 t + \delta\theta(t) \ . \tag{4.4}$$

Substitution into eq. (4.2) yields

$$r_0 \delta\ddot{\theta} + 2\delta\dot{r}\omega_0 = 0 \ , \tag{4.5}$$

$$\delta\dot{\theta} = -\frac{2\omega_0}{r_0}\,\delta r \ , \tag{4.6}$$

$$\delta\ddot{r} - 3\omega_0^2\delta r - 2\omega_0 r_0 \delta\dot{\theta} - \frac{\ddot{a}}{a} r_0 = 0 \tag{4.7}$$

where

$$\frac{\ddot{a}}{a} = -\frac{2}{9t^2} \tag{4.8}$$

in a matter–dominated universe. Combining eqs. (4.6)–(4.8) yields

$$\delta\ddot{r} + \omega_0^2\delta r + \frac{2}{9t^2} r_0 = 0. \tag{4.9}$$

It is easy to show that $\delta\ddot{r}$ is negligible relative to the other two terms in eq. (4.9) when t is of the order of the age of the universe and ω_0^{-1} is of the order of a year. Thus, we find that

$$r(t) \simeq r_0 \left[1 - \frac{2}{9t^2\omega_0^2}\right] \ . \tag{4.10}$$

For the earth–sun system, we take $r_0 \simeq 1.5 \times 10^{11}$ m (although strictly speaking, it is actually the value of r_0 as t approaches infinity). The value of t is taken as the age of the universe, approximately 2×10^{10} years or 6.3×10^{17} sec. The angular frequency ω_0 is taken to be for the earth–year, approximately 2×10^{-7} sec^{-1}.

From eqs. (4.6) and (4.10),

$$\delta\dot{\theta} = \frac{4}{9t^2\omega_0} \ . \tag{4.11}$$

Thus:

1) The angular velocity *decreases* with time: as t approaches infinity, ω approaches ω_0 so strictly speaking, ω_0 is actually the terminal angular velocity.

2) The orbit size *grows* with time: as t approaches infinity, r approaches r_0.
Now consider the fractional rate of change of frequency:

$$\left[\frac{\dot{\omega}}{\omega}\right]_{present} = -8.9 \times 10^{-41}\text{sec}^{-1} = -2.8 \times 10^{-33}\text{yr}^{-1} . \tag{4.12}$$

This can be compared to the observed rate of variation of the orbital period of the moon about the earth, $2.22 \pm 0.35 \times 10^{-11}$ yr^{-1}, which is larger by approximately 22 orders of magnitude. The cosmological effect is not significantly different at the birth of the solar system. For t, we would use the present time minus the age of the solar system which is still of order 10^{17} sec. Thus the rate at birth of the solar system was not significantly different.

For the fractional change in radius of the orbit, we use eq. (4.6) to find

$$\frac{\delta \dot{r}}{r_0} = -\frac{\delta \ddot{\theta}}{2\omega_0} , \tag{4.13}$$

which gives the same kind of insignificant rate of radius growth with the expansion of the universe. Over the life span of the solar system, of order 10^{17} sec, the fractional change in radius was a mere 10^{-24}.

5 Discussion and conclusions

The effect of the cosmic expansion on the dynamics of local spherically symmetric systems is well–known (Einstein & Straus 1945; Dicke & Peebles 1964; Callan *et al.* 1965; Bonnor 1996). In the non–spherical case, it is generally recognized that the expansion of the universe does not have observable effects on local physics, but few discussions of this problem in the literature have gone beyond qualitative statements. A serious problem is that these studies were carried out in coordinate systems that are not easily comparable with the frames used for astronomical observations and thus obscure the physical meaning of the computations. Moreover, different treatments lead to apparently conflicting results, as in the case of the two–body problem. This is the reason why the computations of Secs. 2 and 4, performed in the LIF, are particularly relevant to the problem. While it is reasonable to assume that the time dependence of the scale factor in the FRW metric (2.1) affects lengths at all scales in principle (see the discussion in Anderson 1995; Bonnor 1996), the magnitude of the effect in the LIF is the physically relevant one, and its computation constitutes the essential aspect of this work.

The computation of the cosmological correction to the local equations of motion performed in Sec. 2 allows one to estimate numerically the magnitude of the correction to the acceleration of a particle subject to external forces. The numerical estimates obtained in Sec. 3 suggest that the correction is extremely small and unobservable for galaxy clusters, galaxies and the solar system, and negligible for smaller systems such as stars and even more so for molecules and atoms (cf. Anderson 1995). When the cosmological correction to the local equations of motion is applied to the Newtonian two–body problem, the evolution equations for the perturbation of the orbit can be solved. It is found that the cumulative effect of cosmological expansion on the radius and angular motion of the sun–earth system is also negligible. The cosmic expansion plays an increasingly important role for systems whose sizes and lifetimes become increasingly comparable to the Hubble radius and to Hubble times respectively. In this case, the approximation used in this paper becomes invalid. It is well–known that the cosmological expansion must be taken into account, for example, in the fluid dynamical treatment of the formation of structures in the universe (Weinberg 1972). As a conclusion, it is reasonable to assume that the expansion of the universe affects all scales, but the magnitude of the effect is essentially negligible for local systems, even at the scale of galactic clusters.

