International Journal of Modern Physics A, Vol. 4, No. 18 (1989) 4735–4818
© World Scientific Publishing Company

SOME FUNDAMENTAL ASPECTS OF SEMICLASSICAL AND QUANTUM GRAVITY

T. PADMANABHAN

Theoretical Astrophysics Group, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, INDIA

Received 10 November 1988

Some recent developments in the study of quantum gravity and its semiclassical limit are reviewed. The discussion includes the role of constraint equations in quantization, the definition of 'time' in the semiclassical limit, the various forms of 'backreaction' in semiclassical gravity and the role of vacuum fluctuations in quantum gravity.

Contents

1. Introduction 4736
2. Classical Gravity As A Constrained System 4737
   2.1. Classical mechanics with constraints 4737
   2.2. The electromagnetic field 4740
   2.3. The gravitational field 4744
   2.4. Constraints and dynamics of classical gravity 4745
   2.5. Two clarifications regarding constraint equations 4750
   2.6. Classical superspace and minisuperspace 4753
3. Quantum Gravity and Time 4755
   3.1. Quantization of a constrained system 4755
   3.2. Wheeler-DeWitt equation 4758
   3.3. The concept of time 4760
   3.4. Clocks and time 4762
4. Semiclassical Gravity 4763
   4.1. Semiclassical limit and WKB states 4765
   4.2. Semiclassical limit: Gaussian states 4771
   4.3. The issue of backreaction: WKB states 4774
   4.4. Backreaction with Gaussian states 4782
   4.5. Path integrals and backreaction 4782
   4.6. Critique of semi-classical gravity 4786
5. Examples from minisuperspace 4787
   5.1. Wheeler-DeWitt equation and quantum stationary geometries 4788
   5.2. Mini-superspace of conformally constant three geometries 4791
   5.3. Semiclassical limit: Examples and Issues 4793
   5.4. Initial conditions 4799
   5.5. Mach's principle, time and quantum cosmology 4799

6. Quantum Fluctuations and Gravity 4803
6.1. Zero point energy and gravity 4803
6.2. Vacuum fluctuations in the Rindler frame 4808
6.3. Observability of gravitational effects of vacuum 4812

1. Introduction

The general relativity and quantum theory represent two major intellectual achievements of this century. These two theories are also considered to be 'universally applicable' i.e. all physical systems must obey the principles of quantum theory and general relativity. It is, therefore, essential that we should be able to combine these principles into one consistent set.

There were several attempts, especially in the last twenty years or so, to merge the principles of general relativity and quantum theory. None of these attempts have really succeeded in effecting such a merger; but they have unravelled several fundamental issues related to quantum theory and gravity. In this review, we shall study some of these issues.

Why is it so difficult to combine the principles of general relativity and quantum theory? There seem to be, essentially, three different regions of conflict:

First and foremost, general relativity demands equivalence among all coordinate systems; it does not single out any preferred time coordinate or a set of observers. This is in direct contradiction with the usual description of quantum field theory. These descriptions use the concept of 'virtual quanta' of the fields to study all the interactions. To define these quanta, one has to introduce a privileged class of observers.

The situation gets worse when we attempt to quantize the gravitational interaction. It is impossible to separate the gravitational field from the geometrical aspects of space-time. Therefore, any model which quantizes the gravitational interaction has to tackle the issue of quantizing the space-time structure. In particular, the notion of a 'background space-time'—in which physical interactions take place—is no longer valid. Without a background space-time—and, in particular, without a notion of spacelike surfaces or 'time'—it is impossible to define and study the evolution of the physical systems. This is the second difficulty.

Lastly, certain conceptual issues related to quantum theory assume serious proportions when applied to the space-time structure. Quantum theory, as we understand it today is logically incomplete. The best interpretation available (the 'Copenhagen interpretation') relies on the existence of the classical observers and processes of measurement. It is impossible to apply such concepts directly to gravity. Thus, the semiclassical limit to gravity creates difficulties which are not encountered elsewhere.

This review addresses these questions in five parts, starting from Sec. 2. We give below, a brief summary of the contents and structure of the review:

Section 2 sets the stage by expressing the action for the gravitational field in the canonical (3 + 1) form. This section also describes the dynamics of systems with 'constraints'. The concept of 'constraints' is central in understanding the quantum theory of gravity. Because of this reason, the discussion in Sec. 2 is somewhat longer than what is usual in a review of this kind. Certain subtleties involved in the relationship between the reparametrization invariance and constraints are also clarified in this section.

Section 3 takes up the most peculiar feature of the quantum gravity: lack of a natural time coordinate. This problem can be tackled either by introducing some form of "internal time" or by introducing "clocks" in the space-time. The connection between these two approaches is discussed in this section and the semiclassical nature of 'time' is emphasized.

Section 4 deals with some recent advances in the study of the semi-classical limit to quantum gravity. Since it is generally believed that quantum gravity can be described—formally, at least—by the Wheeler-DeWitt equation, it should be possible to obtain the semi-classical limit of gravity by suitable manipulations of this equation. It turns out, however, that this procedure is not straightforward; several new issues come up. We study the role of WKB states and Gaussian wave packets in describing the semi-classical gravity in this section. We also discuss the problem of backreaction and its connection with the path integral approach to the semi-classical gravity.

Examples from mini-superspace are used in Sec. 5 to illustrate several aspects of quantum gravity in a concrete fashion. A simple model, consisting of a massless scalar field in a closed Friedmann Universe, allows us to clarify the ideas introduced in Secs. 3 and 4. This section also includes a discussion of Mach's principle in the context of quantum cosmology.

The last section is somewhat of a more general nature than the rest. It discusses certain major conceptual problems which are bound to arise in any attempt to put together quantum theory and gravity. The strength of the discussion lies in the fact that these difficulties are completely model independent and are of fundamental nature.

It is is impossible to do justice to all aspects of quantum gravity in any single review. This review is no exception. The subject of Euclidean quantum gravity, the issue of boundary conditions in quantum cosmology and the recent work of Coleman regarding the cosmological constant are notable omissions in this review. Fortunately, all these are covered in several other recent reviews [see Refs. 45, 58 and 59].

Some of the alternative approaches to quantum gravity are not discussed because they fall outside the basic scope of the review. This includes superstrings and Ashtekar variables [see Refs. 60 and 61].

No single approach to quantum cosmology is stressed in this review; the emphasis is more on highlighting the features common to several approaches. Details of specific approaches can also be found in several other reviews cited at various places in the text.

2. Classical Gravity As A Constrained System

2.1. *Classical mechanics with constraints*

We begin with the simplest system imaginable—a point particle in one dimension moving under a potential $V(q)$. This is described by the action:

$$S = \int_{t_1}^{t_2} dt \left\{ \frac{1}{2} \dot{q}^2 - V(q) \right\} . \tag{2.1}$$

arying $q(t)$ we get

$$\delta S = \int_{t_1}^{t_2} dt\{\dot{q}\delta\dot{q} - V'\delta q\} = \dot{q}\delta q\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} dt(\ddot{q} + V')\delta q \,. \tag{2.2}$$

If we want to obtain sensible equations of motion from the action principle "$\delta S = 0$", then we must consider variations δq which satisfy two conditions:

(i) $\delta q = 0$ at $t = t_1$, t_2 and

(ii) $\delta q(t)$ arbitrary for $t_1 < t < t_2$. For such variations we recover the Euler-Lagrange equations:

$$\ddot{q} + V' = 0 \,. \tag{2.3}$$

Note that the action principle itself suggests the nature of the variation $\delta q(t)$.

The same principle can be cast in the Hamiltonian form. We define $p = (\partial L/\partial\dot{q})$, $\dot{q} = \dot{q}(p)$ and

$$H(p, q) = \dot{q}\frac{\partial L}{\partial\dot{q}} - L = \frac{1}{2}p^2 + V(q) \tag{2.4}$$

so that

$$S = \int_{t_1}^{t_2} dt\{p\dot{q} - H\} = \int_{t_1}^{t_2} dt\left\{p\dot{q} - \frac{1}{2}p^2 - V\right\} . \tag{2.5}$$

Varying p and q as independent variables we get

$$\delta S = \int_{t_1}^{t_2} dt\{(\dot{q} - p)\delta p - (\dot{p} + V')\delta q\} + p\delta q\Big|_{t_1}^{t_2} . \tag{2.6}$$

We note that δq should vanish at $t = t_1$, t_2 but δp is entirely arbitrary. The coefficients of δp and δq in (2.6) give rise to Hamilton's equations.

So far there has been no 'constraint' on the system. To introduce the constraints, let us consider an action $S(q, N)$ with an extra degree of freedom N added[1]:

$$S = \int_{t_1}^{t_2} dt\left\{\frac{1}{2}\frac{\dot{q}^2}{N} - NV(q)\right\} = S(q, \dot{q}; N) \,. \tag{2.7}$$

Varying both q and N we get

$$\delta S = \frac{\dot{q}}{N}\delta q\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} dt\left\{\left[\frac{d}{dt}\left(\frac{\dot{q}}{N}\right) + NV'\right]\delta q + \left[\frac{\dot{q}^2}{2N^2} + V\right]\delta N\right\} . \tag{2.8}$$

We see that even though δq should vanish at end-points, δN need not. This gives rise to the following equations

$$\frac{1}{N}\frac{d}{dt}\left(\frac{dq}{Ndt}\right) + V' = 0 \tag{2.9}$$

$$\frac{\dot{q}^2}{2N^2} + V = 0\ . \tag{2.10}$$

Equation (2.10) is a "constraint" equation. It contains only first derivatives and 'constrains', for example, the initial values of $\dot{q}$, q, and N. What is more, Eq. (2.10) implies Eq. (2.9)—a fact which can be verified by differentiating (2.10).

Thus we have only one equation (2.10) for what looks like 2 degrees of freedom $\{q(t), N(t)\}$. Why is there this extra freedom?

The extra freedom is associated with a new invariance possessed by S in (2.7) called 'reparametrization invariance'. Consider the transformation

$$t \to t' = f(t)\ ; \qquad N \to N' = N\left(\frac{df}{dt}\right)^{-1} \tag{2.11}$$

under which $Ndt \to N'dt'$. The action S, in (2.7), and the equations of motion are invariant under this transformation. In other words the equations of motion will only admit solutions which depend on the variable:

$$\tau \equiv \int N\,dt\ . \tag{2.12}$$

It cannot fix $N(t)$ or $q(t)$ individually; only $q(\tau)$ can be determined. From (2.10) we see that

$$\int^{q(\tau)} \frac{dx}{\sqrt{-2V(x)}} = \tau\ . \tag{2.13}$$

[Incidentally, (2.10) has solutions only for $V(q) < 0$.]

The connection between the reparametrization invariance and the constraint can also be seen in a different way. Suppose we start with the unconstrained action (2.5), which may be looked upon as (2.7) in the "gauge" $N = 1$, and demand invariance under the reparametrization $dt \to Ndt$. Under $dt \to Ndt$, (2.5) becomes

$$S \to S' = \int dt\left\{p\dot{q} - N\left(\frac{1}{2}p^2 + V\right)\right\}\ . \tag{2.14}$$

Demanding $(\delta S'/\delta N) = 0$ is equivalent to $(\frac{1}{2}p^2 + V = 0)$ which is the same as the constraint (2.10). Thus invariance implies constraint.

The converse is also true: constraint implies invariance. To see this, we start with (2.5) and impose the constraint $\frac{1}{2}p^2 + V = 0$ via a Lagrange multiplier $\lambda(t)$. We get

$$
\begin{aligned}
S' &= S + \int dt\, \lambda(t)\left(\frac{1}{2}p^2 + V\right) \\
&= \int dt\left(p\dot{q} - \left(\frac{1}{2}p^2 + V\right) + \lambda\left(\frac{1}{2}p^2 + V\right)\right\} \\
&= \int dt\left\{p\dot{q} - N\left(\frac{1}{2}p^2 + V\right)\right\}; \qquad N \equiv (1 - \lambda) \\
&= \int dt\left\{\frac{\dot{q}}{N}\,\dot{q} - N\left(\frac{1}{2}\frac{\dot{q}^2}{N^2} + V\right)\right\} \\
&= \int dt\left\{\frac{1}{2}\frac{\dot{q}^2}{N} - NV\right\} \qquad (2.15)
\end{aligned}
$$

which is the same as (2.7).

All the above 'derivations' are based on the assumption that (q, N) are the basic variables. A very different picture emerges if we put $N = (df/dt)$ and treat (q, f) as the basic variables. This point is discussed later in Sec. (2.5).

2.2. *The electromagnetic field*

There are certain similarities between systems with the reparametrization invariance and the more familiar systems with gauge invariance. In particular, gauge invariance also leads to constraints.

To illustrate these similarities, let us consider the free electromagnetic field with the action:

$$
\begin{aligned}
S &= -\frac{1}{16\pi}\int F_{ik}F^{ik}d^4x = \frac{1}{8\pi}\int dtd^3x\{\underset{\sim}{E}^2 - \underset{\sim}{B}^2\} \\
&= \frac{1}{8\pi}\int dt \int d^3x\{(-\dot{\underset{\sim}{A}} - \nabla\phi)^2 - (\underset{\sim}{\nabla} \times \underset{\sim}{A})^2\}\,. \qquad (2.16)
\end{aligned}
$$

We vary $\underset{\sim}{A}$ and ϕ to get

$$
\begin{aligned}
\delta S &= -\frac{1}{4\pi}\int d^4x F^{ik}\partial_i\delta A_k \\
&= -\frac{1}{4\pi}\left\{\int d^3\underset{\sim}{x}F^{0\alpha}\delta A_\alpha\right\}\Bigg|_{t_1}^{t_2} + \frac{1}{4\pi}\int d^4x(\partial_i F^{ik})\delta A_k
\end{aligned}
$$

$$= \frac{1}{4\pi}\int d^3\underset{\sim}{x}\underset{\sim}{E}\cdot\delta\underset{\sim}{A}\Big|_{t_1}^{t_2} + \frac{1}{4\pi}\int d^4x(-\dot{\underset{\sim}{E}} + \nabla\times\underset{\sim}{B})\cdot\delta\underset{\sim}{A}$$

$$+ \frac{1}{4\pi}\int d^4x(\nabla\cdot\underset{\sim}{E})\delta\phi\ . \tag{2.17}$$

The last term implies that $\delta\phi$ is completely arbitrary and that $\nabla\cdot\underset{\sim}{E} = 0$ everywhere. Using this fact we see that $\delta\underset{\sim}{A}$ must be of the form $\underset{\sim}{\nabla} f$ at $t = t_1, t_2$, if the first term should vanish. This follows from the fact that:

$$\int d^3\underset{\sim}{x}\underset{\sim}{E}\cdot\nabla f = \int d^3\underset{\sim}{x}[\nabla\cdot(\underset{\sim}{E}f) - f\nabla\cdot\underset{\sim}{E}]$$

$$= \int d^3\underset{\sim}{x}\nabla\cdot(\underset{\sim}{E}f) = \int d^2\underset{\sim}{S}\cdot\underset{\sim}{E}f = 0\ . \tag{2.18}$$

Thus $\delta\phi$ is completely arbitrary everywhere and $\delta\underset{\sim}{A}$ is completely arbitrary for $t_1 < t < t_2$, but is a gradient for $t = t_1, t_2$. Therefore $\underset{\sim}{A}$ and $\underset{\sim}{A} + \delta\underset{\sim}{A}$ differ by only a gradient at $t = t_1, t_2$. Put in another way, $\nabla\times\underset{\sim}{A} = \underset{\sim}{B} = \nabla\times(\underset{\sim}{A} + \delta\underset{\sim}{A})$; the variations leave the magnetic field invariant at $t = t_1, t_2$ surfaces. For such variations we recover the equations

$$\dot{\underset{\sim}{E}} = \nabla\times\underset{\sim}{B}\ ; \qquad \nabla\cdot\underset{\sim}{E} = 0\ . \tag{2.19a,b}$$

As a bonus we see what has to be fixed at end-points[2]: it is the $(\nabla\times\underset{\sim}{A})$ or the magnetic field $\underset{\sim}{B}$. We also see that (2.19b) is a constraint equation involving only the first time derivative of the basic variables.

To see the connection between gauge invariance and the constraint equation, let us now write the action (2.16) in a specific gauge: with $\phi = 0$. Then

$$S = \frac{1}{8\pi}\int dt\, d^3\underset{\sim}{x}\{\dot{\underset{\sim}{A}}^2 - (\nabla\times\underset{\sim}{A})^2\}\ . \tag{2.20}$$

We have only $\underset{\sim}{A}$ to vary; varying $\underset{\sim}{A}$ we get

$$\delta S = -\frac{1}{4\pi}\int d^3\underset{\sim}{x}\underset{\sim}{E}\cdot\delta\underset{\sim}{A} + \frac{1}{4\pi}\int d^4x\,\delta\underset{\sim}{A}\cdot(-\dot{\underset{\sim}{E}} + \nabla\times\underset{\sim}{B}) \tag{2.21}$$

$\delta S = 0$ requires $\delta\underset{\sim}{A} = 0$ at $t = t_1$ and t_2. We see that while we recover (2.19a), the constraint Eq. (2.19b) is completely lost. This is to be expected because the constraint equation came up in (2.17) as a condition for vanishing of the coefficient of $\delta\phi$. If ϕ is not varied, then we will lose the constraint equation.

