# Rotating Universes.

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The first systematic approach towards handling the cosmological problem was made with the advent of General Relativity. The approach consists of laying down the form of the line element for the smoothed out universe and then applying Einstein's field equations to it to determine the behaviour of the unknown functions contained in it. The assumptions made to specify the line element include the Weyl Postulate. According to this the world lines of galaxies form a bundle of geodesics orthogonal to a family of spacelike surfaces of 3-dimensions which are labelled by a parameter $t$. This postulate therefore introduces the idea of cosmic time $t$. The next step involves assumptions about the geometry of the surfaces $t = \text{constant}$. It is simplest to assume that they are homogeneous and isotropic. This is not in serious conflict with observation, though there are some indications of anisotropy.

These assumptions lead to the Robertson-Walker line element

$$\text{(1)} \qquad \mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - a^2(t)\left\{\frac{\mathrm{d}r^2}{1-kr^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2)\right\},$$

where $a(t)$ is a function of $t$ only and $k = -1$, $0$ or $+1$. The function $a(t)$ is determined by using the field equations

$$\text{(2)} \qquad R^{ik} - \tfrac{1}{2}g^{ik}R = -KT^{ik} - \lambda g^{ik}$$

and a form of $T^{ik}$ given by

$$\text{(3)} \qquad T^{ik} = \left(p + \varrho c^2 + \frac{4}{3}u\right)\frac{\mathrm{d}x^i}{\mathrm{d}s}\frac{\mathrm{d}x^k}{\mathrm{d}s} - \left(p + \frac{u}{3}\right)g^{ik},$$

where $\varrho$, $u$ are matter and radiation densities $p$, the pressure and

$$\text{(4)} \qquad u^i = \frac{\mathrm{d}x^i}{\mathrm{d}s} = \delta^i_4$$

the flow vector of matter in co-ordinates $x^1 = r$, $x^2 = \theta$, $x^3 = \varphi$, $x^4 = t$.

Neglecting pressure and assuming matter and radiation density to be uncoupled leads to the following equation for $a(t)$:

$$a^2\dot{a}^2 = \tfrac{1}{3}\lambda c^2 a^4 - kc^2 a^2 + A\,a + B\,, \tag{5}$$

where $A$, $B$ are positive constants.

It is immediately seen that in all cases except one we get a universe of singular origin. The exceptional case is that of a universe contracting from infinity to a finite radius and then expanding again. This is obtained by a suitable choice of $\lambda$.

To avoid the singular origin two ways have been suggested. One is to modify the field equations with the introduction of creation terms. The other involves modification of the postulates about the geometry of the universe. I will be concerned here with the latter.

This method consists of replacing the postulate of isotropy by that of universal rotation. The world lines of galaxies are still geodesics along which $t$ is measured and the remaining three co-ordinates are constant. However the surfaces $t = \text{constant}$ are not now normal to the geodesics. This implies the introduction of cross-terms of the type $g_{\mu 4}\,\mathrm{d}t\,\mathrm{d}x^\mu$ $(\mu = 1, 2, 3)$ in the line element, where the geodesic postulate requires $g_{\mu 4}$ to be independent of $t$. The flow vector of matter is given as before by (4).

Define an anti-symmetric tensor $\omega_{ik}$ by

$$\omega_{ik} = \frac{1}{2}\,(u_{i;k} - u_{k;i}) = \frac{1}{2}\left(\frac{\partial g_{i4}}{\partial x^k} - \frac{\partial g_{k4}}{\partial x^i}\right). \tag{6}$$

This is analogous to the classical definition of vorticity $\boldsymbol{\omega} = \nabla \times \boldsymbol{v}$. The magnitude of angular velocity $\boldsymbol{\omega}$ is given by

$$\omega^2 = \tfrac{1}{2}\,\omega_{ik}\,\omega^{ik}\,. \tag{7}$$

Now, using an energy momentum tensor of the form

$$T^{ik} = \varrho\, u^i u^k \qquad (c = 1) \tag{8}$$

it can be shown that the quantity

$$G = (-g)^{\frac{1}{2}}\,; \qquad g = \det\,(g_{ij}) \tag{9}$$

satisfies the differential equation

$$\frac{\ddot{G}}{G} = \frac{1}{3}\left(\lambda - \frac{1}{2}\,k\varrho - \varphi^2 + 2\omega^2\right), \tag{10}$$

where

$$\varphi^2 = \frac{1}{4} g^{\mu\alpha} \dot{g}_{\alpha\lambda} g^{\lambda\nu} \dot{g}_{\nu\mu} - \frac{1}{3} \left[ \frac{\partial}{\partial t} \log \sqrt{-g} \right]^2, \tag{11}$$

vanishes in isotropic expansion. Equation (10) is due to Raychoudhari.