Acknowledgments

We are grateful to Prof. W.B. Bonnor and Prof. E.L. Wright for helpful discussions. This research was supported, in part, by a grant from the Natural Sciences and Engineering Research Council of Canada.

Appendix

In this appendix we find the transformations from FRW coordinates to Fermi normal coordinates. We also find the metric to order $|\vec{x}|^2$ and the equations of motion to lowest order.

Consider an observer whose world line is the geodesic $r = 0$. To find the Fermi normal coordinates of a point $P = (t_{FRW}, \vec{x}_{FRW})$ we find the unique spacelike geodesic which goes through the point P and intersects $r = 0$ orthogonally. For a sufficiently small region about $r = 0$, such a unique geodesic is guaranteed to exist. Let Q be the point of intersection between $r = 0$ and this geodesic and let the geodesic parameter τ be zero at Q. The initial velocity vector $T^\mu = dx^\mu/d\tau|_{\tau=0}$ is chosen so that the geodesic reaches P at $\tau = 1$. The Fermi normal time t_F is taken to be the proper time from the initial cosmological singularity to the point Q along the geodesic $r = 0$. The Fermi normal spatial coordinates are given by the projection of T^μ onto the orthonormal triad $e^\mu_{(a)}$, where $e^\mu_{(1)}, e^\mu_{(2)}$, and $e^\mu_{(3)}$ point in the x, y, and z directions respectively.

The geodesic equation has solutions of the form

$$x^k = x^k(\tau, c_m) \qquad t = t(\tau, c_m) \tag{A.1}$$

where $\{c_m\}$ is a set of eight constants. From the above discussion we have the following conditions

$$x^k(0, c_m) = 0 \qquad t_F = t(0, c_m)\ , \tag{A.2}$$

$$\frac{\partial t}{\partial \tau}(0, c_m) = 0 \qquad x^k_F = a(t_F)\frac{\partial x^k}{\partial \tau}(0, c_m)\ . \tag{A.3}$$

This set of equations allows us to solve for $c_m = c_m(x^k_F, t_F)$. Substituting this into

$$x^k_{FRW} = x^k(1, c_m) \qquad t_{FRW} = t(1, c_m) \tag{A.4}$$

gives the required transformations

$$x^k_{FRW} = x^k_{FRW}(x^m_F, t_F) \qquad t_{FRW} = t_{FRW}(x^m_F, t_F)\ . \tag{A.5}$$

The geodesic equations are

$$\frac{d^2t}{d\tau^2} + a\dot{a}\left[\left(\frac{dx}{d\tau}\right)^2 + \left(\frac{dy}{d\tau}\right)^2 + \left(\frac{dz}{d\tau}\right)^2\right] = 0\ , \tag{A.6}$$

and

$$\frac{d^2x^k}{d\tau^2} + 2\frac{\dot{a}}{a}\frac{dt}{d\tau}\frac{dx^k}{d\tau} = 0\,. \tag{A.7}$$

From eq. (A.7) we have

$$\frac{dx^k}{d\tau} = \frac{C_1^k}{a^2}\,, \tag{A.8}$$

where the C_1^k are constants. From equation (A.6) we have

$$\frac{dt}{d\tau} = \sqrt{C_2 + \frac{|\vec{C}_1|^2}{a^2}}\,, \tag{A.9}$$

where C_2 is a constant.

We now specialize to FRW spacetimes with

$$a(t) = (\alpha t)^n \tag{A.10}$$

where α and n are constants. The above differential equations have the power series solutions

$$t(\tau) = t_0 + \sqrt{C_2 + \frac{|\vec{C}_1|^2}{(\alpha t_0)^{2n}}}\,\tau - \frac{n|\vec{C}_1|^2}{2(\alpha t_0)^{2n} t_0}\,\tau^2 + \frac{(2n+1)n|\vec{C}_1|^2}{6(\alpha t_0)^{2n} t_0^2}\sqrt{C_2 + \frac{|\vec{C}_1|^2}{(\alpha t_0)^{2n}}}\,\tau^3 + O(\tau^4)\,, \tag{A.11}$$

$$\begin{aligned} x^k(\tau) &= x_0^k + \frac{C_1^k}{(\alpha t_0)^{2n}}\,\tau - \frac{nC_1^k}{(\alpha t_0)^{2n} t_0}\sqrt{C_2 + \frac{|\vec{C}_1|^2}{(\alpha t_0)^{2n}}}\,\tau^2 \\ &\quad + \frac{nC_1^k\left[(2n+1)C_2 + (3n+1)\frac{|\vec{C}_1|^2}{(\alpha t_0)^{2n}}\right]}{3(\alpha t_0)^{2n} t_0^2}\,\tau^3 + O(\tau^4)\,, \end{aligned} \tag{A.12}$$