All the same, we can recover the constraint equation by explicitly demanding that S should be gauge invariant: i.e. S should be invariant under the transformation $\underset{\sim}{A} \to \underset{\sim}{A} + \underset{\sim}{\nabla} f$. This gives

$$\begin{aligned} 0 = \delta S|_{\delta \underset{\sim}{A} = \underset{\sim}{\nabla} f} &= \frac{1}{4\pi} \int dt \int d^3\underset{\sim}{x} \dot{\underset{\sim}{A}} \cdot \nabla \dot{f} \\ &\equiv -\frac{1}{4\pi} \int dt \int d^3x \underset{\sim}{E} \cdot \underset{\sim}{\nabla} g \\ &= -\frac{1}{4\pi} \int dt \int d^2\underset{\sim}{x} [\nabla \cdot (\underset{\sim}{E} g) - g \nabla \cdot \underset{\sim}{E}] \\ &= \frac{1}{4\pi} \int dt \int d^3\underset{\sim}{x} g (\nabla \cdot \underset{\sim}{E}) \end{aligned} \tag{2.22}$$

implying $\nabla \cdot \underset{\sim}{E} = 0$. Thus we can look upon the constraint equation as a consequence of the gauge invariance of S. The converse is also true: imposing the constraint by a Lagrange multiplier will ensure gauge invariance. To see this, note that the canonical momentum corresponding to $\underset{\sim}{A}$ is

$$\underset{\sim}{P} = \frac{\partial L}{\partial \dot{\underset{\sim}{A}}} = \frac{1}{4\pi} \dot{\underset{\sim}{A}} = -\frac{1}{4\pi} \underset{\sim}{E} \tag{2.23}$$

So that the 'Hamiltonian' form of the action is

$$\begin{aligned} S &= \int dt\, d^3\underset{\sim}{x} \{\underset{\sim}{P} \cdot \dot{\underset{\sim}{A}} - H(\underset{\sim}{P}, \underset{\sim}{A})\} \\ &= \int dt\, d^3\underset{\sim}{x} \left\{ \underset{\sim}{P} \cdot \dot{\underset{\sim}{A}} - \frac{1}{8\pi} (\nabla \times \underset{\sim}{A})^2 - 2\pi \underset{\sim}{P}^2 \right\} \\ &= \int dt\, d^3\underset{\sim}{x} \frac{1}{4\pi} \left\{ -\underset{\sim}{E} \cdot \dot{\underset{\sim}{A}} - \frac{1}{2} (\underset{\sim}{E}^2 + \underset{\sim}{B}^2) \right\}. \end{aligned} \tag{2.24}$$

We add a term with Lagrange multiplier $\lambda(\underset{\sim}{x}, t)$ to impose the constraint $\nabla \cdot \underset{\sim}{E} = 0$. We get

$$S' = \frac{1}{4\pi} \int dt\, d^3\underset{\sim}{x} \left\{ -\underset{\sim}{E} \cdot \dot{\underset{\sim}{A}} - \frac{1}{2} (\underset{\sim}{E}^2 + \underset{\sim}{B}^2) + \lambda(\underset{\sim}{x}, t) \nabla \cdot \underset{\sim}{E} \right\}. \tag{2.25}$$

We now write

$$\int dt \int d^3\underset{\sim}{x}\lambda\nabla\cdot\underset{\sim}{E} = \int dt\, d^3\underset{\sim}{x}[\nabla\cdot(\lambda\underset{\sim}{E}) - \underset{\sim}{E}\cdot\nabla\lambda]$$

$$= -\int dt\, d^3\underset{\sim}{x}\underset{\sim}{E}\cdot\nabla\lambda \tag{2.26}$$

getting

$$S' = \frac{1}{4\pi}\int dt\, d^3\underset{\sim}{x}\left\{-\underset{\sim}{E}\cdot(\dot{\underset{\sim}{A}} + \nabla\lambda) - \frac{1}{2}(\underset{\sim}{E}^2 + \underset{\sim}{B}^2)\right\}. \tag{2.27}$$

Variation of $\delta\underset{\sim}{E}$ identifies $\underset{\sim}{E}$ with $(-\dot{\underset{\sim}{A}} - \nabla\lambda)$ so that the Lagrangian form of S' is just

$$S' = \frac{1}{4\pi}\int dt\, d^3\underset{\sim}{x}\left\{\frac{1}{2}(-\dot{\underset{\sim}{A}} - \nabla\lambda)^2 - \frac{1}{2}(\nabla\times\underset{\sim}{A})^2\right\} \tag{2.28}$$

which is identical to (2.26).

Equation (2.22) can also be interpreted in a different manner, which is sometimes very useful. The action S', evaluated along the extremum trajectory, can be treated as a function(al) of the variables at the end point. We know that the canonical momentum can be defined as the derivative of such an S_{ext} with respect to the variables at the end-point. [In mechanics e.g. $p = (\partial S_{\text{ext}})/\partial q$.] Here, at $t = t_2$

$$\delta S_{\text{ext}} = \int d^3\underset{\sim}{x}\,\frac{\delta S_{\text{ext}}}{\delta\underset{\sim}{A}}\cdot\delta\underset{\sim}{A} = \int d^3x\cdot\underset{\sim}{P}\cdot\delta\underset{\sim}{A}$$

$$= -\frac{1}{4\pi}\int d^3x\underset{\sim}{E}\cdot\delta\underset{\sim}{A}\,. \tag{2.29}$$

We demand that $S_{\text{ext}}(\underset{\sim}{A})$ must be gauge invariant. That is $\delta S_{\text{ext}} = 0$ for $\delta\underset{\sim}{A} = \nabla f$ at $t = t_1$. This immediately gives

$$0 = \int d^3\underset{\sim}{x}\underset{\sim}{E}\cdot\delta\underset{\sim}{A} = \int d^3\underset{\sim}{x}E^\alpha\partial_\alpha f$$

$$= \int d^3\underset{\sim}{x}[\partial_\alpha(E^\alpha f) - f\partial_\alpha E^\alpha]$$

$$= -\int d^3\underset{\sim}{x}f(\underset{\sim}{x})\partial_\alpha E^\alpha \tag{2.30}$$

implying the constraint, $\nabla\cdot\underset{\sim}{E} = 0$. We will see later that a similar situation arises in the gravity.

The materials of these two sections provide the 'warming up' to study the dynamics of gravity as a constrained system. It is therefore useful to summarize the conclusions of these two sections.

We notice that constraints are signalled by the existence of basic variables in the action without their time derivative (in (2.7) N appears without $\dot{N}$ and in (2.16) ϕ appears without $\dot{\phi}$.) The variation of such a variable leads to the constraint equation (Eq. (2.10) arises from coefficient of δN in (2.8) and (2.19b) arises from coefficient of $\delta\phi$ in (2.17)). Clearly the constraint equation will only have the first time derivatives of basic variables.

The redundant variables exist in the action because of certain extra invariances possessed by the system. [Reparametrization in (2.7) and gauge transformation in (2.16).] Because of this fact, the equations of motion cannot determine all the basic variables which appear in the action. Only certain invariant combinations can be observed (like τ of (2.12) rather than q and N; or $\underset{\sim}{E}$ and $\underset{\sim}{B}$ rather than $\underset{\sim}{A}$ and ϕ).

We can use the extra invariances to set the redundant variables to some definite values (e.g. setting $N = 1$ reduces (2.7) to (2.1); setting $\phi = 0$ reduces (2.16) to (2.20).). This will not change the dynamical equations of the system (e.g. with $N = 1$, (2.9) reduces to (2.3); (2.19a) is obtained in (2.21) as well). However, this procedure makes us lose all information about the constraint equation.

We can recover the constraint equations—even in a chosen 'gauge'—if we explicitly demand the invariance of the action [e.g.: starting with S in the $N = 1$ 'gauge' and demanding invariance under $t \to t'$, $\delta S/\delta N = 0$, we recover the constraint—see (2.14); starting with $\phi = 0$ gauge, (2.20) and demanding gauge invariance of S we recover the constraint: (2.22)]. Equivalently, by starting with an action in a fixed 'gauge' and adding the constraint by a Lagrange multiplier, one can recover the full gauge invariant action. [e.g. Eq. (2.15), and Eqs. (2.25) to (2.28)].

We shall see that all these features come up in the study of gravitational action as well.

2.3. *The gravitational field*

Einstein's equations are usually derived from the action principle based on[3]

$$S = \frac{1}{16\pi G}\int R\sqrt{-g}\, d^4x + S_{\text{matter}}\,. \tag{2.31}$$

On varying g_{ik} they lead to the equations

$$R^i_k - \frac{1}{2}\,\delta^i_k R = 8\pi G T^i_k\,. \tag{2.32}$$

It may seem that these ten equations determine the ten variables $g_{ik}(\underset{\sim}{x}, t)$, provided $g_{ik}(\underset{\sim}{x}, 0)$ and $\dot{g}_{ik}(\underset{\sim}{x}, 0)$ are specified. However, this is not the case. First of all, the ten equations in (2.32) are all not independent but connected by 4-Bianchi identities ($G^i_{k;i} = 0$), leaving only $10 - 4 = 6$ independent equations. Secondly, not all equations in (2.32) are dynamical equations. The $\binom{0}{0}$ and $\binom{0}{\alpha}$ components ($\alpha = 1, 2, 3$) do not

contain any second time derivative. Further, even $\binom{\alpha}{\beta}$ equations—which contain second time derivatives—contains only $\ddot{g}_{\mu\nu}(\underline{x}, t)$ and not $\ddot{g}_{00}$ or $\ddot{g}_{0\alpha}$. Thus $\ddot{g}_{00}$ and $\ddot{g}_{0\alpha}$ never appear in (2.32).

To see these, note that the second time derivative enters R_{iklm} only through $R_{0\alpha 0\beta}$, in the form $(-\frac{1}{2}\ddot{g}_{\alpha\beta})$. Since R^i_k is formed by contraction, it cannot contain any other ($\ddot{g}_{00}$ or $\ddot{g}_{0\alpha}$) second time derivative. Next, note that the Bianchi identity can be written as

$$\left(R^0_i - \frac{1}{2}\,\delta^0_i R\right)_{;0} = -\left(R^\alpha_i - \frac{1}{2}\,\delta^\alpha_i R\right)_{;\alpha} . \tag{2.33}$$

The highest time derivative on the right-hand side is a second time derivative. Therefore, the highest time derivative on the left-hand side also should be of order 2; since there is an explicit differentiation with respect to t, $(R^0_i - \frac{1}{2}\delta^0_i R)$ can at best contain only the first time derivatives $\dot{g}_{ik}$. Furthermore, they contain only the time derivatives $\dot{g}_{\alpha\beta}$ and not $\dot{g}_{0\alpha}$ or $\dot{g}_{00}$. (Again $\dot{g}_{0\alpha}$ and $\dot{g}_{00}$ enter via $\Gamma_{\alpha,00}$ and $\Gamma_{0,00}$ only into $R_{0\alpha 0\beta}$ which do not contribute to $\binom{0}{0}$ and $\binom{0}{\alpha}$ equations.)

Thus $\binom{0}{0}$ and $\binom{0}{\alpha}$ equations contain only $(g_{00}, g_{0\alpha}, \dot{g}_{\alpha\beta})$ while $\binom{\alpha}{\beta}$ equations contain $(g_{00}, \dot{g}_{00}, g_{0\alpha}, g_{\alpha\beta}, \dot{g}_{\alpha\beta}, \ddot{g}_{\alpha\beta})$. The structure of these equations suggest that the six variables $g_{\alpha\beta}$ are genuine dynamical variables, while g_{00} and $g_{0\alpha}$ are redundant variables leading to constraints. In the variation δS_g, it is the coefficients of δg_{00} and $\delta g_{0\alpha}$ which lead to the constraints, just as the coefficient of $\delta\phi$ led to the constraint $\nabla \cdot \underline{E} = 0$ in electromagnetism. We also suspect that the constraints are related to the coordinate reparametrizations $x^i \to x^{i'}(x)$.

We shall now cast S_g in a different form in which these aspects become transparent.

2.4. *Constraints and dynamics of classical gravity*

We begin by noticing that the quantity $R\sqrt{-g}$ can be written in the form

$$R\sqrt{-g} = \sqrt{-g}\,g^{ab}(\Gamma^l_{ab}\Gamma^m_{lm} - \Gamma^l_{ak}\Gamma^k_{bl}) + \partial_a v^a \tag{2.34}$$

where

$$v^a = \sqrt{-g}\,g^{ik}\Gamma^a_{ik} - \sqrt{-g}\,g^{ia}\Gamma^l_{il} . \tag{2.35}$$

We can ignore the four-divergence $\partial_a v^a$ in studying the dynamics. The first term in (2.34) requires some algebraic transmutations which can be best performed in a specific 'gauge'. Let us choose our coordinates such that $g_{00} = 1$ and $g_{0\alpha} = 0$ making

$$ds^2 = dt^2 + g_{\alpha\beta}dx^\alpha dx^\beta . \tag{2.36}$$

We define the 'extrinsic curvature' of the t = constant hypersurface by

$$K_{\alpha\beta} \equiv n_{\alpha;\beta} = \frac{1}{2}\,\dot{g}_{\alpha\beta} \tag{2.37}$$

where $n_\alpha = n^\alpha = (1, \underset{\sim}{0})$ is the normal to the hypersurface. It is easy to verify that

$$\Gamma^0_{\alpha\beta} = \frac{1}{2} g^{00}\Gamma_{0\alpha\beta} = -\frac{1}{2}\dot{g}_{\alpha\beta} = -K_{\alpha\beta} \tag{2.38}$$

$$\Gamma^\alpha_{0\beta} = \frac{1}{2} g^{\alpha\mu}\Gamma_{\mu 0\beta} = g^{\alpha\mu}K_{\mu\beta} \equiv K^\alpha_\beta \,. \tag{2.39}$$

Therefore we can write the set $\Gamma^i_{kl} = (\Gamma^0_{\alpha\beta}, \Gamma^\alpha_{0\beta}, \Gamma^\alpha_{\beta\gamma})$ in terms of $(K_{\alpha\beta}, \Gamma^\alpha_{\beta\gamma})$. (Note that $\Gamma^0_{0i} = \Gamma^\alpha_{00} = 0$.) We now have

$$\begin{aligned} g^{ab}\Gamma^l_{ab}\Gamma^m_{lm} &= g^{\alpha\beta}\Gamma^l_{\alpha\beta}\Gamma^m_{lm} = g^{\alpha\beta}\Gamma^0_{\alpha\beta}\Gamma^m_{0m} + g^{\alpha\beta}\Gamma^\sigma_{\alpha\beta}\Gamma^m_{\sigma m} \\ &= g^{\alpha\beta}\Gamma^0_{\alpha\beta}\Gamma^\mu_{0\mu} + g^{\alpha\beta}\Gamma^\sigma_{\alpha\beta}\Gamma^\mu_{\sigma\mu} \\ &= -(\mathrm{Tr}\, K)^2 + (3\text{-space}) \end{aligned} \tag{2.40}$$

where "3-space" stands for the expression with all indices going 1, 2, 3. Similarly

$$\begin{aligned} g^{ab}\Gamma^l_{ak}\Gamma^k_{bl} &= \Gamma^l_{0k}\Gamma^k_{0l} + \Gamma^l_{\alpha k}\Gamma^k_{\beta l}g^{\alpha\beta} \\ &= \Gamma^l_{0k}\Gamma^k_{0l} + g^{\alpha\beta}\Gamma^0_{\alpha k}\Gamma^k_{\beta 0} + g^{\alpha\beta}\Gamma^\sigma_{\alpha k}\Gamma^k_{\beta\sigma} \\ &= K^\mu_\nu K^\nu_\mu + g^{\alpha\beta}(-K_{\alpha\mu})K^\mu_\beta + g^{\alpha\beta}\Gamma^\sigma_{\alpha 0}\Gamma^0_{\beta\sigma} + (3\text{-space}) \\ &= (\mathrm{Tr}\, K^2) - (\mathrm{Tr}\, K^2) + g^{\alpha\beta}K^\sigma_\alpha(-K_{\beta\sigma}) + (3\text{-space}) \\ &= -\mathrm{Tr}(K^2) + (3\text{-space}) \,. \end{aligned} \tag{2.41}$$

Substituting (2.40) and (2.41) into the first term of (2.34) we find that

$$(R\sqrt{-g} - \partial_a v^a) = \sqrt{-g}\{\mathrm{Tr}\, K^2 - (\mathrm{Tr}\, K^2)\} + (3\text{-space}) \,. \tag{2.42}$$

However, an equation with identical structure as (2.34) holds for 3-dimensional space-time as well. That is we can write (note that $\sqrt{-g} = \sqrt{-{}^3g}$ since $g_{00} = 1$, $g_{0\alpha} = 0$):

$${}^3R\sqrt{-g} = (3\text{-space}) + \partial_\mu u^\mu \tag{2.43}$$

where u^μ is same as (2.36) with indices restricted to 3-space and ${}^3R$ is scalar curvature made from $g_{\alpha\beta}$. Substituting for "3-space" from (2.43) into (2.42) we get

$$R\sqrt{-g} = \sqrt{-g}\{\mathrm{Tr}\, K^2 - (\mathrm{Tr}\, K)^2\} + \sqrt{-g}\,{}^3R + \partial_a v^a - \partial_\mu u^\mu \,. \tag{2.44}$$

Ignoring the total derivatives, we see that the Lagrangian for the gravity can be written in the form

$$L = \sqrt{-g}\{\mathrm{Tr}\, K^2 - (\mathrm{Tr}\, K)^2 + {}^3R\}\ . \tag{2.45}$$

Expanding the first two terms as

$$\begin{aligned}\mathrm{Tr}\, K^2 - (\mathrm{Tr}\, K)^2 &= g^{\alpha\mu}g^{\beta\nu}K_{\alpha\beta}K_{\mu\nu} - g^{\alpha\beta}g^{\mu\nu}K_{\alpha\beta}K_{\mu\nu} \\ &= (g^{\alpha\mu}g^{\beta\nu} - g^{\alpha\beta}g^{\mu\nu})K_{\alpha\beta}K_{\mu\nu} \\ &= \frac{1}{4}(g^{\alpha\mu}g^{\beta\nu} - g^{\alpha\beta}g^{\mu\nu})\dot{g}_{\alpha\beta}\dot{g}_{\mu\nu}\end{aligned} \tag{2.46}$$

we can write the gravitational action as

$$S_g = \frac{1}{16\pi G}\int dt\, d^3x\sqrt{-g}\left\{\frac{1}{4}(g^{\alpha\mu}g^{\beta\nu} - g^{\alpha\beta}g^{\mu\nu})\dot{g}_{\alpha\beta}\dot{g}_{\mu\nu} + {}^3R\right\} \tag{2.47}$$

showing a separation between the "kinetic" and "potential" parts. The dynamical coordinates are clearly $g_{\alpha\beta}$ and the corresponding conjugate momentum is

$$\begin{aligned}\pi^{\alpha\beta} \equiv \frac{\partial L}{\partial \dot{g}_{\alpha\beta}} &= \frac{1}{2}\sqrt{-g}(g^{\alpha\mu}g^{\beta\nu} - g^{\alpha\beta}g^{\mu\nu})\dot{g}_{\mu\nu} \\ &= \sqrt{-g}(K^{\alpha\beta} - g^{\alpha\beta}K)\ ; \qquad K = K^{\alpha}{}_{\alpha}\ .\end{aligned} \tag{2.48}$$

Inverting this relation we can find $K^{\alpha\beta}$ (and thus $\dot{g}_{\alpha\beta}$) in terms of $\pi^{\alpha\beta}$: (with $\pi \equiv \pi^{\alpha}_{\alpha}$)

$$K_{\alpha\beta} = \frac{1}{\sqrt{-g}}\left(\pi_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}\pi\right); \tag{2.49}$$

$$\begin{aligned}\dot{g}_{\alpha\beta} &= \frac{1}{\sqrt{-g}}\left(\pi_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}\pi\right); \\ &= \frac{2}{\sqrt{-g}}\left\{g_{\mu\alpha}\dot{g}_{\nu\beta}\pi^{\mu\nu} - \frac{1}{2}g_{\alpha\beta}g_{\mu\nu}\pi^{\mu\nu}\right\} \\ &= \frac{2}{\sqrt{-g}}\left\{g_{\alpha\mu}g_{\beta\nu} - \frac{1}{2}g_{\alpha\beta}g_{\mu\nu}\right\}\pi^{\mu\nu} \\ &= \frac{1}{\sqrt{-g}}\{g_{\alpha\mu}g_{\beta\nu} + g_{\alpha\nu}g_{\beta\mu} - g_{\alpha\beta}g_{\mu\nu}\}\pi^{\mu\nu} \\ &\equiv G_{\alpha\beta\mu\nu}\pi^{\mu\nu}\ .\end{aligned} \tag{2.50}$$