From (11) we see that the rotation introduces a force of the same nature as that given by the $\lambda$ term, *i.e.* one of repulsion. This helps in preventing the collapse to a point under gravitational forces. On the other hand there are forces introduced by the anisotropy of expansion which act so as to help the gravitational forces. It is not therefore immediately obvious that in general the singularity will be prevented.

Attention was first drawn towards rotating solutions by Gödel who obtained a solution of the field equations which showed uniform rotation and no shear. The metric is given by

$$ds^2 = dt^2 + 2 e^{x'} dt\, dx^2 - (dx')^2 + \tfrac{1}{2} e^{2x'} (dx^2)^2 - (dx^3)^2 . \tag{12}$$

Such a model, because of its stationary character, would not have a red shift. Even so it is of interest since it exhibits a universal angular velocity and hence does not fulfil Mach's principle.

Heckmann and Schücking have extended these ideas to nonstatic models, with a view to obtaining universes with nonsingular origin. Oscillating universes with minimum and maximum radii would be interesting from this point of view.

One of the models considered by Heckmann and Schücking has the following line element:

$$ds^2 = dt^2 + 2 e^{x'} dt\, dx^2 - c_{11}(dx')^2 - 2 c_{12} e^{x'} dx' dx^2 + \alpha c_{11} e^{2x'} (dx^2)^2 - \dot{S}^2 (dx^3)^2 , \tag{13}$$

where $c_{11}$, $c_{12}$, $s$ are functions of time and $\alpha$ = constant. For this model

$$\varrho = \frac{\alpha^2}{RS}, \qquad \omega = \frac{1}{\sqrt{2}R}, \qquad R^2 = c_{11} - \alpha c_{11}^2 - c_{12}^2, \tag{14}$$

and $c_{11}$, $c_{12}$, $S$ satisfy the equations

$$\left\{ \begin{aligned} & \frac{(\dot{c}_{11}+1)^2 + \alpha c_{11}^2 - 4\alpha c_{11}}{4R^2} + \frac{\dot{S}}{S}\left( \frac{2c_{12} + \dot{c}_{11}}{2R^2} - \frac{\dot{R}}{R} \right) = -\lambda - \frac{\alpha^2}{RS}, \\ & \dot{c}_{12} c_{11} - \dot{c}_{11} c_{12} - c_{11} = -\alpha \frac{R}{S}, \\ & \frac{\ddot{R}}{R} + \frac{\ddot{S}}{S} - \frac{1 - \alpha \dot{c}_{11}^2 - \dot{c}_{12}^2}{2R^2} = \lambda - \frac{\alpha^2}{2RS}. \end{aligned} \right. \tag{15}$$

The geometry is that of a cylinder with a cross-section of constant negative curvature at any given $t$, which rotates and shears about its axis which is parallel to $x^3$. HECKMANN and SCHÜCKING have shown that for $\alpha > 0$ this universe makes small oscillations about the Gödel universe.

However, the case $\alpha > 0$ involves an indefinite metric for the surfaces $t = \text{constant}$. This would give rise to closed time-like lines with the possibility of an observer meeting himself in the past. A more natural case would appear to be to consider $\alpha < 0$. It is then easy to show that the universe does not oscillate between finite limits. For, if it did, we could write $R_1 < c_{11} < R_2$ and then the second of (15) gives

$$(16)\qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{c_{12}}{c_{11}}\right) = \frac{1}{c_{11}} - \alpha\,\frac{R}{c_{11}^2 S} > \frac{1}{R_2}, \qquad i.e.,\ \frac{c_{12}}{c_{11}} > \frac{t - t_0}{R_2}; \qquad t_0 = \text{constant}.$$

But the signature condition requires $c_{12}^2 < -\alpha c_{11}^2$, *i.e.*; $c_{12}/c_{11} < (-\alpha)^{\frac{1}{2}}$ which is impossible with $t \to \infty$. At best this model, with suitable choice of $\lambda$, could shrink from infinity to a finite limit and then expand again. The situation is similar to that in the isotropic case.