where t_0 and x_0^k are constants. Now $t_F = t(\tau = 0)$ gives $t_F = t_0$, $x^k(\tau = 0) = 0$ gives $x_0^k = 0$, $x_F^k = a(t_0)dx^k/d\tau|_{\tau=0}$ gives $C_1^k = (\alpha t_F)^n x_F^k$, and $dt/d\tau|_{\tau=0} = 0$ gives $C_2 = -|\vec{x}_F|^2$. Thus, using $t_{FRW} = t(\tau = 1)$ and $x_{FRW}^k = x^k(\tau = 1)$, we have

$$t_{FRW} = t_F - \frac{n|\vec{x}_F|^2}{2t_F} + \mathrm{O}(|\vec{x}_F|^4) \tag{A.13}$$

and

$$x_{FRW}^k = \frac{x_F^k}{(\alpha t_F)^n}\left[1 + \frac{n^2|\vec{x}_F|^2}{3t_F^2}\right] + \mathrm{O}(|\vec{x}_F|^4)\,. \tag{A.14}$$

Note that to lowest order $x_F^k = a(t_{FRW})x_{FRW}^k$ and $t_F = t_{FRW}$, so that to lowest order Fermi normal coordinates are just "physical" coordinates in FRW spacetime.

The spatial components of the geodesic equation, to lowest order in x^k and $\dot{x}^k$, are

$$\frac{d^2\vec{x}_F}{dt^2} - \frac{\ddot{a}}{a}\vec{x}_F = 0 \tag{A.15}$$

which is identical to (2.8). The metric in Fermi normal coordinates is

$$ds^2 = -\left[1 - \frac{n(n-1)|\vec{x}_F|^2}{t_F^2}\right] dt_F^2 + \left[\delta_{kl}\left(1 - \frac{n^2|\vec{x}_F|^2}{3t_F^2}\right) + \frac{n^2 x_k^F x_l^F}{3t_F^2}\right] dx_F^k dx_F^l \,. \tag{A.16}$$

This can be written, to lowest order in x_F, as

$$ds^2 = -(1 + R_{0l0m}^F x_F^l x_F^m)dt_F^2 - \left(\frac{4}{3}R_{0ljm}^F x_F^l x_F^m\right) dt_F dx_F^j + \left[\delta_{ij} - \frac{1}{3}R_{iljm}^F x_F^l x_F^m\right] dx_F^i dx_F^j \tag{A.17}$$

since, to lowest order, the nonzero components of the Riemann tensor are

$$R_{0x0x}^F = R_{0y0y}^F = R_{0z0z}^F = -\frac{\ddot{a}}{a} = -\frac{n(n-1)}{t_F^2} \tag{A.18}$$

and

$$R_{xyxy}^F = R_{xzxz}^F = R_{yzyz}^F = \left(\frac{\dot{a}}{a}\right)^2 = \frac{n^2}{t_F^2} \tag{A.19}$$

(plus components related to these by symmetry). This expression is identical to the metric in Fermi normal coordinates given by Manasse and Misner (1963).

References

Anderson, J. L. 1995, Phys. Rev. Lett. 75, 3602
Binney, J. & Tremaine, S. 1987, Galactic Dynamics (Princeton: Princeton University Press)
Bonnor, W. B. 1996, MNRAS 282, 1467
Callan, C. *et al.* 1965, Am. J. Phys. 33, 105
Dicke, R. H. & Peebles, P. J. E. 1964, Phys. Rev. Lett 12, 435
Einstein, A. & Straus, E. G. 1945, Rev. Mod. Phys. 17, 120
Ferraris, M. *et al.* 1996, Nuovo Cimento 111B, 1031
Irvine, W. M. 1965, Ann. Phys. (NY) 32, 322
Järnefelt, G. 1940, Ann. Acad. Sci. Fenn. Ser. A, 55, Paper 3
——— 1942, Ann. Acad. Sci. Fenn. Ser. A, 1, Paper 12
Landau L. D. & Lifshitz E. M. 1989, The Classical Theory of Fields, fourth revised edition (Oxford: Pergamon Press)
Lightman A. P., Press W. H., Price R. H. and Teukolsky S. A. 1975, Problem Book in Relativity and Gravitation (Princeton: Princeton University Press)
Manasse, F. K. and Misner, C. W. 1963, J. Math Phys. 4, 735
McVittie, G. C. 1933, MNRAS 93, 325
Misner, C. W. *et al.* 1973, Gravitation (San Francisco: Freeman)
Noerdlinger, P. D. & Petrosian, V. 1971, ApJ 168, 1
Pachner, J. 1963, Phys. Rev. 132, 1837
Schücking, E. 1954, Z. Phys. 137, 595
Van Flandern, T. C. 1975, MNRAS 170, 333
Weinberg, S. 1972, Gravitation and Cosmology (New York: J. Wiley & Sons)