The tensor $G_{\alpha\beta\mu\nu}$ converts $\pi_{\alpha\beta}$ to $\dot{g}_{\alpha\beta}$. Its 'inverse' is $H_{\alpha\beta\mu\nu}$ defined via the relation:

$$\pi^{\alpha\beta} = \frac{1}{2}\sqrt{-g}(g^{\alpha\mu}g^{\beta\nu} - g^{\alpha\beta}g^{\mu\nu})\dot{g}_{\mu\nu}$$

$$= \frac{1}{4}\sqrt{-g}(g^{\alpha\mu}g^{\beta\nu} + g^{\alpha\nu}g^{\beta\mu} - 2g^{\alpha\beta}g^{\mu\nu})\dot{g}_{\mu\nu}$$

$$= H^{\alpha\beta\mu\nu}\dot{g}_{\mu\nu} \,. \tag{2.51}$$

It can be directly verified that

$$H^{\alpha\beta\mu\nu}G_{\mu\nu\rho\sigma} = \delta^{\alpha}_{\rho}\delta^{\beta}_{\sigma}\,; \qquad H^{\alpha\beta\mu\nu}\dot{g}_{\alpha\beta}\dot{g}_{\mu\nu} = G_{\alpha\beta\mu\nu}\pi^{\alpha\beta}\pi^{\mu\nu}\,; \tag{2.52}$$

as it should. We are now in a position to define the Lagrangian and Hamiltonian for $g_{\alpha\beta}$'s. From (2.47) and (2.51) it follows that the Lagrangian (ignoring $(16\pi G)^{-1}$) is:

$$L = \left\{\frac{1}{2}H^{\alpha\beta\mu\nu}\dot{g}_{\alpha\beta}\dot{g}_{\mu\nu} + \sqrt{-g}^{3}R\right\}. \tag{2.53}$$

The canonical momentum is, of course, $H^{\alpha\beta\mu\nu}\dot{g}_{\mu\nu} = \pi^{\alpha\beta}$. So the Hamiltonian is

$$H = \pi^{\alpha\beta}\dot{g}_{\alpha\beta} - L = \pi^{\alpha\beta}G_{\alpha\beta\mu\nu}\pi^{\mu\nu} - \frac{1}{2}G_{\alpha\beta\mu\nu}\pi^{\alpha\beta}\pi^{\mu\nu} - \sqrt{-g}^{3}R$$

$$= \frac{1}{2}G^{\alpha\beta\mu\nu}\pi_{\alpha\beta}\pi_{\mu\nu} - \sqrt{-g}^{3}R \,. \tag{2.54}$$

The action can be written in the Hamiltonian form as[4]

$$S_g = \frac{1}{l^2}\int R\sqrt{-g}\; d^4x$$

$$= \int dt\, d^3x\left\{\pi^{\alpha\beta}\dot{g}_{\alpha\beta} - \left(l^2 G_{\alpha\beta\mu\nu}\pi^{\alpha\beta}\pi^{\mu\nu} - \frac{1}{l^2}\sqrt{-g}^{3}R\right)\right\} \tag{2.55}$$

where we have reintroduced $l^2 \equiv (16\pi G)$ which scales $\pi_{\alpha\beta}$ and H.

This form suggests a condensed notation. Let us indicate by g_A the variable $g_{\alpha\beta}$ and let G_{AB} stand for $G_{\alpha\beta\mu\nu}$; we will also assume that repeated capital letters stand for both summation over α, β etc. and integration over the 3-space. For example, we can write

$$\int d^3\underline{x}\, G_{\alpha\beta\mu\nu}(\underline{x})\pi^{\alpha\beta}(\underline{x})\pi^{\mu\nu}(\underline{x})$$

$$= \int d^3\underset{\sim}{x}\left\{\frac{1}{\sqrt{-g}}(g_{\alpha\mu}g_{\beta\nu} + g_{\alpha\nu}g_{\beta\mu} - g_{\alpha\beta}g_{\mu\nu})\big|_{\underset{\sim}{x}}\pi^{\alpha\beta}(\underset{\sim}{x})\pi^{\mu\nu}(\underset{\sim}{x})\right\}$$

$$= \int d^3\underset{\sim}{x}d^3\underset{\sim}{x}'\,\delta(\underset{\sim}{x} - \underset{\sim}{x}')\left\{\frac{1}{\sqrt{-g}}(g_{\alpha\mu}(\underset{\sim}{x})g_{\beta\nu}(\underset{\sim}{x}') + g_{\alpha\nu}(\underset{\sim}{x})g_{\beta\mu}(\underset{\sim}{x}') - g_{\alpha\beta}(\underset{\sim}{x})g_{\mu\nu}(\underset{\sim}{x}'))\right\}\pi^{\alpha\beta}(\underset{\sim}{x})\pi^{\mu\nu}(\underset{\sim}{x})$$

$$= \sum_{AB} G_{AB}\pi^A\pi^B = G_{AB}\pi^A\pi^B. \tag{2.56}$$

The summation over A, B takes care of both

(i) summation over (α, β) and (μ, ν) and

(ii) integration over $\underset{\sim}{x}$, $\underset{\sim}{x}'$. We have defined G_{AB} with a delta function $(\delta(\underset{\sim}{x} - \underset{\sim}{x}')G_{\alpha\beta\mu\nu}(\underset{\sim}{x}))$ in order to reduce the double summation to a single integration. In this compact notation,

$$S_g = \int dt\{\pi^A\dot{g}_A - (l^2G_{AB}\pi^A\pi^B - l^{-2}\sqrt{-g}^3R)\}$$

$$= \int dt\{\pi^A\dot{g}_A - H(\pi^A, g_B)\}\ . \tag{2.57}$$

The variation of S_g with respect to π^A and g_A reproduces the equations

$$R^\alpha_\beta - \frac{1}{2}\delta^\alpha_\beta R = 0\ . \tag{2.58}$$

(This is easily seen by noticing that the variation is equivalent to varying $g_{\alpha\beta}$ in the original action.) There is nothing more left to vary; we have 'lost' the $\binom{0}{0}$ and $\binom{0}{\alpha}$ equations!

From our discussion in Sec. 2 we know why we have lost these equations; it is because we set the gauge as $g_{00} = 1$, $g_{0\alpha} = 0$ (similar to taking $N = 1$ in (2.7) or $\phi = 0$ in (2.20)). We also know how to "recover the lost equations": we need only to impose the invariances 'by hand' on S_g. Let us do that.

We know that the original action in (2.31) is invariant under

(i) $t \to t' = f(t)$—reparametrization of time coordinate and

(ii) $x^\alpha \to x^\alpha + \xi^\alpha(\underset{\sim}{x})$ $(\alpha = 1, 2, 3)$—the reparametrizations of the spacelike hypersurface. We have to, therefore, demand that

(i) $[\delta S_g/\delta N) = 0$ if $dt' = Ndt$ and

(ii) $\delta S_g = 0$ for $\delta g_{\alpha\beta}$ of the form $(\xi_{\alpha;\beta} + \xi_{\beta;\alpha})$. Under the transformation $dt \to N\,dt$, S_g becomes

$$S'_g = \int dt\{\pi^A \dot{g}_A - NH\} \,. \tag{2.59}$$

Thus the condition (i) $\delta S_g/\delta N = 0$ implies the constraint

$$\begin{aligned} H &= l^2 G_{AB}\pi^A\pi^B - l^{-2}\sqrt{-g}^3 R \\ &= l^2 G_{\alpha\beta\mu\nu}\pi^{\alpha\beta}\pi^{\mu\nu} - l^{-2}\sqrt{-g}^3 R = 0 \,. \end{aligned} \tag{2.60}$$

The second constraint ($\delta S = 0$ for $\delta g_{\alpha\beta} = \xi(\alpha;\beta)$) is equivalent to

$$\begin{aligned} 0 = \delta S_g^{\text{ext}}|_{\delta g_{\alpha\beta}=\xi(\alpha;\beta)} &= 2\int d^3\underset{\sim}{x}\left(\frac{\delta S_g}{\delta g_{\alpha\beta}}\right)\xi_{(\alpha;\beta)} \\ &= 2\int d^3\underset{\sim}{x}\pi^{\alpha\beta}\xi_{\alpha;\beta} \\ &= 2\int d^3\underset{\sim}{x}[(\pi^{\alpha\beta}\xi_\alpha)_{;\beta} - \xi_\alpha\pi^{\alpha\beta}_{;\beta}] \\ &= -2\int d^3\underset{\sim}{x}\xi_\alpha\pi^{\alpha\beta}_{;\beta} \,. \end{aligned} \tag{2.61}$$

That is

$$\pi^{\alpha\beta}_{;\beta} = 0 \,. \tag{2.62}$$

(Note the similarity between this derivation and that of Eq. (2.60) giving $\underset{\sim}{\nabla}\cdot E = 0 = E^{\alpha}_{;\alpha}$ in electromagnetism.) Equation (2.58) [6 equations], Eq. (2.50) [1 equation; $\binom{\upsilon}{0}$ part] and Eq. (2.62) [3 equations; $\binom{0}{\alpha}$ part] constitute the Einstein's equations. The difference between dynamical and constraint equations is now clearly exhibited. The constraint (2.60) is clearly due to the reparametrization invariance ($t \to t'$) of the theory; Eq. (2.62) may be interpreted in two ways: we may think of them as due to the invariance under reparametrization of the spatial coordinates $x^\alpha \to x^{\alpha'}$; or we may say that the theory must be invariant under the gauge transformation of $g_{\alpha\beta}$: $g_{\alpha\beta} \to g_{\alpha\beta} + \xi_{(\alpha;\beta)}$. Either view leads to (2.62).

One can recover the full action (2.31) by pulling the constraints (2.62) and (2.60) to (2.57) with suitable Lagrange multipliers. The analysis is similar to the one performed in (2.25) to (2.27) and is not very illuminating.

2.5. *Two clarifications regarding constraint equations*

The 'reparametrizations' which we considered are actually general coordinate transformations. Standard text books, however, show that the infinitesimal coordinate

transformations ($x^i \to x^i + \xi^i(t, \underline{x})$) lead to the Bianchi identities [$(R^i_k - \frac{1}{2}\delta^i_k R)_{;i} = G^i_{k;i} = 0$] rather than to the constraint equations $G^i_0 = 0$. It is necessary to clarify this point.

Under an infinitesimal transfomation $x^i \to x^i + \xi^i(t, \underline{x})$, the metric tensor changes by $\delta g^{ik}(\underline{x}, t) = \xi^{i;k}(\underline{x}, t) + \xi^{k;i}(\underline{x}, t)$. The change in the action is

$$16\pi G\,\delta S_g = \int_V G_{ik}\delta g^{ik}\sqrt{-g}\,d^4x$$

$$= \int_V [(G^k_i\xi^i)_{;k} - G^k_{i;k}\xi^i]\sqrt{-g}\,d^4x\,. \tag{2.63}$$

We now convert the first term to a surface integral around the (4-dim) volume V. This boundary consists of two spacelike surfaces $t = t_1$ and $t = t_2$ ("bottom" and "top") and a timelike ($|\underline{x}| \to \infty$) surface at spatial infinity ("sides"). We will assume that $\xi(\underline{x}, t) \to 0$ as $|\underline{x}| \to \infty$ for all t. Then only the 'top' and 'bottom' spacelike surfaces contribute, giving,

$$\int_V (G^k_i\xi^i)_{;k}\sqrt{-g}\,d^4x = \int_{\partial V} d^3\underline{x}\sqrt{-{}^3g}(G^0_i(\underline{x}, t)\xi^i(\underline{x}, t))\Big|^{t=t_2}_{t=t_1}. \tag{2.64}$$

Therefore

$$16\pi G\delta S_g = -\int_V \sqrt{-g}\,d^4x(G^k_{i;k}\xi^i)$$

$$+ \int_{\partial V} d^3\underline{x}\sqrt{-{}^3g}(G^0_i(\underline{x}, t)\xi^i(\underline{x}, t))\Big|^{t=t_2}_{t=t_1}. \tag{2.65}$$

We want δS_g to vanish for arbitrary $\xi^i(t, \underline{x})$. Consider a class of $\xi^i(t, \underline{x})$ such that $\xi^i(t_1, \underline{x}) = 0 = \xi^i(t_2, \underline{x})$ but $\xi^i(t, \underline{x})$ is arbitrary in the interval $t_1 < t < t_2$. (This is the kind of ξ^i usually considered in text books.) Then the second term of (2.66) vanishes and we are left with the Bianchi identities:

$$G^k_{i;k} = 0\,. \tag{2.66}$$

Now consider $\xi^i(t, \underline{x})$ which is nonzero for $t = t_1, t_2$ as well. Since (2.66) must be identically satisfied (or, since the values of $\xi^i(\underline{x}, t)$ for $t_1 < t < t_2$ and for $t = t_1, t_2$ are independent) we now have

$$16\pi G\delta S_g = 0 = \int d^3\underline{x}\sqrt{-{}^3g}(G^0_i\xi^i)\Big|^{t_2}_{t_1}$$

implying (since $\xi^i(t_1, \underline{x}) \neq 0$, and t_1, t_2 are arbitrary)

$$G_i^0 = 0 . \tag{2.67}$$

Thus both constraint and 'conservation' (Bianchi identities) can be obtained if we allow for transformations with $\xi^i(t_1, \underline{x}) \neq 0$, $\xi^i(t_2, \underline{x}) \neq 0$.

The fact that $\xi^i(\underline{x}, t)$ is nonzero at end-points implies two facts: Since $x'^i = x^i + \xi^i(\underline{x}, t)$, we see that the spatial coordinates x^α of the $t = t_1$ (say) surface are relabelled: $x'^\alpha = x^\alpha + \xi^\alpha(\underline{x}, t_1)$. What is more, the time label of the very surface is changed. The surface originally labelled $x^0 = t_1$, will now be changed to $x^{0'} = t_1 + \xi^0(\underline{x}, t_1)$. Demanding the invariance under such relabelling leads to the constraints.

This result is not a pecularity of gravity. In (2.17) we can consider δA^i which are of the form $(\partial^i f)$. This will give both charge conservation (if a source term is added) and the constraint.

The situation is similar in mechanics as well. From (2.7) and (2.11) we see that, if $\delta N = (df/dt)$,

$$\begin{aligned}\delta S &= \int_{t_1}^{t_2} (-1)\left\{\frac{1}{2}\frac{\dot{q}^2}{N^2} + V\right\}\frac{df}{dt}\,dt \\ &= -\left(\frac{1}{2}\frac{\dot{q}^2}{N^2} + V\right) f(t)\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} dt\, f\,\frac{d}{dt}\left(\frac{1}{2}\frac{\dot{q}^2}{N^2} + V\right).\end{aligned} \tag{2.68}$$

The values of $f(t)$ at $t = t_1$, t_2 and its value for $t_1 < t < t_2$ are, of course, independent. Thus if we allow $f(t_1) \neq 0$ etc. we get both the conservation (of energy)

$$\frac{d}{dt}\left\{\frac{1}{2}\frac{\dot{q}^2}{N^2} + V\right\} = 0 \tag{2.69}$$

and the constraint

$$\frac{1}{2}\frac{\dot{q}^2}{N^2} + V = 0 . \tag{2.70}$$

Thus the usual conservation laws for G^{ik} (equivalent to T^{ik}), charge (J^i), energy etc. can be obtained along with constraints if the theory possesses the reparametrization invariance.

The second (related) point which requires clarification is the connection between the reparametrization invariance and the constraints.[5] We argued all along, in conformity with the conventional wisdom, that they are deeply related and in particular, a theory with reparametrization invariance will lead to constraints.

This is indeed true, but one has to be sure that we are working with the "correct" basic variables. Consider, for example, the action

$$S = \int_{t_1}^{t_2} dt \left\{ \frac{1}{2} \frac{\dot{q}^2}{\dot{\tau}} - \dot{\tau} V(q) \right\} ; \qquad \dot{\tau} = \frac{d\tau}{dt} . \tag{2.71}$$

This is identical to S in (2.7) if we identify $N(t)$ with $(d\tau/dt)$. Naively, we would expect the same results here. Let us vary $\tau(t)$ and $q(t)$; we get:

$$\delta S = \int_{t_1}^{t_2} dt \left[- \left\{ \frac{d}{dt} \left(\frac{\dot{q}}{\dot{\tau}} \right) + \dot{\tau} V' \right\} \delta q + \frac{d}{dt} \left\{ \frac{1}{2} \frac{\dot{q}^2}{\dot{\tau}^2} + V \right\} \delta \tau \right]$$

$$+ \frac{\dot{q}}{\dot{\tau}} \delta q \Big|_{t_1}^{t_2} - \left(\frac{1}{2} \frac{\dot{q}^2}{\dot{\tau}^2} + V \right) \delta \tau \Big|_{t_1}^{t_2} . \tag{2.72}$$

What are the "correct" variations to consider? There are two options. We may say that q and τ are variables to be treated at the same footing and assume δq and $\delta \tau$ to vanish at $t = t_1, t_2$. Then we get the equations

$$\frac{1}{\dot{\tau}} \frac{d}{dt} \left\{ \frac{\dot{q}}{\dot{\tau}} \right\} + V' = 0 . \tag{2.73}$$

$$\frac{d}{dt} \left\{ \frac{1}{2} \frac{\dot{q}^2}{\dot{\tau}^2} + V \right\} = 0 . \tag{2.74}$$

It is trivial to see that (2.73) implies (2.74) ("energy conservation"). Thus we do not get any constraint equation if we treat both q and τ at equal footing. Note that the action in (2.71) is perfectly reparametrization invariant. Here is an example in which the invariance does not imply constraint.

The second option, of course, is to treat $\dot{\tau}$ as N "in disguise" and demand $\delta \tau$ to be arbitrary at the end-points. This will give rise to the constraints. This option looks more natural when N is treated as a basic variable rather than τ.

The above discussion emphasizes the hidden assumptions which go into the conclusion "invariance implies constraints". It is possible to get around this conclusion, by a suitable choice of end-point variations.