Since anisotropy of expansion works in the opposite sense to rotation it might be profitable to consider models where this is small. An approach somewhat different to the one described above will be adopted. This consists of assuming a specific form of line element with certain unknown time-dependent functions. Since it is known that $u^i = \delta^i_4$ the flox-vector of matter is the eigenvector of $T^{ik}$ with eigenvalue $\varrho$, using (2) gives

$$(17)\qquad \left|\begin{array}{ll} & T^i_k u^k = \varrho u^i\,, \\ \dot{u}\,, & T^i_4 = \varrho\,\delta^i_4\,, \\ \dot{u}\,, & R^i_4 - \frac{1}{2} R\delta^i_4 = -K\varrho\delta^i_4 - \lambda\delta^i_4\,, \\ \dot{u}\,, & R_{i4} = (\frac{1}{2}R - K\varrho - \lambda)g_{i4}\,. \end{array}\right.$$

Eliminating $\varrho$ gives

$$(18)\qquad \frac{R_{14}}{g_{14}} = \frac{R_{24}}{g_{24}} = \frac{R_{34}}{g_{34}} = \frac{R_{44}}{g_{44}}\,.$$

These are the field equations to determine the unknown functions. Once these are determined, the eigenvalues of the Riemann tensor would lead to determination of density and pressures. Any assumed anisotropy of space would result in inequality of pressures along different directions.

Starting with a completely isotropically expanding space $t = \text{constant}$ it can be shown that no cross terms with $\mathrm{d}t\,\mathrm{d}x^\mu$ can be introduced to define $\omega$

as a function of $t$ only (to be consistent with homogeneity) such that the field eqs. (18) are also satisfied. It is therefore not possible to have isotropic expansion with rotation.

The next best case would be that of a cylindrical space rotating without twist about a direction parallel to its generators. The expansion would differ along and perpendicular to this direction. The cross-action of the cylinder can be taken to be a 2-space of constant curvature. The metric then takes the form

$$(19)\qquad ds^2 = dt^2 + \frac{2f\,dr\,dt}{\sqrt{1-kr^2}} + 2gr\,d\varphi\,dt - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\,d\varphi^2\right] - s^2(t)\,dz^2\,, \qquad (k = -1, 0, 1)\,.$$

Since we want $\omega$ to be along the $z$-direction and depending only on time, $f$, $g$ must depend on $r$, $\varphi$ only and must satisfy

$$(20)\qquad f^2 + g^2 = 1 \qquad \text{(without loss of generality)},$$

$$(21)\qquad \sqrt{1-kr^2}\,\frac{\partial}{\partial r}(gr) - \frac{\partial f}{\partial \varphi} = \alpha r \qquad (\alpha = \text{constant})\,.$$

With $x^1 = r$, $x^2 = \varphi$, $x^3 = z$, $x^4 = t$ the field eqs. (18) lead to

$$(22)\qquad \frac{\alpha f}{g} - \frac{1}{fg}\sqrt{1-kr^2}\,\frac{\partial g}{\partial r} = \beta \qquad (\beta = \text{constant})\,.$$

Equations (20), (21), (22) are compatible provided

$$(23)\qquad \alpha^2 + \beta^2 + k = 0\,.$$

(We get the same equations for $f$, $g$ by considering the Killing equations for space-like Killing vectors.)

From (23) we see that only the hyperbolic case permits the rotation. Further the singularity is not prevented in this case.

This conclusion is similar to what one would expect from a naive Newtonian argument. For in such a case if we assume a Newtonian time $t$ and Euclidean co-ordinates $(x, y, z)$ with a universal angular velocity $(0, 0, \omega)$ and expansion factors $a$, $S$ as above, we get the velocity of a galaxy relative to the observer as

$$(24)\qquad \boldsymbol{v} = \left(\frac{\dot a}{a}x,\ \frac{\dot a}{a}y,\ \frac{\dot S}{S}z\right) + \left\|\begin{matrix} 0, & 0, & \omega \\ x, & y, & z\end{matrix}\right\| = \left(\frac{\dot a}{a}x - \omega y,\ \ \frac{\dot a}{a}y + \omega x,\ \ \frac{\dot S}{S}z\right).$$

Substitution in hydrodynamic equations leads to the following equations for $a$, $S$, $\omega$, $\varrho$:

$$a^2\omega = h \text{ (constant)}, \qquad a^2 S\varrho = M \text{ (constant)},$$

$$(25) \qquad \begin{cases} \dfrac{\ddot{a}}{a} - \omega^2 + \dfrac{4\pi G\varrho}{3} = 0\,, \\ \dfrac{\ddot{S}}{S} + \dfrac{4\pi G\varrho}{3} = 0\,. \end{cases}$$

From these we see that although the centrifugal force would prevent matter from collapsing to the $z$-axis there is no such force to prevent the collapse along the $z$-axis. Thus instead of a point singularity we have a disc singularity. A similar situation would be expected to exist in more complicated velocity fields where collapse would result along the direction of the angular velocity vector.

## REFERENCES

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