2.6. *Classical superspace and minisuperspace*

We have seen in Sec. 2.4 that the gravitational field can be described by the Lagrangian

$$L = \int l^{-2} \left\{ \frac{1}{2} H^{\alpha\beta\mu\nu} \dot{g}_{\alpha\beta} \dot{g}_{\mu\nu} + \sqrt{-g}^3 R \right\} d^3x d^3x' \tag{2.75}$$

which, in a 'condensed notation' can be written as:

$$L = l^{-2} \left\{ \frac{1}{2} H^{AB} \dot{g}_A \dot{g}_B - V(g_A) \right\} . \tag{2.76}$$

Variation of $g_{\alpha\beta}$ (or g_A) will lead to the 'space-space' component of Einstein's equation. In order to obtain the full set of Einstein's equation, we have to supplement these dynamical equations by the two constraint equations (2.60) and (2.62).

The solution to the Einstein's equations describes the evolution of g_A's in time. However g_A's stand for the metric tensor of 3-space, $(g_{\alpha\beta})$ in some chosen coordinate system and hence can be changed by relabelling the spatial coordinates. We define an equivalence class of $g_{\alpha\beta}$'s as the set $\{g_{\alpha\beta}\}$ of metrics, obtained by coordinate relabelling from a given $g_{\alpha\beta}$. This set describes a particular 3-geometry. It is usual to define an abstract, infinite-dimensional space called ''superspace'' in which each point is a 3-geometry;[6] i.e. each point in superspace corresponds to an equivalence class of all g_A's related to each other by relabelling of coordinates. (This terminology was used by Wheeler, DeWitt and others in the sixties and seventies. In more recent years particle physicists have chosen to call an entirely different space as superspace. We will stick to the original terminology.] With this definition of superspace, the evolution of space-time geometry can be described by a curve in the superspace.

The quantities H^{AB} introduced in (2.76) can be treated as a metric in this superspace. It is possible to introduce formal and aesthetically pleasing geometrical structures in the superspace.[7] (For example, it is possible to parametrize the curves in such a way that Einstein's equations are geodesic equations in the superspace.) We refer the reader to original literature[6,7] for these developments. We will confine our attention here to a simpler structure, usually called mini-superspace.

The idea of mini-superspace can be understood as follows: In general, $g_{\alpha\beta}$'s depend on x^α as well as t. Translated as g_A's, this means that we need an infinite number of A's to describe the situation (Remember that A, B etc. stand for the discrete indices $\alpha\beta$ as well as the space coordinate x^α; thus A runs over $6\infty^3$ values! This is why superspace is infinite-dimensional.) But suppose that we are only interested in a class of space-times in which $g_{\alpha\beta}$'s can be chosen to be functions of time alone. Then our superspace is, at best, 6 dimensional! The Lagrangian in (2.76) can now be considered as describing the motion of a particle in a finite-dimensional space called 'mini-superspace'.

For this idea to be useful, it is essential that evolution via Einstein's equations preserves this property. This is indeed ensured in a wide class of space-times which are spatially homogeneous.[8] Such space-times have been classified[8,9] into nine different types known as 'Bianchi types I to IX'. For example, a Bianchi Type I space-time has the metric:

$$ds^2 = dt^2 - (q_1^2(t)dx^2 + q_2^2(t)dy^2 + q_3^2(t)dz^2) \,. \tag{2.77}$$

At any instant, the geometry can be described as a point in a 3-dimensional space with coordinates (q_1, q_2, q_3). A more familiar system will be the closed Friedmann universe described by the metric:

$$ds^2 = dt^2 - q^2(t)(d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2)) \,. \tag{2.78}$$

The mini-superspace corresponding to this sytem is one-dimensional with the coordinate q. The Friedman universe belongs to Bianchi Type IX. In both the above cases, as well as

in any other Bianchi model, the evolution via Einstein's equations preserves the form of the metric. We will make extensive use of the concept of 'mini-superspace' in later sections.

3. Quantum Gravity and Time

Given the Hamiltonian of the classical gravity, the quantum theory can be constructed by writing the Schrodinger equation with the Hamiltonian operator:

$$H(\pi_{\alpha\beta}, g_{\alpha\beta}) = H\left(-i\frac{\delta}{\delta g_{\alpha\beta}}, g_{\alpha\beta}\right).$$

For gravity, just as for any other field theory, this Schrodinger equation will be a functional differential equation. We do not know how to solve such equations exactly. The only technique available—viz. a perturbation expansion in some small parameter—works only for a handful of Hamiltonians (called 'perturbatively renormalizable').[9] Gravity does not oblige us by belonging to this set.[10] Several attempts to change the Hamiltonian of gravity in order to fit this scheme has not yet produced any success. (The latest in this saga being superstrings.)

To make any further progress, it is therefore necessary to resort to some form of approximation. Two such approximations seem to give tractable models.

The first uses the special relationship between cosmology and gravity and has led to the birth of an area called "quantum cosmology". This subject attempts to give quantum mechanical meaning to classical cosmological models. Since classical cosmological models are often described by a mini-superspace with finite number of functions of time, $q_i(t)$ $[i = 1, \ldots N]$, one can reduce the problem of quantum cosmology to one of quantum mechanics—which can be solved. We will discuss some specific models for quantum cosmology in Sec. 5.

The second area in which some progress can be made is that of "semiclassical gravity". Here we seek formal techniques which will allow us to compute quantum corrections to classical solutions in a systematic manner. It is hoped that this road will eventually, take us to quantum gravity.

Investigations in the above areas have served one very important purpose: They have unearthed a host of conceptual problems which are special to gravity. The most important among these are the following three issues:

(a) How do we introduce and use the concept of time in quantum gravity?

(b) Can one define a consistent semiclassical theory of gravity?

(c) How should one interpret the solutions to the equations of quantum cosmology?

We shall discuss the first question in Sec. 3. Since the issue of time is intimately related to that of constraints, we start by reviewing the quantization of a simple toy model with constraints.

3.1. *Quantization of a constrained system*

To understand the complexities which are involved, let us consider a simple system described by the Lagrangian

$$L = \frac{1}{2}\frac{(\dot{x}^2 + \dot{y}^2)}{N} - V(x, y)N \,. \tag{3.1}$$

This is a generalization of the system described by (2.7). The classical equations of motion obtained by varying x, y and N are:

$$\frac{1}{N}\frac{d}{dt}\left(\frac{\dot{x}}{N}\right) = -\frac{\partial V}{\partial x}\,; \qquad \frac{1}{N}\frac{d}{dt}\left(\frac{\dot{y}}{N}\right) = -\frac{\partial V}{\partial y} \tag{3.2}$$

$$\frac{1}{2}\left\{\frac{\dot{x}^2 + \dot{y}^2}{N^2}\right\} + V = 0 \,. \tag{3.3}$$

It is clear that:

(i) only the combination $d\tau = Ndt$ appears in the equations and

(ii) any one of (3.2) along with (3.3) implies the second equation of (3.2).

Thus the solution will be some functions:

$$x = x(\tau) = x\left(\int Ndt\right)\,; \qquad y = y(\tau) = y\left(\int Ndt\right) \tag{3.4}$$

with $N(t)$ undetermined. These solutions—just as the equations—are invariant under the reparametrizations $t \to t' = f(t)$, $N(t) \to N' = N(\dot{f})^{-1}$.

We are, of course, allowed to choose any $N(t)$. This choice will change the explicit functional form of $x(t)$ and $y(t)$. But this choice does *not* affect the functional form of $x(\tau)$ or $y(\tau)$; neither does it affect the relation $x = x(y)$ obtained by eliminating τ between the two relations.

Let us now try to quantize the system described by (3.1). The classical Hamiltonian is:

$$H = N\left\{\frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + V\right\} . \tag{3.5}$$

The Schrodinger equation corresponding to this Hamiltonian will be

$$i\frac{\partial \psi}{\partial t} = N\left\{-\frac{1}{2}\frac{\partial^2 \psi}{\partial x^2} - \frac{1}{2}\frac{\partial^2 \psi}{\partial y^2} + V\psi\right\} \tag{3.6}$$

or, equivalently

$$i\frac{\partial \psi}{\partial \tau} = -\frac{1}{2}\frac{\partial^2 \psi}{\partial x^2} - \frac{1}{2}\frac{\partial^2 \psi}{\partial y^2} + V\psi \,. \tag{3.7}$$

Notice that this equation—just like its classical counterpart—is reparametrization invariant (under $t \to t'$, $N \to N' = N\,(dt'/dt)^{-1}$). The solution $\psi = \psi(x, y, \tau)$ depends only on τ; N is not determined and can be chosen as we wish.

The trouble begins when we try to answer the question: Does (3.7) represent a quantum version of the classical system described by (3.1) to (3.3)? To answer this question we need a rule which sets up a 'correspondence' between a quantum theory and classical one. Let us suppose, for the moment, that such a rule is given in terms of some prescription for taking the 'semiclassical limit'' (We will say more about this later). Then we can ask: Does the theory described by (3.7) leads to (3.2) and (3.3) in the semiclassical limit?

One way of approaching the semi-classical limit is to treat the $x(\tau)$, $y(\tau)$ in (3.2) and (3.3) as the expectation values of $\hat{x}$ and $\hat{y}$ in a state described by $\psi(x, t, \tau)$. Then, a simple analysis based on the Ehrenfest's theorem tells us that the expectation values $\langle x\rangle$ and $\langle y\rangle$ will evolve according to Eq. (3.3). This, in turn, implies that the quantity on the left-hand side of (3.3) is a constant; but this constant need not necessarily be zero. Thus, for an arbitrary $\psi(x, y, \tau)$, we will only recover (3.2) and not (3.3).

So, if we want to ensure that we obtain (3.2) and (3.3) in the classical limit, then (3.7) is not enough; we must put more restrictions on $\psi(x, t, \tau)$. There are several ways of doing this:[5] The most straightforward method is to restrict oneself to those $\psi(x, y, \tau)$ for which $\langle x\rangle$ and $\langle y\rangle$ satisfy the constraint (3.3) at some instant, say at $\tau = 0$. Then the equations of motion guarantee that (3.3) will be valid at all later times. In order words, we use the constraint equation as an initial condition, determining the choice of $\psi(x, y, \tau = 0)$. (This process has a clear parallel in classical theory. Suppose we find an arbitrary solution to the coupled equations (3.2). This solution $(x(\tau), y(\tau))$ will ensure that the left-hand side of (3.3) is a constant but not necessarily zero. We now choose the constants of integration in $x(\tau)$, $y(\tau)$ such that this constant is zero. Dynamics guarantees that this condition is satisfied automatically at later times.)

More formally, we can restrict ourselves to the class of $\psi(x, y, \tau)$ which satisfies the condition:

$$\langle H\rangle = \int dxdy\psi^*\hat{H}\psi = 0 \ . \tag{3.8}$$

In the classical limit, this is equivalent to looking at trajectories with zero energy—which is what the constraint (3.3) amounts to.

While this approach is perfectly sound as far as the classical limit is concerned, it suffers from an apparent shortcoming in the quantum domain. It violates the principle of the superposition of states: Suppose ψ_1 and ψ_2 are two solutions to (3.7) with $\langle x\rangle$ and $\langle y\rangle$ (in each state) constrained by the condition (3.3). In general, the $\langle x\rangle$ and $\langle y\rangle$ constructed from the superposed state $(\psi_1 + \psi_2)$ will not satisfy the constraints. Similarly if ψ_1 and ψ_2 individually satisfy (3.8), there is no guarantee that $(\psi_1 + \psi_2)$ will also do so.

One way of ensuring superposition is to impose (3.8) at a stronger level, viz. as $H\psi = 0$. In other words, we restrict ψ to be the zero eigenvalue solution of the Schrödinger equation

$$-\frac{1}{2}\frac{\partial^2\psi}{\partial x^2} - \frac{1}{2}\frac{\partial^2\psi}{\partial y^2} + V\psi = 0 \ . \tag{3.9}$$

If ψ_1 and ψ_2 are two solutions to this equation, so is $(\psi_1 + \psi_2)$. In standard literature, (3.9) has always been suggested as the method of quantizing our constrained system. It is

even claimed, at times, that the reparametrization invariance implies that ψ should vanish, leading to (3.9). This is, of course, completely wrong. Both (3.7) and (3.9) are reparametrization invariant. The only difference is in the classical limit: (3.7) leads to (3.2) while (3.9) could in principle, lead to (3.2) and (3.3). One should regard (3.9) merely as a way of quantizing our classical system.

There is, however, a price to be paid if we use (3.9). In using (3:7) we could identify the classical trajectory with the expectation value $\langle x\rangle$, $\langle y\rangle$ etc. This is not possible in (3.9) because $\psi(x, y)$ is now a stationary state. All expectation values will be independent of time and cannot describe 'classical trajectories' as functions of time. The classical limit has to be obtained in a very nontrivial manner, which we will discuss in a later section.

We have thus found two inequivalent quantum versions for our classical system:

(i) We may use (3.7) and restrict ourselves to solutions in which $\langle x\rangle$ and $\langle y\rangle$ satisfy (3.3).

(ii) We can use (3.9) directly.

Let us compare these two approaches briefly:

Both the approaches, as we have stressed, are reparametrization invariant. In both approaches, we confine our attention to a subset of all possible solutions to the Schrödinger equation (3.7) (In (i), this is done using $\langle x\rangle$ and $\langle y\rangle$; in (ii) by taking $\dot{\psi} = 0$.) In approach (i), we abandon the principle of superposition. In approach (ii), this is formally retained; but it is not of much significance unless the $E = 0$ state is degenerate in a nontrivial manner. The major difference between the two approaches, however, is in the realization of the classical limit. In (i) we obtain the classical limit by taking the expectation values of x and y. This cannot be done in (ii). Since $\psi(x, y)$ in (ii) is independent of time, all expectation values will be time independent. The semiclassical limit has to be interpreted in terms of the WKB wavefunctions which we will discuss later.

Everything which has been said here about the system in (3.1) has a relevant parallel in the study of quantum gravity to which we shall now direct our attention.

3.2. *Wheeler-DeWitt equation*

We saw in Sec. 2.4 that the Lagrangian for classical gravity can be written in the form

$$L = \frac{1}{l^2}\left\{\frac{1}{2}H^{AB}\dot{g}_A\dot{g}_B + \sqrt{-{}^3g}\,{}^3R\right\} \tag{3.10}$$

$$= \frac{1}{l^2}\left\{\frac{1}{2}G^{AB}\dot{g}_A\dot{g}_B + \sqrt{-{}^3g}\,{}^3R\right\}; \qquad l^2 = 16\pi G\,. \tag{3.11}$$

(In arriving at (3.11) we have used the fact that H is the inverse of G; see (2.52).) This Lagrangian can be made invariant under reparametrization of t by introducing the constraint variable N, which changes L to

$$L_0 = \frac{1}{l^2}\left\{\frac{1}{2}G^{AB}\frac{\dot{g}_A\dot{g}_B}{N} + \sqrt{-{}^3g}\,{}^3RN\right)\,. \tag{3.12}$$

Variation of N leads to the constraint

$$\frac{1}{2} G^{AB} \dot{g}_A \dot{g}_B N^{-2} - \sqrt{-^3g}\,{}^3R = 0 \tag{3.13}$$

which, on using (2.50) and (2.60) becomes

$$\frac{l^2}{2} G_{PQ} \pi^P \pi^Q - l^{-2} \sqrt{-^3g}\,{}^3R = N^{-1} H = 0 \,. \tag{3.14}$$

Thus the Lagrangian in (3.12) takes care of both the dynamics and the Hamiltonian constraint (or equivalently, it reproduces, in the classical limit, the $\binom{0}{0}$ and $\binom{\alpha}{\beta}$ components of Einstein's equations). Strictly speaking we have to worry about the three constraints in (2.62) as well, which embody the $\binom{0}{\alpha}$ equations. However, we will ignore them right now and pretend that our system is completely described by the Lagrangian in (3.12). It will turn out that the remaining equations do not really add to our understanding; in fact, they are trivially satisfied in any mini-superspace model.[7]

We want to quantize this constrained system. From the discussion in the last section, we know that there are two different approaches to this problem. Naively, one would have written down the Schrödinger equation for (3.11) to be

$$i \frac{\partial \psi}{\partial t} = H\psi = N\left(\frac{l^2}{2} G_{PQ} \pi^P \pi^Q - l^{-2} \sqrt{-^3g}\,{}^3R \right) \psi \tag{3.15}$$

where we have used the form of (3.11) to compute H. When we convert H to an operator, π^P will become $(-i\partial/\partial q_P)$. Since G_{PQ} are functions of g's we run into factor-ordering problem in $G_{PQ}\pi^P\pi^Q$. This problem can only be resolved if some additional physical criterion is introduced. We take a simple-minded choice and replace $G_{PQ}\pi^P\pi^Q$ by the 'Laplacian' $(-\nabla^2)$ made from the metric G_{PQ}. Other criteria are certainly possible, but this choice has the virtue of being invariant under the "coordinate transformations" $(g_A \to g'_A)$ in the mini-superspace. With this choice, and the notation $V(g_A) \equiv -\sqrt{-^3g}\,{}^3R$, (3.15) becomes:

$$iN^{-1} \frac{\partial \psi}{\partial t} = i \frac{\partial \psi}{\partial \tau} = \left(-\frac{l^2}{2} \nabla^2 + l^{-2} V(g_A) \right) \psi \,. \tag{3.16}$$

In doing all these we have considered only the gravitational part of the action. The full theory will, of course, be described by the action $S_{\text{total}} = S_{\text{gravity}} + S_{\text{matter}}$. It is easy to see that, when matter variables are introduced, the H in (3.15) should be replaced by the total Hamiltonian $(H_{\text{gravity}} + H_{\text{matter}})$. Thus, in a full theory, with matter, Eq. (3.16) will be replaced by

$$i \frac{\partial \psi}{\partial \tau} = \left[-\frac{l^2}{2} \nabla^2 + l^{-2} V(g_A) + H_m\left(q_i, -i \frac{\partial}{\partial q_i} \right) \right] \psi \tag{3.17}$$

where we have denoted the matter variables symbolically by the set $\{q_i\}$.

Equation (3.17) is completely equivalent to the dynamical part (viz. the $\binom{\alpha}{\beta}$ part) of the Einstein's equations. In the classical limit, (3.17) will correctly reproduce this "space-space" part of the Einstein's equations. The wave function $\psi(\tau, g_A, q_i)$ is also reparametrization invariant; it depends only on the combination (Ndt) and not individually on $N(t)$ and t.

The only trouble with (3.17) is that it does not reproduce the constraint equation $H = 0$ in the classical limit. It can only reproduce the weaker condition $H =$ constant. (The situation is identical to the one encountered in last section.) If we want to claim that we are actually quantizing the system described by (3.12), then we have to put additional restrictions on ψ. We have already discussed such restrictions in Sec. 3.1. There are essentially two different ways of doing it:

(a) We can choose $\psi(\tau, g_A, q_i)$ in such a way that the expectation values $\langle g_A \rangle$ and $\langle q_i \rangle$ satisfy the constraint equation at some instant τ. The evolution will preserve this condition thereafter.

(b) We can impose on ψ the 'zero-energy condition' $H|\psi\rangle = 0$. This will ensure the validity of the constraint equations automatically. From (3.17), we see that this condition amounts to

$$\left\{-\frac{l^2}{2}\nabla^2 + l^{-2}V(g_A) + H_m\left(q_i, -i\frac{\partial}{\partial q_i}\right)\right\}\psi(g_H, q_i) = 0\,. \tag{3.18}$$

This equation is called the 'Wheeler-DeWitt equation' and forms the basis for several investigations in quantum gravity.[11] We shall therefore, take a closer look at (3.18).

3.3. *The concept of time*

One of the most striking features about (3.18) is the absence of any time derivative. Suppose we manage to solve (3.18) getting ψ as a functional of $g_{\alpha\beta}(x)$ $(=g_A)$ and some fields $\phi(x)$ $(=q_i)$. How do we interpret $\psi(g_A, q_i)$? All that we can say is that $|\psi|^2$ represents the probability for the values g_A and q_i. We can, of course, compute the expectation value $\langle g_A \rangle = \langle \psi | g_A | \psi \rangle$ as a function of q_i (or the expectation value of q_i as a function of g_A). For an arbitrary solution ψ these values will be hard to interpret. The only situation in which a probabilistic interpretation for $|\psi|^2$ makes sense is when $|\psi|^2$ is peaked around some 'curve' $g_A = g_A(q_i)$. This will not happen for a generic solution $\psi(g_A, q_i)$. What is more, lack of time dependence signals the lack of any (conventional) evolution. How can we then understand the origin of time evolution in physical systems?

To decide on the issue of evolution, we need some concept of time. Several suggestions have been made in the literature regarding this issue. The most natural idea was based on Wheeler's philosophy[12] that "3-geometry is a carrier of information about time". This requires one to identify a particular degree of freedom among g_A's as 'special' and consider it to represent 'time'. There is indeed one such degree of freedom: the overall volume of 3-space. This degree of freedom is 'special' because it contributes negatively to the kinetic energy term in (3.10). To see this, consider a class of 3-metrics $g_{\alpha\beta}$ which are conformal to some static 3 metric $\bar{g}_{\alpha\beta}$:

$$g_{\alpha\beta} = \Omega^2 \bar{g}_{\alpha\beta} \; ; \qquad \dot{g}_{\alpha\beta} = 2\Omega\dot{\Omega}\bar{g}_{\alpha\beta} \; ; \qquad \Omega > 0 \; . \tag{3.19}$$

From (2.47) we find that

$$L_{\text{kinetic}} = \frac{1}{2} G^{AB}\dot{g}_A\dot{g}_B = \frac{1}{2} G^{\alpha\beta\mu\nu}\dot{g}_{\alpha\beta}\dot{g}_{\mu\nu} \propto -6\Omega\dot{\Omega}^2 \; . \tag{3.20}$$

The minus sign indicates that Ω contributes 'negative kinetic energy'. In other words, the operator ∇^2 in Wheeler-DeWitt equation is not positive definite. Just as in flat space—where the negative part of $\Box^2 = -(\partial^2/\partial t^2) + \nabla_3^2$ is identified with time—we can 'identify' Ω with a time variable. More rigorously, it can be shown[7] that the signature of $G^{\alpha\beta\mu\nu}$ in superspace is $(-+++++)$; the Ω-mode contributes the negative mode. With suitable coordinate choice, there will be three independent components in $g_{\alpha\beta}$, of which one can be taken to be Ω representing time, and the other two as the "transverse" degrees of freedom of gravity.

This programme has been discussed in detail in the literature.[13] Unfortunately, there are several technical and conceptual difficulties with this idea. In the Schrödinger picture which we are using, the time derivative should occur in first order. (It is in the Heisenberg picture field theory that we use the second-order time derivatives.) The identification of Ω as time is contrary to this spirit. The probabilistic interpretation of $|\psi|^2$ faces all the usual problems which are associated with the solutions of the Klein-Gordon equation. It has also been found that there are examples of mini-superspace in which Ω is not the most natural choice for 'time'. Thus we have to abandon, though with regret, the idea of defining time intrinsically using the variables in $g_{\alpha\beta}$.

Let us, therefore, explore the alternatives. The simplest one, of course, is to go back to Eq. (3.17) with an explicit $(i\partial\psi/\partial\tau)$ on the left-hand side.[14] This allows us to completely bypass the problem of time; we have cut the Gordian knot by putting in 'time' by hand! In order to recover the constraint equations in the classical limit, we have to suitably restrict the choice of ψ, as discussed earlier in Sec. 3.1. There are, indeed, several advantages to this approach.

First of all, $\psi(g_A, q_i, \tau)$ can now be interpreted exactly as in standard quantum mechanics. (There are no new conceptual problems.) In the classical limit, the variables will automatically satisfy the dynamical equations. The constraint equations, on the other hand, can be ensured by a suitable choice of ψ.

Secondly, this approach helps us to understand the nature of physical observables in the theory better. It is usual to ssume that only operators which commute with the constraints of the theory are observables. If we use the Wheeler-DeWitt equation, $H\psi = 0$, then the constrained quantity is the Hamiltonian itself. The physical observables have to be those variables which commute with H, viz. the constants of motion. This runs contrary to our ability to measure several 'time dependent' quantities including the metric components themselves. If we use (3.17) instead, we can avoid this difficulty.

Lately, operational methods of defining 'time', using 'clocks', lead to (3.17) automatically. We will say more about this in the next section.

The price we pay for these conveniences is the following: The time variable τ has no meaning in the quantum domain and has to be "propped up" by semiclassical

considerations. While this is somewhat discouraging, this could very well be a property of time which we should learn to live with. It is possible that 'Time'—as we know it—is a semi-classical concept. Several pieces of evidence point in this direction.

3.4. *Clocks and time*

Let us next explore an entirely operational point of view towards time: "Time is what the clocks measure". If we can define a system as a clock in some sensible manner then we can arrive at an operational notion of time.

This can be done easily. We will define an 'ideal clock' to be a quantum variable $\hat{c}$ governed by a Hamiltonian $\hat{H}_c$ such that the following conditions are satisfied:

(i) $[\hat{c}, \hat{H}_c] = i$ and

(ii) $\hat{H}_c$ is not strongly coupled to the "rest of the world", especially to gravity.[15]

The first condition ensures that $\dot{c} = i[\hat{H}_c, \hat{c}] = 1$ making c proportional to the time coordinate. The second condition keeps the clock unaffected by fluctuations in the fields which we are trying to measure. Ideal clocks can exist only if we can arrange this condition to be satisfied.

For any quantum system we have the relation $[\hat{c}, \hat{p}_c] = i$ where $\hat{p}_c$ is the momentum conjugate to the variable $\hat{c}$. Comparing this condition with (i) above, we conclude that we can choose the Hamiltonian H_c to be the same as the momentum p_c:

$$\hat{H}_c = \hat{p}_c = -i\frac{\partial}{\partial c} . \tag{3.21}$$

Consider now a system consisting of clock and the gravitational field. The combined Hamiltonian for the system is $(H_g + H_c)$. We will take the conventional attitude and quantize this system using Wheeler-DeWitt equation, which will now read as:

$$\hat{H}_{\text{total}}\psi(g_{\alpha\beta}, c) = [H_g(g_{\alpha\beta}) + \hat{H}_c]\psi(g_{\alpha\beta}, c) = 0 . \tag{3.22}$$

Or, on using $H_c = i\partial/\partial c$, we get

$$i\frac{\partial\psi}{\partial c} = \hat{H}_g\psi(g_{\alpha\beta}, c) . \tag{3.23}$$

This is precisely what we wrote down earlier, ignoring the constraint and introducing an *adhoc* time variable. One significant difference, however, must be noted. In the last section we had no dynamical basis for the time variable τ. Thus we had to impose the constraint equations by hand by suitably restricting the choice of ψ. Here we no longer need to do it. The constraint equations (and the dynamical equations) for the system containing the clock and the gravitational field are automatically ensured by (3.23). The solution to (3.23) also allows one to interpret $|\psi(c, g_{\alpha\beta})|^2$ as the probability for the existence of a 3 geometry $g_{\alpha\beta}$ when our 'clock reading' is c.

It might seem that we have solved all our problems. It is not so. The difficulty is that the ideal clocks, as defined above, do not exist. To see this, notice that the Hamiltonian for

the ideal clock, $H = p_c$, is unbounded from below. Realistic physical systems cannot be constructed (from known forms of matter) with this property. All that we can do is to construct clocks which are, say, very massive so that the quantum fluctuations intrinsic to the clock remain negligible over the domain of interest. This works in the semi-classical limit but not in the full quantum theory. Jim Hartle has made a detailed study of the dynamics of clocks under various circumstances.[16] He concludes that the ideal clocks capable of surviving quantum gravitational fluctuations do not exist.

Thus we have not really succeeded in introducing 'time' in the quantum gravitational domain. But we have succeeded in providing a fully operational definition for the variable τ which we introduced in the last section. This variable τ is nothing but the proper time measured by a classical clock.

The above discussion also suggests that the semi-classical limit of gravity must contain certain unusual features. The fully quantum mechanical version of gravity contains no reference to any time-coordinate; but a concept of 'time' seems to emerge in the classical limit! The transition region of semi-classical gravity must contain the mechanism which produces this 'time' coordinate. We shall next study this 'no-man's land' of semi-classical gravity.

4. Semi-classical Gravity

The interaction of gravity with other fields can be described at three different levels. The first level is the classical one in which both the gravitional field (g_A) and the other fields (q_i) are treated as classical c-numbers, obeying the classical equations. In the second level, one would like to provide a fully quantum mechanical description of both q_i and g_A by means of, say, a wave function $\psi(g, q)$. We know that $\psi(g, q)$ obeys the Wheeler-DeWitt equation (at least, in a formal sense) but we have no means of solving it. This inability has opened up the third level of description, viz. the one in which gravity is treated as a classical c-number object and the fields are quantized in this, given, background gravitational field ($\bar{g}_A$). In this level of description, $\bar{g}_A$ obeys some c-number equation and the fields are described by some wavefunction $\chi(\bar{g}_A, q_i)$ in a given background. It is this third level, in which gravity is classical but the fields are not, that concerns us in this part. The description in this level needs determination of $\chi(\bar{g}, q_i)$ and $\bar{g}_A$. Of these, $\chi(\bar{g}_A, q_i)$ falls in the domain of 'quantum field theory in curved space-time'. It is usually determined by the (functional) Schrödinger equation:

$$[i(\partial/\partial t) - H(\bar{g}_A, q_i]\chi(\bar{g}_A, q_i) = 0 \tag{4.1}$$

where $H(\bar{g}_A, q_i)$ is the Hamiltonian for the field q_i in a given background metric $\bar{g}_A$. (Usually, one uses the equivalent Heisenberg picture version of (4.1).) To complete the description, we need a c-number equation for $\bar{g}_A$. The major question in the study of 'semi-classical gravity' is the determination of the correct form of this equation in a well-defined order of approximation.

Several choices for this equation have been suggested in the literature. To the "lowest order" it seems reasonable to assume Einstein's equation for g. This would require using

$$R_{ik} - \frac{1}{2} g_{ik} R = 8\pi G T_{ik}(\langle q_i \rangle) \tag{4.2}$$

where the right-hand side is the purely classical object constructed from the, say, expectation value of q_i. Such a description is adequate when both gravity and the quantum field are in a highly excited, coherent, near classical state.

Now suppose we want to study quantum fluctuations in $q_i[O(\hbar)]$, around $\langle q_i \rangle$. Can we use (4.1) to study these fluctuations while retaining (4.2) for the description of g? That is, can we suppress $O(\hbar)$ effects of gravity compared to $O(\hbar)$ effects of f? To bring this question sharply into focus, consider a situation in which $\langle q_i \rangle$ vanishes classically ("vacuum"). Does it now make sense to use simultaneously the equations

$$\bar{R}_{ik} - \frac{1}{2} \bar{g}_{ik} \bar{R} = 0 \tag{4.3}$$

$$\left[i\left(\frac{\partial}{\partial t}\right) - H(g_A, q_i) \right] \chi(q_i, g_A) = 0 \tag{4.4}$$

The usual answer is 'yes'. This allows the study of the quantum fields in curved space-times. (There have been, however, some dissenting views; see for e.g. Ref. 17.) Lapchinsky and Rubakov have derived (4.3) and (4.4) (Ref. 18) from the full Wheeler-DeWitt equation—in a consistent order of approximation—by extending the earlier work[19] of Gerlach. (We will discuss this derivation in Sec. 4.) Their approximation scheme was effectively an expansion in the powers of G and corresponds to a weak coupling limit. This adds to our confidence in (4.3) and (4.4).

Several workers have tried to go beyond (4.3) and (4.4). There is a popular belief that, again in some consistent approximation procedure, (4.3) and (4.4) may be replaced by (4) and the following equation:

$$R_{ik} - \frac{1}{2} g_{ik} R = 8\pi G \langle T_{ik} \rangle \ . \tag{4.5}$$

In (5) the mystic symbol $\langle T_{ik} \rangle$ stands for a "suitable quantum average". Two choices are popular for $\langle T_{ik} \rangle$:

(i) The "in"-"out" matrix element of T_{ik} defined by the path integral methods and

(ii) The expectation value of T_{ik} in some quantum state, $\langle \psi | T_{ik} | \psi \rangle$. In particular $|\psi\rangle$ could be the vacuum state. There was a large industry[20] computing this object $\langle T_{ik} \rangle$ with the pious hope that it has something to do with quantum gravity. We will call this term, $\langle T_{ik} \rangle$—rather loosely—as 'backreaction'.

If correct, it must be possible to derive (5) from the Wheeler-DeWitt equation just as one could obtain (3) and (4). Such an attempt was recently made by Hartle (see Ref. 21). He used the same approximation method as Lapchinsky etc. and obtained equations which seem to suggest a backreaction. However, a reanalysis[22] of his calculation shows that Eq. (4.5) cannot be obtained by this approximation alone. Several other considerations enter into the picture. We shall try to disentangle these issues in Sec. 4.3.

There is another fact to the issue of semi-classical gravity which needs to be commented upon. This arises from the recent interest in the study of inflationary universe.[23] Several people have attempted to use the quantum fluctuations in the inflation scalar field as a source for gravity. The general belief is that these quantum fluctuations "lead to" classical density perturbations.[24] It is not clear whether this is really the case and—even if it is—under what circumstances such an approximation could be valid. Even if we grant a backreaction of the form (4.5) we cannot obtain the classical density perturbations in an inflationary universe. (In standard models, the inflation field is taken to be in the vacuum state; then $\langle T_{ik} \rangle$ will be homogeneous.) We need to set up some kind of correspondence between quantum fluctuations and classical stochastic averages. This is usually done in an *adhoc* manner in the inflationary models. In a more complete theory, we should either be able to derive this correspondence or disprove its existence.

The plan for the rest of the discussion in this part is as follows. In Sec. 4.1 we will review the WKB approach to the semi-classical limit. We will use a simple toy model and translate the results to pure gravity without matter. This will be based on the Wheeler-DeWitt equation. In Sec. 4.2 we will discuss an alternative approach to the semi-classical limit which uses Gaussian wave packets. This will be more appropriate when we have an explicit time coordinate supplementing the Wheeler-DeWitt equation (for example, in the form of a clock). In Secs. 4.3 and 4.4, we study the issue of backreaction when matter fields are also present. Section 4.5 describes the connection between the path integrals and the semi-classical limit.

4.1. *Semi-classical limit and WKB states*

We will begin our discussion with the Lagrangian:

$$L = \frac{1}{2} M(\dot{Q}^2 - V(Q)) \tag{4.6}$$

for a system with one degree of freedom Q. This system is modelled after the Lagrangian for pure gravity in a mini-superspace. Comparing (3.11) and (4.6) we see that pure gravity can be obtained by the following extensions:

(i) Instead of one variable Q introduces a set of gravitation degrees of freedom $\{g_A\}$. The kinetic energy term becomes a general quadratic $\frac{1}{2}G_{AB}\dot{g}^A\dot{g}^B$.

(ii) The potential $V(Q)$ is replaced by the potential in $V(g_A)$ (see Eq. (3.16)).

Notice that we have explicitly kept the mass M in (4.6) as an overall multiplicative factor. This is crucial if (4.6) should mimic the features of pure gravity. In gravity, there is no relative coupling between kinetic and potential terms. The classical equations of motion for the Lagrangian in (4.6) are

$$M(\ddot{Q} + V'(Q)) = 0 \,. \tag{4.7}$$

In the other extreme limit, the quantum dynamics of Q is described by the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2M} \frac{\partial^2 \psi}{\partial Q^2} + MV\psi = E\psi \,. \tag{4.8}$$

Our task is to set up a systematic approximation scheme which will take us from (4.8) to (4.7).

It turns out that there are several ways in which this could be done. All these methods are related to each other and each method could be appropriate in different physical situations. We will explore some of these methods in this and coming sections.

We begin with the most conventional one called the WKB approximation.[25] To motivate this approximation and to decide when it will be appropriate, let us first rewrite (4.8) as 2 real equations. We put

$$\psi(Q) = R(Q) \exp i[\theta(Q)/\hbar] \tag{4.9}$$

where R and θ are real. Substituting (4.9) into (4.8) we get

$$\frac{(\theta')^2}{2M} + MV - E = \frac{\hbar^2}{2M} \frac{R''}{R} \tag{4.10}$$

$$R\theta'' + 2R'\theta' = 0 \,. \tag{4.11}$$

Assuming that $\theta' \neq 0$, we can integrate (4.11) to obtain R and substitute it in (4.10). This gives us

$$R\sqrt{\theta'} = \text{constant} \tag{4.12}$$

and

$$\frac{(\theta')^2}{2M} + MV - E = \frac{\hbar^2}{2M} \sqrt{\theta'} \left\{ \frac{d^2}{dQ^2} \left(\frac{1}{\sqrt{\theta'}} \right) \right\} . \tag{4.13}$$

Equations (4.12) and (4.13) are equivalent to the exact Schrödinger equation. Nothing has yet been lost. We shall now assume that the classical limit corresponds to $\hbar$ becoming vanishingly small. This allows us to iterate Eq. (4.13) in an order by order expansion in $\hbar^2$. To the lowest order, ignoring $\theta(\hbar^2)$ term in the right-hand side of (4.13) we get the equations:

$$\frac{\theta_0'^2}{2M} + MV - E = \mathbb{O}(\hbar^2) \approx 0 \tag{4.14}$$

and

$$R(Q) = \frac{\text{constant}}{\sqrt{\theta_0'}} \,. \tag{4.15}$$

The resulting wave function

$$\psi_E^{(0)}(Q) = \frac{N}{\sqrt{\theta_0'(Q)}} \exp \frac{i}{\hbar} \theta_0(Q) \tag{4.16}$$

is called the WKB wave function. We notice that $\theta_0(Q)$ is just the classical action for the energy E. Equation (4.14) is the Hamilton-Jacobi equation for θ_0.

In what sense does (4.16) represent the classical limit described by (4.7)? The classical equation (4.7) integrates to give a trajectory $Q = Q(t, E)$ when the energy E is specified. It is impossible to obtain such a "trajectory" out of (4.16). Since (4.16) is a stationary state, all expectation values—including that of Q—will be independent of time. It will not be possible to interpret $|\psi|^2$ as "peaked around" a classical trajectory with energy E.

The quantity $|\psi^{(0)}(Q)|^2 dQ$ gives a "classical probability" in a very different sense. Note that the classical momentum p

$$P_{cl}(Q) = \sqrt{2M(E - MV(Q))} \tag{4.17}$$

is just $\theta_0'(Q)$ [see (4.14)]. Therefore, $|\psi^0(Q)|^2 dQ$, which is proportional to $(\theta_0')^{-1} dQ$, represents the time taken by the system to traverse the range $(Q, Q + dQ)$.

$$|\psi_0(Q)|^2 dQ \propto \frac{dQ}{\theta_0(Q)} = \frac{dQ}{p_{cla}(Q)} = dt_Q \,. \tag{4.18}$$

Thus the probability $|\psi_0(Q)|^2$ is high near those values of Q, around which the classical system lingers for long periods of time.[25] This is the only connection we have between $\psi^{(0)}(Q)$ and the classical limit, in the Q-space.

A more suggestive interpretation for $\psi^{(0)}$ can be given if we use the Wigner function and classical phase space.[26,27] It can then be easily shown that the Wigner function will be peaked around the curve $p = p_{cl}(Q)$; however, that is still not the same as probability being peaked around a classical trajectory $Q(t; E)$.

These results suggest that we can identify the derivative of the phase of WKB wave function $\theta'(Q)$, as the classical momentum. (This is a "rule of correspondence" which we stipulate). If this is done, then it is also possible to arrive at a notion of 'time'. Since $\theta(Q)$ is given, the equation $p = \theta'(Q)$ defines a curve in the phase space. We can parametrize this curve by some variable t, such that the tangent vector (d/dt) is just $(p(Q)/M)$:

$$\frac{d}{dt} = \frac{p(Q)}{M} \frac{\partial}{\partial Q} = v(Q) \frac{\partial}{\partial Q} \tag{4.19}$$

where $v(Q)$ is the velocity. (This is merely another way of saying $p(Q) = MdQ/dt$). The parameter t defines semi-classical notion of time.[15]

The next question which arises is about the domain of validity of the WKB approximation.[25] This approximation will be valid if the momentum varies slowly over one 'de-Broglie wave length' of the system i.e. when $(p'/p) \ll p/\hbar$ or, equivalently, when

$$|p^3| \gg \hbar M^2 |V'| \,. \tag{4.20}$$

This condition can also be rewritten in the form:

$$E \gg MV(Q) + (M\hbar^2 V'^2)^{1/3} \equiv f(Q) \,. \tag{4.21}$$

Equations (4.20) and (4.21) emphasize two different aspects of WKB approximation. If E is fixed, then (4.20) implies that the approximation is valid far away from the turning points at which $p(Q) = 0$. Suppose, on the other hand, we are interested in obtaining a WKB approximation around some value $Q = Q_1$. Equation (4.21) then gives the limitation on E for this to be feasible. If $V(Q)$ is a rapidly increasing function of Q (near Q_1) then we could satisfy (4.21) only for a small region near Q_1.

When necessary, we can also consider the higher order corrections to WKB approximation.[25,22] These can be obtained by writing

$$\theta(Q) = \theta_0(Q) + \hbar^2 \theta_1(Q) + \cdots$$

and substituting into (4.13). We find that, for example,

$$\theta_1'(Q) = \frac{1}{2\sqrt{\theta_0'}} \frac{d^2}{dQ^2} \left\{ \frac{1}{\sqrt{\theta_0'}} \right\} . \tag{4.22}$$

It should be emphasized that the first quantum correction to WKB solution is of $\mathcal{O}(\hbar^2)$ and not of $\mathcal{O}(\hbar)$.

The Schrödinger equation (4.8) depends only on the ratio $(\hbar/M)$ and not on the individual values of $\hbar$ and M. (This can be easily seen by dividing (4.8) throughout by M.) Thus an expansion in $\hbar$ is identical to an expansion in M^{-1}. We could have, for example, taken $\psi(Q)$ to be

$$\psi(Q) = (A(Q) + M^{-1}A_1(Q) \cdots) \exp \frac{i}{\hbar} (M\sigma_0(Q) + \sigma_1(Q) + M^{-1}\sigma_2(Q) \cdots) \tag{4.23}$$

and equated the coefficients of equal powers of M in the Schrödinger equation. To $\mathcal{O}(M)$ and $\mathcal{O}(M^0)$ this will give

$$\frac{1}{2} M(\sigma_0')^2 + MV - E = 0 \,; \qquad \sigma_1' = 0 \tag{4.24}$$

and

$$2A'(Q)\sigma_0' + \sigma_0'' A = 0 \,. \tag{4.25}$$

Except for trivial scalings with respect to M, this leads to the same result as (4.16).

Let us now translate the above analysis to the mini-superspace describing gravity.[22,27] The Lagrangian is (see (3.11))

$$L = M\left\{\frac{1}{2} G^{AB} \frac{\dot{g}_A \dot{g}_B}{N} - NV(g_A)\right\} \tag{4.26}$$

where $M = (16\pi G)^{-1}$ sets the overall scale. We shall quantize this system using the Wheeler-DeWitt equation (Eq. (3.17), with matter variables absent):

$$\left(-\frac{\hbar^2}{2M}\nabla^2 + MV\right)\psi(g_A) = 0 \,. \tag{4.27}$$

Separating ψ into the modulus and phase

$$\psi(g_A) = R(g_A)e^{i\theta(g_A)/\hbar} \tag{4.28}$$

and substituting into (4.27) we get the equations:

$$\frac{1}{2M}(\nabla\theta)^2 + MV = \frac{\hbar^2}{2M}\left\{\frac{\nabla^2 R}{R}\right\} \tag{4.29}$$

$$\nabla \cdot (R^2 \nabla\theta) = 0 \,. \tag{4.30}$$

(The notation is self explanatory: ∇ stands for $(\partial/\partial g^A)$ and all 'dot products' are calculated using G^{AB} and its inverse.)

Equations (4.29) and (4.30) are exact and are completely equivalent to the Wheeler-DeWitt equation. We can approximate them in the same way as we approximated (4.10) and (4.11). The WKB wave function for gravity will be

$$\psi^{(0)}(g_A) = R(g_A)\exp\frac{i}{\hbar}\theta_0(g_A) \tag{4.31}$$

where

$$\frac{1}{2M}(\nabla\theta_0)^2 + MV(g_A) = 0 \tag{4.32}$$

and R satisfies (4.30) with θ replaced by θ_0. The only difference between gravity and the system considered earlier is the following: In (4.14), E was a free parameter which we can choose at will; here Wheeler-DeWitt equation automatically sets its value to zero.

It is possible to solve for R in terms θ_0. When there is only one variable this is trivial and R is proportional to $(\theta')^{-1/2}$. If there are more variables then R can be expressed as[28]

$$R = \left\{ \det\left(-\frac{\partial^2 \theta_0}{\partial g_A^{(i)} \partial g_B^{(f)}} \right) \right\}^{1/2} \equiv F(g_A) \tag{4.33}$$

where θ_0 is the classical action, evaluated for a trajectory with end-points $g_A^{(i)}$ and $g_A^{(f)}$. Thus (4.31) can be expressed entirely in terms of θ_0.

The semi-classical behavior of (4.31) follows in the same way as that of (4.16). The wavefunction $\psi(g_A)$ is not peaked about any classical trajectory in the configuration space. The only way classical limit can be obtained is by using the rule: "derivative of WKB phase is equal to the classical momentum". This rule will give

$$p^A = \frac{\partial \theta_0}{\partial g_A} = p^A(g_A) \ . \tag{4.34}$$

This relation defines a vector field in the mini-superspace:

$$\underset{\sim}{p} = \underset{\sim}{\nabla} \theta = \{ p^A(g_A) \} \ . \tag{4.35}$$

Equation (4.32) now becomes

$$\frac{1}{2M} G^{AB} p_A p_B + MV = 0 \tag{4.36}$$

which is just the constraint equation $H = 0$. Differentiating this relation with respect to, say, g_c we get,

$$\frac{1}{2M} G^{AB}_{,C} p_A p_B + \frac{1}{M} G^{AB} p_A p_{B,C} + MV_{,C} = 0 \tag{4.37}$$

which, on using (4.34), becomes

$$\frac{1}{2M} G^{AB}_{,C} \frac{\partial \theta_0}{\partial g^A} \frac{\partial \theta_0}{\partial g^B} + \frac{1}{M} G^{AB} \frac{\partial \theta_0}{\partial g^A} \frac{\partial^2 \theta_0}{\partial g^B \partial g^C} + MV_{,C} = 0 \ . \tag{4.38}$$

We can rewrite this equation in a more meaningful form. To do this we first introduce the concept of time, as in Eq. (4.19). Equation (4.35) defines a vector field, $\nabla \theta_0$ in superspace to which integral curves can be drawn. These integral curves can be parametrized by a variable τ such that

$$\frac{d}{d\tau} \equiv \frac{p^A}{M} \frac{\partial}{\partial g^A} = \frac{1}{M} \left\{ G^{AB} \frac{\partial \theta_0}{\partial g^B} \right\} \frac{\partial}{\partial g^A} \ . \tag{4.39}$$

Using (4.39) in the third term of (4.38), we get

$$\frac{dp_c}{d\tau} + \frac{1}{2M} G^{AB}_{,c} p_A p_B = -MV_{,c} \,. \tag{4.40}$$

It can be easily shown that (4.40) is identical to the dynamical equations[19] obtained by varying g_A in the Lagrangian (4.26), once we make the identification $Ndt = d\tau$.

The semi-classical interpretation of ψ depends on two identifications represented by Eq. (4.34) (for p_A) and Eq. (4.39) (for τ). The above discussion once again emphasizes the semi-classical nature of 'time'.

The conditions of validity for the WKB approximation can be worked out in a manner similar to the one-dimensional case. The higher order corrections can also be computed in essentially the same way. In particular, the first quantum corrections to the WKB wavefunctions will be of order $\hbar^2$ and not of order $\hbar$.

4.2. *Semi-classical limit: Gaussian states*

In the last section we saw that WKB states allow a particular kind of approximation to the Schröding equation. By identifying the derivative of the phase of the WKB wave function with the classical momentum, we can provide a semi-classical interpretation to the WKB states.

There are, however, other means of approaching the semi-classical limit. In particular, it is possible to construct approximate solutions to the Schrödinger equation which are actually peaked around some trajectories in the configuration space. (This is in contrast with the WKB wavefunction, for which $|\psi|^2$ is not peaked around any trajectory). These solutions are specially relevant when we consider the time-dependent Wheeler-DeWitt equation.

We shall illustrate the construction with a quantum mechanical system described by the Schrödinger equation:

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2M} \frac{\partial^2 \psi}{\partial Q^2} + MV\psi \,. \tag{4.41}$$

The connection between (4.41) and the gravitational system coupled to a clock, discussed in Sec. 3.4, should be obvious. The Schrödinger equation for the 'clock + gravity' system

$$i \frac{\partial \psi}{\partial c} = -\frac{1}{2} l^2 \nabla^2 \psi + \frac{V}{l^2} \psi \tag{4.42}$$

is structurally same as (4.41). Nothing is lost by introducing the clock variable c; we have seen that in any approach 'time' appears only as a semi-classical construct. The clocks do exist in the semi-classical limit.

We are interested in the solutions to (4.41) which are sharply peaked around some real trajectory in the Q-space. This suggests the following ansatz:

$$\psi(Q, t) = N(t) \exp\left(-\frac{1}{4} B(Q - x)^2 + \frac{i}{\hbar} PQ + \varepsilon\right) \tag{4.43}$$

where B is complex and x, p, ε and N are real. All these quantities are taken to be functions of time. It is clear that $|\psi|^2$ is peaked around x and has a width inversely related to the real part of B. We are interested in the conditions under which ψ will be an approximate solution to the Schrödinger equation.

Our choice of ψ corresponds to the following choices for the amplitude and phase of the wavefunction:

$$R = N \exp\left\{-\frac{1}{4} B_R(Q - x)^2\right\} \tag{4.44}$$

and

$$\theta = -\frac{1}{4} B_I(Q - x)^2 + PQ + \hbar\varepsilon \tag{4.45}$$

where B_R and B_I are the real and imaginary parts of B. We have earlier derived the equations satisfied by R and (see (4.11) and (4.1)). In the time dependent case (4.11) is replaced by

$$2R\dot{R} + \frac{\partial}{\partial Q}(R^2\theta') = 0 \tag{4.46}$$

we substitute (4.44) and (4.45) into (4.46). Both the terms in (4.46) will be quadratic expressions in Q. Equating the coefficients of Q^2, Q and Q^0 we get 3 equations which can be simplified to give the following conditions

$$\hbar B_I = M \frac{\dot{B}_R}{B_R} \tag{4.47}$$

$$B_R\left(\dot{x} - \frac{P}{M}\right) = 0 \tag{4.48}$$

$$N = (\text{const}) \cdot B_R^{1/4} . \tag{4.49}$$

Of these, (4.49) is merely the normalization condition for the wave function and (4.47) expresses B_I in terms of B_R. The real physical content is in (4.48) which says that the parameter P is the classical momentum corresponding to the trajectory $x(t)$. This is encouraging because the momentum distribution corresponding to ψ will be peaked around P. The relation (4.48) shows that our wave function is peaked around classical trajectory in the Q-space and classical momentum in the P-space. This is precisely what we expect in the semi-classical limit.

So far, we have not made any approximations. Equation (4.46) was solved exactly. We now have to handle Eq. (4.10). It is clear that some form of approximation will be required here. Rewriting (4.10) in the form

$$MV = \frac{\hbar^2}{2M}\frac{R''}{R} - \left\{\dot{\theta} + \frac{(\theta')^2}{2M}\right\} \tag{4.50}$$

we see that all the terms in the right-hand side will be of quadratic order or lower in $(Q - x)$. However, the potential V was entirely arbitrary and could contain any function of Q. Thus it will not be possible to satisfy this equation identically for all values of Q.

Here is where the approximation comes in. We will assume that $V(Q)$ can be expanded in a Taylor series around x and that we can retain terms up to quadratic order in $(Q - x)$. If this is done, the left-hand side can indeed be made approximately equal to an expression quadratic in Q. The coefficients of Q^2, Q and Q^0 can again be equated. This gives us the following relations after some algebra:

$$\dot{P} = -MV'(x) \tag{4.51}$$

$$\dot{\varepsilon} = -\frac{p^2}{2M} + MV + MV' - \frac{\hbar^2}{4M}B_R \tag{4.52}$$

$$\hbar\dot{B}_I = 2MV'' - \frac{\hbar^2}{2M}B_R^2 - B_I^2 . \tag{4.53}$$

Of these, the first equation guarantees our classical limit. This equation combined with (4.48) identifies $x(t)$ as the classical trajectory. Equation (4.53) describes the evolution of the time-dependent phase and is not of particular interest. Equation (4.52) describes the evolution of quantum fluctuations around the classical trajectory. By introducing a variable $J(t)$ defined by the relation

$$B = \frac{2M}{i\hbar}\frac{\dot{J}}{J}$$

and using (4.47), Eq. (4.53) can be cast in the form

$$\ddot{J} + V''J = 0 . \tag{4.54}$$

This shows that the fluctuations are essentially driven by the second-order term in the potential.

We have thus succeeded in producing an approximate semi-classical solution to the Schrödinger equation. We next ask: when is this approximation reliable? Notice that the only approximation which went in was the quadratic Taylor expansion for $V(Q)$. This is valid whenever the fluctuations are small. More quantitatively, we can say that the deviation from the classical trajectory, which should be of order $\sigma = (\frac{1}{2}B_R)^{-1/2}$ must be

$\mathcal{O}(\hbar)$. In any potential, not violently changing with time, the fluctuations wil remain bounded and can be taken to be of $\mathcal{O}(\hbar)$. In other situations, the validity has to be ascertained by a more detailed study.

It must be emphasized that the WKB approximation also relies indirectly on the smallness of the fluctuations for its semi-classical interpretation. The 'rule' of identifying the derivative of the phase with the classical momentum is meaningful only when the fluctuations are not too large.

For higher order corrections around the classical trajectory, we can proceed by retaining larger deviations from the classical trajectory. Quantitatively, this can be done by using in the exponent of the wave function terms higher than the quadratic. This is similar to the approach used for obtaining higher order corrections to the WKB approximation.

All that has been said here can be easily translated to the gravitational variables. Since we had already gone through a similar exercise in the previous section, we will not repeat it here. It should be clear that one can use Gaussian states with as much ease as the WKB states to study the semi-classical gravity.[29] In fact, the semi-classical limit for Gaussian states arises in a very natural manner without we having to perform any gymnastics in the phase space.

The contrast between Gaussian states and WKB states is very apparent in the case of a quantum harmonic oscillator in a potential $\frac{1}{2}M\omega^2Q^2$. The WKB approximation corresponds to taking a stationary state with large energy E. The exact wave function $\psi_E(Q)$ oscillates rapidly and has a profile which follows classical trajectory.[30] The Gaussian approximation—on the other hand—corresponds to using a coherent state solution with amplitude $a = (2E/M\omega^2)^{1/2}$. The trajectories are now actually peaked around the classical path.

4.3. *The issue of backreaction: WKB states*

So far we have been considering the semi-classical limit of pure gravity. While this is of some mathematical interest, we almost never encounter such a situation in a realistic physical scenario. There will always be matter present, the quantum fluctuations of which needs to be considered.

When both gravity and matter are treated classically, the situation can be handled by Einstein's equations. When both matter and gravity are quantum mechanical, then we need to solve some form of Wheeler-DeWitt equation. (We shall assume in this section that the relevant Wheeler-DeWitt equation is the time independent one (3.18); the modifications which are needed when we use (3.17) will be mentioned in the next section.)

As indicated in the introduction to this part, there is a widespread belief that a third level of description is possible for a system involving gravity and matter fields, viz. the one in which gravity is treated semi-classically and matter is treated quantum mechanically. The crucial word in the last sentence is 'semi-classical', which suggests something more than classical. There is absolutely no difficulty in devising a sensible approximation scheme in which gravity is classical and matter fields are quantum mechanical (''quantum fields in curved space''). In this scheme, gravity is unaffected by the quantum fluctuations of matter. What one hopes to do is to proceed to the next level in

which some kind of backreaction of matter fields on gravity can be introduced. We shall discuss this issue in detail and show that this hope can be realized only under very strigent conditions.[22,31] By and large, it is incorrect to develop a scheme in which quantum fluctuations of matter fields are included but those of gravity are excluded.

Our system consisting of gravity and matter will be described by the Lagrangian $L_{\text{total}} = L_{\text{gravity}} + L_{\text{matter}}$ where L_{gravity} is given by (3.11) and L_{matter} is the matter Lagrangian. Once again we will resort to a toy quantum mechanical model to illustrate the concepts. In the end, we can easily translate the results to the context of gravity. This model consists of two point 'particles' described by the Lagrangian:

$$L = \frac{1}{2} M(\dot{Q}^2 - V(Q)) + \frac{1}{2}\dot{q}^2 - U(Q, q) \tag{4.55}$$

where Q and q represent the coordinates of the two particles. The degree of freedom Q is analogous to gravity and q is analogous to the quantum field. In the 'level one' description both Q and q are classical and the system is described by the following equations:

$$M\ddot{Q} + MV' = -\partial U(Q, q)/\partial Q \tag{4.56}$$

$$\ddot{q} = -\partial U(Q, q)/\partial q \,. \tag{4.57}$$

In 'level two' Q and q are quantum mechanical and the system is described by the wave function $\psi(Q, q)$ which satisfies the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2M}\frac{\partial^2 \psi}{\partial Q^2} - \frac{\hbar^2}{2}\frac{\partial^2 \psi}{\partial q^2} + MV\psi + U\psi = E\psi \,. \tag{4.58}$$

In 'level three' we demand an intermediate description in which Q is described by a c-number equation while q is described by a wave function $\chi(q, \bar{Q})$ in a given background $\bar{Q}$.

The first two terms in (4.6) are identical to the terms in the 'pure-gravity' model (4.6), with M playing the role of G^{-1} (inverse of Newton's constant). This suggests that one should not really think of M as the mass of Q-particle. This may be done, it is not very illuminating. A better feel for M in (4.55) can be obtained by writing (4.55) as

$$\frac{L}{M} = \frac{1}{2}\dot{Q}^2 - V(Q) + M^{-1}\left(\frac{1}{2}\dot{q}^2 - U(Q, q)\right) . \tag{4.59}$$

Clearly, M^{-1} (which is analogous to G) determines the coupling between the variables Q and q. In particular, the potential felt by Q is

$$V_{\text{total}} = V(Q) + M^{-1}U(Q, q) \,. \tag{4.60}$$

Note that as $M \to \infty$ the effect of q on Q is suppressed. In what follows, it is useful to keep in mind the correspondence: $Q \leftrightarrow$ gravity, $q \leftrightarrow$ field and

$$(M \to \infty) \leftrightarrow (G \to 0) \ . \tag{4.61}$$

For the system described by the Lagrangian of (4.55) we can work out two kinds of approximations. We can attempt a power series expansion in $\hbar$ or in M. Since L in (4.55) does not scale out as $(M/\hbar)$ [in contrast to the L of (4.6)] these two approximations are inequivalent, and describe different physical situations.

In a $\hbar$-expansion, $\mathcal{O}(1)$ term will give the classical equation of motion for both Q and q—i.e. Eqs. (4.56) and (4.57). The $O(\hbar)$ term will give a determinant factor analogous to $(\theta')^{-1/2}$ term in (4.15). This expansion does not give a dichotomas result with Q-classical and q-quantum mechanical. Instead, $\hbar$-expansion treats the quantum nature of both Q and q at the same footing. Since our primary interest is not in this situation, we will not consider this expansion. Instead, we will start with the Schrödinger equation

$$-\frac{\hbar^2}{2M}\frac{\partial^2\psi}{\partial Q^2} - \frac{\hbar^2}{2}\frac{\partial^2\psi}{\partial q^2} + MV\psi + U\psi = E\psi \tag{4.62}$$

and seek a power series solution in M^{-1}. As in (4.23), we write

$$\psi \equiv Ae^{(i/\hbar)S} \cong (A_0 + M^{-1}A_1 + \cdots)\exp\frac{i}{\hbar}(M\sigma_0 + \sigma_1 + M^{-1}\sigma_2 + \cdots) \tag{4.63}$$

and combine terms which are of the same order in M. This can be easily done by pulling out A_0 and expanding $\exp(i/\hbar)(M^{-1}\sigma_2 + \cdots)$ in a Taylor series: We get

$$\psi = \exp\left(\frac{i}{\hbar}M\sigma_0\right)\cdot\left(A_0\exp\frac{i}{\hbar}\sigma_1\right)\left(1 + M^{-1}\frac{A_1}{A_0} + \cdots\right)\left(1 + M^{-1}\frac{i\sigma_2}{\hbar} + \cdots\right)$$

$$= \exp\left(\frac{i}{\hbar}M\sigma_0\right)\cdot\chi\cdot(1 + M^{-1}B_1 + \cdots) \ . \tag{4.64}$$

Here χ is $O(M^0) = O(1)$, and B_n involves (A_n/A_0) and terms of order M^{-n} arising from the exponential. This calculation shows that it is redundant to expand both S and A in a power series in M. It is enough to consider

$$\psi = e^{(i/\hbar)M\sigma_0}\cdot B \tag{4.65}$$

where $B = \chi + M^{-1}B_1 + M^{-2}B_2 + \cdots$ etc. We will take $\sigma_0 = \sigma_0(Q)$ and $\chi = \chi(Q, q)$, $B_i = B_i(Q, q)$. We can also assume—without any loss of generality—what σ_0 is real and B is complex. Substituting (4.65) into (4.62) and retaining the leading order terms we get:

$$M\left\{\frac{\sigma_0'^2}{2} + V - \varepsilon\right\}\chi - \left\{\frac{\hbar^2}{2}\frac{\partial^2\chi}{\partial q^2} - U\chi + i\hbar\sigma_0'\chi' + \frac{i\hbar}{2}\sigma_0''\chi\right\} = 0 \tag{4.66}$$

(Prime denotes differentiation with respect to Q; as we did before, we have assumed E to be $\mathcal{O}(M)$, i.e. $E = M\varepsilon$, rather than expand it; this only changes the phase in an insignificant manner).

Equation (4.66) is one complex—or, two real—equation for 3 unknown real functions: σ_0 and real and imaginary parts of χ; it is underdetermined unless some extra criterion is used. We tackle this indeterminacy in the usual way: We demand the coefficients of each power of M to vanish individually. To $\mathcal{O}(M)$ and $\mathcal{O}(1)$ we obtain:

$$\frac{1}{2}\sigma_0'^2 + V - \varepsilon = 0 \tag{4.67}$$

$$i\hbar\sigma_0'\chi' + \frac{i\hbar}{2}\sigma_0''\chi = -\frac{\hbar^2}{2}\frac{\partial^2\chi}{\partial q^2} + U\chi \,. \tag{4.68}$$

In (4.68), we note that

$$i\hbar\sigma_0'\chi' + \frac{i\hbar}{2}\sigma_0''\chi = i\hbar\sigma_0'\chi\left\{\frac{\partial}{\partial Q}\ln(\sqrt{\sigma_0'}\chi)\right\}$$

$$= i\hbar\sqrt{\sigma_0'}\frac{\partial f}{\partial Q} \tag{4.69}$$

where we have defined $f \equiv \sqrt{\sigma_0'}\chi$, or, equivalently

$$\chi(Q, q) = \frac{1}{\sqrt{\sigma_0'(Q)}} f(Q, q) \,. \tag{4.70}$$

Using (4.48) and (4.49) in (4.47) we get

$$i\hbar\sigma_0'\frac{\partial f}{\partial Q} = -\frac{\hbar^2}{2}\frac{\partial^2 f}{\partial q^2} + U(Q, q)f \,. \tag{4.71}$$

Equation (4.71) can be written in a more suggestive fashion. We note that (4.67) determines a classical trajectory $Q_c(t)$ satisfying

$$\frac{1}{2}\left(\frac{dQ_c}{dt}\right)^2 = \varepsilon - V(Q_c) \tag{4.72}$$

so that, in (4.71), $U(Q_c, q) = U(t, q)$ and

$$i\hbar\sigma_0'(Q_c)\frac{\partial}{\partial Q} = i\hbar\frac{\partial}{\partial t} \,. \tag{4.73}$$

Therefore (4.71) can be written as

$$i\hbar \frac{\partial f}{\partial t} = -\frac{\hbar^2}{2}\frac{\partial^2 f}{\partial q^2} + U(Q_c(t), q)f \ . \tag{4.74}$$

The message of (4.74) and (4.67) is loud and clear: To $\mathcal{O}(1)$, Q is determined by the classical equations (with q ignored) and the wave function $\chi(q, Q_c)$ represent a quantum theory in a given curved background $Q_c(t)$ via (4.70) and (4.74). To this order, the wave function $\psi(Q, q)$ has the following interpretation:

$$\begin{aligned}\psi(Q, q) &= \left\{\begin{matrix}\text{amplitude} \\ \text{for } Q,\ q\end{matrix}\right\} \\ &= \left\{\begin{matrix}\text{WKB amplitude for} \\ Q,\ \text{ignoring } q\end{matrix}\right\} \times \left\{\begin{matrix}\text{amplitude for } q \\ \text{given a } Q\end{matrix}\right\} \\ &= \frac{1}{\sqrt{\sigma_0'(Q)}} \cdot \exp\left(\frac{i}{\hbar} M\sigma_0(Q)\right) \cdot f(q, Q) \ .\end{aligned} \tag{4.75}$$

Thus to $\mathcal{O}(M^{-1})$ we may say that there is no 'backreaction'.

A crucial ambiguity rears its head now. There is no question that (4.75) correctly describes an approximate wave function $\sim\mathcal{O}(1)$. But getting a classical interpretation from the WKB wave function is—as we have seen in Sec. (4.1)—nontrivial. The "rule" of WKB physics is the following: "The derivative of the phase of the WKB wavefunction is the classical C-number momenta". If we want the classical equation to $\mathcal{O}(M)$, then the phase is uniquely determined. It is the $M\sigma_0(Q)$ given in (4.67). Identifying its derivative with a classical momentum we will get (4.56) with right-hand side set to zero. There is no backreaction. This, of course, is no surprise. But suppose we want to write the classical equations correct to not to $\mathcal{O}(M)$ but to $\mathcal{O}(1)$. To do this we have to take the phase of the wave function up to $\mathcal{O}(1)$ and consider it to be "classical action".

Equation (4.75) signals that this desire is dangerous and in general cannot be fulfilled. To see this, consider what happens if we try to evaluate the phase up to $\mathcal{O}(1)$. We can separate f into amplitude and phase and write

$$f(Q, q) = a(\theta, q) \exp(ib(Q, q)) \ . \tag{4.76}$$

This done, the total phase of ψ to $\mathcal{O}(1)$ is $\{(M\sigma_0/\hbar) + b(Q, q)\}$. This phase is completely useless for a semi-classical description of Q because it depends—in general—on the quantum variable q explicitly. It is easy to see that, for a general state, b must depend on q. If we substitute f into (4.74) and separate the real and imaginary parts we will get

$$-\sigma_0' \frac{a'}{a} = \frac{\hbar}{2}\left[\frac{\partial^2 b}{\partial q^2} + \frac{2}{a}\left(\frac{\partial a}{\partial q}\right)\left(\frac{\partial b}{\partial q}\right)\right] \tag{4.77}$$

$$-\hbar\sigma_0' b' = \frac{\hbar^2}{2}\left[\left(\frac{\partial b}{\partial q}\right)^2 - \frac{1}{a}\left(\frac{\partial^2 a}{\partial q^2}\right)\right] + U(Q, q) \ . \tag{4.78}$$

Thus if b is independent of q, a should be independent of Q. What is more we must have

$$-\hbar\sigma_0' b' \equiv F(Q) = -\frac{\hbar^2}{2}\frac{1}{a(q)}\left(\frac{d^2 a(q)}{dq^2}\right) + U(Q, q) \ . \tag{4.79}$$

This condition is impossible to satisfy because U, in general, depends on both the variables.

We are therefore forced to conclude that—in general—there is no c-number backreaction to $\mathbb{O}(1)$. Of course there was no *a priori* reason for such a backreaction to exist. It is just that one would have been happier if such a prescription had existed.

It is possible to introduce an artificial phase in (4.75) and mimic a backreaction. There is however no way of determining this phase without invoking additional principles. We refer the reader to original literature for discussion of this point (Refs. 21, 22 and 32).

Having said that in general we cannot obtain a $\mathbb{O}(1)$ backreaction we would like to comment on a special situation in which this may be possible.[31,33] Consider a particular case in which

$$U(Q, q) = \frac{1}{2}\omega^2(Q(t))q^2 \ . \tag{4.80}$$

Our function f in this case is determined by the equation

$$i\frac{\partial}{\partial t}f(q, t) = -\frac{1}{2}\frac{\partial^2 f}{\partial q^2} + \frac{1}{2}\omega^2(Q(t))q^2 f \tag{4.81}$$

(we have set $\hbar = 1$). Consider the following ansatz for $f(q, t)$:

$$f(q, t) = e^{-\alpha - \eta q^2} \tag{4.82}$$

where α and η are functions of Q and hence of time. Since the potential in (4.80) is quadratic, the Schrödinger equation propagates an initial Gaussian into another Gaussian.

If f is of the form (4.82), the Schrödinger equation determines the phase $\alpha(Q)$ and hence the backreaction. Substituting (4.82) in (4.81), and comparing the coefficients of different powers of q gives

$$-i\dot{\alpha} = \eta \ , \qquad \dot{\eta} = -2i\eta^2 + \frac{i}{2}\omega^2 \ . \tag{4.83}$$

Equation (4.83) implies that $\alpha = i\int \eta(t)dt$. Splitting η into its real and imaginary parts $-\eta \equiv R + iI$,—and comparing with (4.82) shows that f has a Q-dependent phase: $\theta \equiv -\int R(t)dt$. The full wave function $\psi(Q, q)$ of (4.75) may be rewritten as

$$\Psi(Q.\ q) = \frac{1}{\sqrt{\sigma_0'}} \exp\left(iM\sigma_0(Q) + i\theta(Q) + \int I(t)dt - \eta q^2 \right) . \tag{4.84}$$

The classical equation for Q- to $\mathcal{O}(1)$- will be now determined by the total Q-dependent phase given by

$$\sigma_{\text{total}}(Q) = M\sigma_0(Q) + \theta(Q) . \tag{4.85}$$

It is easy to see that this phase defines a classical effective potential:

$$V_{\text{eff.}} = MV - \sigma_0'(Q)\theta' = MV - \dot{\theta} . \tag{4.86}$$

In other words, the phase of f provides a backreaction $(-\dot{\theta}) = R(Q)$ in the $\mathcal{O}(1)$ level. It should be stressed that we obtained this backreaction only because both our potential U and the quantum state f had very special forms. In general, such a neat separation may not occur. In this simple example, one can proceed further and determine the form of the backreaction explicitly. In particular the correction term θ in (4.86) is just the expectation value of the Hamiltonian for q, if the background field Q varies slowly.[31]

To conclude this section, we briefly indicate how the result can be translated for the Wheeler-DeWitt equation. We consider the case in which the matter field is a scalar field ϕ with its conjugate momentum denote Π_ϕ. The Wheeler-Dewitt equation will then be:

$$\left\{ -\frac{1}{2M} \nabla^2 + MV(g_A) + H_m(g_A, \phi, \Pi_\phi) \right\} \psi(g, \phi) = 0 . \tag{4.87}$$

We have used the symbol M to denote 1^{-2}. In (4.87) we can consider the limit of $M \to \infty$, which would correspond to treating g_A's as "heavy" degrees of freedom and ϕ as a "light" degree of freedom. In this approximation, one can look for a WKB-type solution

$$\begin{aligned} \psi(g, \phi) &= \exp\{iMS_0(g)\} \cdot A(g) \cdot F(g, \phi) . \\ &= F(g, \phi) \exp[iMS_0(q) + iS_1(g) + \mathcal{O}(M^{-1})] . \end{aligned} \tag{4.88}$$

Note that F and S_1 are $\mathcal{O}(1)$ and that S_1 is pure imaginary (since A is real). Substituting (4.88) into (4.87) and equating coefficients of $\mathcal{O}(M)$ and $\mathcal{O}(1)$ we get

$$\frac{1}{2} (\nabla S_0)^2 + V(g) = 0 \tag{4.89}$$

$$\left\{ \nabla S_0 \cdot \nabla S_1 - \frac{i}{2} \nabla^2 S_0 \right\} F - i(\nabla S_0) \cdot \nabla F + \hat{H}_m F = 0 \tag{4.90}$$

(all the dot products are defined using G^{AB}; e.g. $\nabla S_0 \cdot \nabla S_1$ is $G^{AB} \partial_A S_0 \partial_B S_1$ etc.). We will now rewrite these equations in a more tractable form.

We note that (4.89) involves only the gravitational modes and is just the Hamilton-Jacobi equation for pure gravity. Given any solution $S_0(g)$ of this equation, we can obtain a vector field $\nabla S_0 = (\partial S_0/\partial g_A) = p^A(g)$. The integral curves to this vector field will be parametrized by the propertime τ: i.e. $p^A = (dg^A/d\tau)$. Thus we can write

$$\frac{\partial}{\partial \tau} = \frac{\partial S_0}{\partial g^A} \cdot \frac{\partial}{\partial g^A} . \tag{4.91}$$

Using this (4.90) can be written as

$$i \frac{\partial F}{\partial \tau} = \hat{H}_m F + \left\{ \nabla S_0 \cdot \nabla S_1 - \frac{i}{2} \nabla^2 S_0 \right\} F . \tag{4.92}$$

Let us now define the 'dot product' between two functions f and g by

$$(f, g) = \int_{-\infty}^{\infty} d\phi f^* g . \tag{4.93}$$

Since we need $(F, F) = 1$, it follows that $i(F, \dot{F})$ is a real quantity. Consider the dot product of (4.92) with F. We have

$$i(F, \dot{F}) = \langle \hat{H}_m \rangle + \left\{ \nabla S_0 \cdot \nabla S_1 - \frac{i}{2} \nabla^2 S_0 \right\} . \tag{4.94}$$

Since the matter Hamiltonian is Hermitian, $\langle H_m \rangle$ is real; so is the left-hand side. But since S_1 is pure imaginary and S_0 is real both the terms in the curly bracket { } are imaginary. Thus equating the real and imaginary part of (4.94) we get

$$\nabla S_0 \cdot \nabla S_1 = \frac{i}{2} \nabla^2 S_0 \tag{4.95}$$

$$i \frac{\partial F}{\partial \tau} = \hat{H}_m(q, \phi, \pi) F . \tag{4.96}$$

Equations (4.89), (4.95) and (4.96) determine S_0, S_1 and F in the wave function $\phi(g, \phi)$. In this form these equations have a simple interpretation.

We see that (4.89) and (4.95) fix the amplitude and phase (i.e. $A = \exp iS_1$, and S_0) of the gravitational part of the wave function. These equations are the standard WKB equations for pure gravity. In other words, we see no "backreaction" effect of ϕ on g_A at this order of approximation.

Once the geometry is determined in the WKB limit, Eq. (4.96) represents the Schrödinger equation for the field in this background geometry g_A. This equation leads to what is usually called the 'quantum field theory in curved spacetime' limit.

4.4. *Backreaction with Gaussian states*

In Secs. 4.1 and 4.2, we introduced two possible approaches towards the semi-classical limit. The WKB limit was suitable when Wheeler-DeWitt equation is used and the Gaussian states turned out to be more appropriate if time-dependent Wheeler-DeWitt equation is used.

The issue of backreaction, introduced in the last section, can also be studied using Gaussian states. This will be appropriate when a clock variable is present, making Wheeler-DeWitt equation time-dependent. The analysis proceeds in a manner identical to the discussion in the last section, and hence will not be repeated here.

The results are also broadly the same.[34] To the lowest order one obtains classical equations for the gravity and the equations for the quantum field theory in curved space-time. In general, one cannot obtain 'backreaction' from the quantum fields alone. However, there are some special situations—similar to the ones described towards the end of Sec. 4.3—in which such a backreaction is possible.

4.5. *Path integrals and backreaction*

It is usual to derive the 'backreaction' in a couple of lines using the path integrals: We start with the full path integral (PI):

$$K = \int \mathcal{D}Q\mathcal{D}q \exp\left(\frac{i}{\hbar}\right)[A_0(Q) + A_1(Q, q)] \tag{4.97}$$

and perform the q integration, obtaining:

$$K = \int \mathcal{D}Q \exp\frac{i}{\hbar}[A_0(Q) + A_{\text{eff}}(Q)] \tag{4.98}$$

where we have defined

$$e^{iA_{\text{eff}}(Q)/\hbar} = \int \mathcal{D}q e^{iA_1(Q,q)/\hbar}\,. \tag{4.99}$$

We now do the Q-integration in the saddle-point approximation (SPA), determining Q as a solution to

$$\frac{\delta A_0}{\delta Q} = -\frac{\delta A_{\text{eff}}}{\delta Q} = \frac{\int \mathcal{D}q \dfrac{\delta A_1}{\delta Q} e^{iA_1(Q,q)/\hbar}}{\int \mathcal{D}q e^{iA_1/\hbar}}\,. \tag{4.100}$$

The right-hand side is the "backreaction". This is in fact the deviation most familiar to everybody. It is therefore necessary to look at the derivation of (4.100) closely and

compare it with our results in previous sections. We shall now address ourselves to this task.

We shall first show that the expansion in the powers of M^{-1} in the path integral leads to exactly the same result as in Sec. 4.3 viz. that there is no unique backreaction. Having settled this, we will discuss under what circumstances a backreaction can be obtained.

Consider the path integral in (4.97) with the boundary conditions: $Q = Q_1$, $q = q_1$ at $t = t_1$ and $Q = Q_2$, $q = q_2$ at $t = t_2$. The action for our system is

$$\begin{aligned} A &= A_0(Q) + A_1(Q, q) \\ &= \int dt\, M\left(\frac{1}{2}\dot{Q}^2 - V(Q)\right) + \int dt\left(\frac{1}{2}\dot{q}^2 - U(Q, q)\right). \end{aligned} \tag{4.101}$$

Performing the q-integration in (4.97), we get

$$K(Q_2 q_2 t_2;\, Q_1 q_1 t_1) = \int \mathcal{D}Q \exp\left(\frac{i}{\hbar} M \int dt\left(\frac{1}{2}\dot{Q}^2 - V(Q)\right)\right) \mathcal{G}(q_2 t_2;\, q_1 t_1 \,|\, Q)\,. \tag{4.102}$$

We want to evaluate this expression in the limit $M \to \infty$. The phase of $(\mathcal{G}e^{iA_0/\hbar})$ oscillates rapidly as $M \to \infty$ while the amplitude $\mathcal{G}$ (which is of $O(1)$) varies slowly. Clearly a saddlepoint approximation will pick the solution $Q = \bar{Q}$ so that

$$\left[\frac{\delta A_0}{\delta Q}\right]_{Q=\bar{Q}} = 0 \qquad \text{i.e.} \quad \ddot{\bar{Q}} + V'(\bar{Q}) = 0\,. \tag{4.103}$$

In this $(M \to \infty)$ limit the Kernel is

$$\begin{aligned} K(Q_2, q_2, t_2;\, Q_1, q_1, t_1) &= N\left(\frac{\partial^2 \bar{A}_0}{\partial Q_1 \partial Q_2}\right)^{-1/2} \exp\left(\frac{i}{\hbar}\bar{A}_0(\bar{Q})\right) \mathcal{G}(q_2 t_2;\, q_1 t_1 \,|\, \bar{Q}) \\ &= \left\{\begin{array}{l}\text{saddle-point propagator for} \\ Q \text{ ignoring } q\end{array}\right\} \\ &\quad \times \left\{\begin{array}{l}\text{amplitude to go from} \\ (q_1 t_1) \text{ to } (q_2 t_2) \text{ in a} \\ \text{background } \bar{Q}\end{array}\right\}. \end{aligned} \tag{4.104}$$

This result corresponds to Eqs. (4.71) and (4.75) of Sec. 4.3. The propagator $\mathcal{G}(q_2 t_2;\, q_1 t_1 \,|\, Q)$ is equivalent to the Schrödinger equation in a background potential $U(Q, q)$; it can be shown that:

(a) the saddlepoint-Kernel for Q is exactly the one we will obtain from the WKB wavefunctions, and

(b) this Kernel propagates the WKB wavefunctions to WKB wavefunctions. (This is a standard result; see e.g. Ref. 35.)

The phase ambiguity encountered in Sec. 4.3 also has a simple interpretation in terms of Kernels. Suppose we rewrite (4.104) introducing $1 = e^{iR}e^{-iR}$. If we redefine $\mathcal{G}$ as $(\mathcal{G}e^{-iR}) = \mathcal{G}'$ and combine $(iA_0/\hbar)$ and iR we will reach the situation with an $\mathcal{O}(1)$ phase; $\mathcal{G}'$ will satisfy a modified Schrödinger equation and iR term will give a "backreaction" to A_0. It is quite clear that this term is spurious unless supplemented by some other argument.

We now ask the question: When does the PI give a 'backreaction?'

The natural temptation to write the PI in (4.102) as

$$\exp\frac{i}{\hbar}\left[A_0(Q) - i\hbar \ln \mathcal{G}(q_2t_2, q_1t_1|Q)\right] \tag{4.105}$$

and do a saddlepoint-integration on the whole phase must be resisted if we are interested in the strict $M \to \infty$ limit. To see this point clearly consider the ordinary integral

$$I = \int_{-\infty}^{+\infty} dx g(x) e^{iMf(x)} \tag{4.106}$$

in the limit of $M \to \infty$. We fix the saddlepoint $x = x_0$ by obtaining the stationary point for the phase, at which

$$f'(x_0) = 0 \tag{4.107}$$

and evaluate the integral as

$$I \cong g(x_0)e^{iMf(x_0)}\left[\frac{2\pi i}{f''(x_0)}\right]^{1/2}. \tag{4.108}$$

Instead, if we write I as

$$I = \int_{-\infty}^{+\infty} dx\, e^{i[Mf(x) - i\ln g(x)]} \tag{4.109}$$

and treat the whole phase as stationary we would have obtained:

$$I \cong g(x_1)\, e^{iMf(x_1)} \cdot \left[\frac{2\pi i}{f''(x_1) - \frac{i}{M}\left(\frac{g'}{g}\right)'_{x_1}}\right]^{1/2} \tag{4.110}$$

where x_1 is the solution to

$$f'(x_1) = \frac{i}{M}\frac{g'(x_1)}{g(x_1)}. \tag{4.111}$$

The right-hand side of (4.111) represents the "backreaction" of g on f. If we are interested in the strict $M \to \infty$ limit, (4.108) is the right result and (4.111) is wrong. The fact that the phase is $\mathcal{O}(M)$ while the amplitude is $\mathcal{O}(1)$ uniquely fixes the form of the exponential. For the same reason (4.102) is correct and (4.105) is wrong.

Let us now consider a different integral:

$$I_1 = \int_{-\infty}^{+\infty} dx g(x, \varepsilon) \exp\left(i \frac{M}{\varepsilon} f(x)\right) \tag{4.112}$$

where we have introduced an extra parameter ε. Suppose we want to evaluate I_1 in the limit of $\varepsilon \to 0$. To do this we need to know the behavior of $g(x, \varepsilon)$ as $\varepsilon \to 0$. Suppose g is analytic near $\varepsilon = 0$. Then we can again calculate I_1 by the saddlepoint. Since the $\varepsilon \to 0$ limit is identical to the $M \to \infty$ limit, as far as the phase is concerned, we get the same saddlepoint as in (4.107).

A more interesting case is when $g(x, \varepsilon)$ has the expansion:

$$g(x, \varepsilon) = e^{ig_0(x)/\varepsilon}(1 + \varepsilon g_1(x) + \cdots) \tag{4.113}$$

near $\varepsilon = 0$. Equivalently

$$\ln g(x, \varepsilon) = \frac{ig_0(x)}{\varepsilon} + O(\varepsilon) . \tag{4.114}$$

So that to the leading order, $\ln g$ is $O(\varepsilon^{-1})$. Here we must write I_1 as

$$I_1 = \int_{-\infty}^{+\infty} dx \exp \frac{i}{\varepsilon}(Mf(x) + g_0(x)) \tag{4.115}$$

and the saddle-point (for $\varepsilon \to 0$) is at $x = x_1$ where

$$f'(x_1) = -\frac{1}{M} g_0'(x_1) = -\frac{i\varepsilon}{M} \frac{g'(x_1)}{g(x_1)} . \tag{4.116}$$

This is identical to (4.111) when $\varepsilon = 1$, and represents a backreaction. Note that this evaluation is correct in the $\varepsilon \to 0$ limit irrespective of the value of M. Similarly (4.108) is correct in the $M \to \infty$ limit irrespective of the value of ε. These two correspond to different approximations.

The connection between the above example and the $Q - q$ system should be clear. The $\hbar$ plays the role of ε. We do know that $\mathcal{G}(q_2 t_2; q_1 t_1 | Q)$ has an expansion in $\hbar$ with a leading term $O(\exp iF/\hbar)$ near $\hbar \to 0$. Writing $\mathcal{G}$ as

$$\mathcal{G} \equiv D \exp \frac{i}{\hbar} \theta \tag{4.117}$$

our part integral in (4.102) becomes

$$K(Q_2q_2t_2;\, Q_1q_1t_1) = \int \mathcal{D}Q\; e^{i/\hbar(A_0+\theta)}D\;. \tag{4.118}$$

If we evaluate K for $M \to \infty$ limit, then $(De^{i\theta/\hbar})$—which is $O(M^0 = 1)$—acts as an amplitude and we get (4.104). However, if we evaluate K in the $\hbar \to 0$ limit, we obtain the equations

$$\frac{\delta A_0}{\delta Q} = -\frac{\delta\theta}{\delta Q}\,. \tag{4.119}$$

We seem to have got a backreaction in (4.119) but this is not what we want. The trouble is that this is just the classical equation for $Q - q$ system! Note that θ is independent of $\hbar$, so we have not obtained a quantum backreaction. This is precisely the situation we will have in a classical theory itself.

To get a quantum backreaction we have to calculate higher orders in $\hbar$. Can't we just calculate to $O(\hbar)$ in (4.118) and use it in (4.119)? In fact we can, and it is often done. Since D has an expansion in $\hbar$ as $(D_0 + \hbar D_1 + \cdots)$ we can incorporate $O(\hbar)$ terms in (4.118) by a saddlepoint integration which includes D. There are, however, two problems with this approach.

The first one is purely technical. As it stands, $\mathcal{G}$ depends on the end-points q_1 and q_2 of q. We have to do something about this if our backreaction has to be independent of the boundary conditions on q. This is related to the correspondence between expectation values and path integral averages. We refer the reader to original literature for this aspect.

The more important conceptual point is the following. We are actually trying to evaluate the A_{eff} of (4.99) in a series expansion in $\hbar$ with the leading term being θ. This calculation requires an expansion of q around some classical value and some assumption regarding the smallness of the fluctuations. But if we do a similar analysis on $A_0(Q)$ we will pick up $O(\hbar)$ correction from the quantum fluctuations of Q. It is conceptually inconsistent to ignore these while retaining the $O(\hbar)$ terms from q. Since the $O(\hbar)$ effective action contains an effective potential[36]

$$V_{\text{total,eff}} \simeq \frac{1}{2}\hbar\left[\left\{\frac{\partial^2 V}{\partial Q^2}\right\}^{1/2} + \left\{\frac{\partial^2 U}{\partial q^2}\right\}^{1/2}\right] \tag{4.120}$$

it is clear that both U and V contribute in the $\mathcal{O}(\hbar)$. It is—in general—inconsistent to include one and ignore the other.

4.6. *Critique of semi-classical gravity*

We saw in Sec. 3 that the full fledged quantum gravity poses severe interpretational problems, especially as regards the question of time. In the last few sections we described semi-classical gravity in which these problems seem to have been amicably resolved. We

would like to dispel this illusion and caution the reader about the assumptions which have gone under the rug.

To begin with, consider the case of pure gravity. The Wheeler-DeWitt equation uses the operator ∇^2 which has second derivatives with respect to all the basic variables. The Wheeler-DeWitt equation is a real equation because there is no $i\dot{\psi}$ term on the right-hand side. No positive definite conserved probability can be defined for such equations. It is mathematically impossible to obtain from this equation an approximate equation which is:

(a) First order in some chosen intrinsic time variable and

(b) has an exactly conserved probability. Even if we achieve (a), we cannot achieve (b) except as an approximation.

The situation is no better when the matter fields are present. All known forms of matter Hamiltonian are quadratic in their canonical momenta. The Wheeler-DeWitt equation including the matter fields will also be a second order differential equation in both the gravitational and matter variables. What is more, it will be a real equation (it would correspond to the $E = 0$ situation in our toy model in (4.8)). Once again we do not have any positive definite, conserved probability.

From such an equation, we obtain an approximate Schrödinger equation in Sec. 4.1. This approximate equation was complex and had a conserved probability. However, in order to obtain this we had made a specific ansatz for the WKB wave-function describing gravity. Since the Hamilton-Jacobi equation is quadratic in the classical action (when $E = 0$) the phase of the WKB function can have either sign. Thus the general solution to the WKB equation will be a linear superposition of two complex conjugate solution. In Secs. 4.1 and 4.3 we explicitly chose one of these branches and threw away the other. This choice is motivated and dictated by the fact that we want the Wigner function to be paked on a unique trajectory. It is this assumption which allowed us to obtain a Schrödinger equation with a $i\partial\psi/\partial t$ term. While this may be justifiable, it definitely shows that a very special choice has been made in the solution to the WKB equation.

Since the exact equation does not have a conserved positive definite probability, our approximate Schrödinger equation will produce one such, only to some order in the expansion parameter. The situation here is mathematically similar to the procedure by which the nonrelativistic Schrödinger equation is derived from the Klein-Gordon equation. There are situations like Klein's paradox in which probability conservation is badly violated. In a similar way exact quantum gravity model will violate the approximate probability conservation (and unitarity) present in our semi-classical limits. In the case of Klein-Gordon equation, we do know how to quantize this system correctly and obtain a probability interpretation. We lack this luxury in the case of quantum gravity and hence it is difficult to decide how serious is the breakdown of unitarity.

5. Examples from Mini-superspace

The last two sections we introduced several techniques relevant to the study of the quantum gravity. In Sec. 5, we will illustrate these techniques in some specific contexts. Since we have no way of handling the full functional Schrödinger equation, the specific examples have to be necessarily based on minisuperspace models.

5.1. *Wheeler-DeWitt equation and quantum stationary geometries*

In Sec. 3.2, we introduced the Wheeler-DeWitt equation for the quantum gravity. The simplest 3-space geometry to which this equation can be applied is that of a closed 3-sphere:

$$dL^2 = a^2(t)R^2(d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta\, d\phi^2)) \,. \tag{5.1}$$

The radius of the 3-sphere is taken to be $a(t)R$ where $a(t)$ is the dimensionless dynamical variable and R is the scaled length. The gravitational action for the mode $a(t)$ will be

$$A = \frac{1}{l^2}\int dt(2\pi^2R^3)\left(-\frac{1}{2}a\dot{a}^2 + \frac{a}{R^2}\right); \qquad l^2 = \frac{4\pi G}{3} \,. \tag{5.2}$$

This corresponds to the classical Hamiltonian

$$H = -\left\{\left(\frac{l^2}{2\pi^2R^3}\right)\frac{1}{2}\frac{p^2}{a} + \frac{2\pi^2R}{l^2}\,a\right\} \,. \tag{5.3}$$

Classically, the Hamiltonian constraint $H = 0$ leads to the condition:

$$\frac{1}{2}\frac{p^2}{a} + \left(\frac{2\pi^2R^2}{l^2}\right)^2 a = 0 \tag{5.4}$$

which is impossible to satisfy (all the terms in (5.4) have the same sign). Empty, closed, Friedmann universes do not exist classically.

Quantum mechanically, the Schrödinger equation corresponding to (5.3) can be written down, if the factor ordering between p^2 and a^{-1} is decided. A somewhat general approach will be to write $a^{-1}p^2$ as $(a^{-n}pa^{n-1}p)$ and then substitute $(p = -i\partial/\partial a)$. This gives:

$$\frac{1}{a^n}\frac{d}{da}\left(a^{n-1}\frac{\partial\psi}{da}\right) = \lambda^2 a\psi\,; \qquad \lambda^2 = 2\left\{\frac{2\pi^2R^2}{l^2}\right\}^2 \,. \tag{5.5}$$

When $n = 1$, this is the Schrödinger equation for a harmonic oscillator with energy $E = 0$. We know that no normalizable solutions exist if we take the range $(-\infty, +\infty)$ for a. However, if we take the range of a to be $(0, \infty)$ the following solution exists:

$$\psi(a) = Na^{1/2}K_{1/4}\left(\frac{\lambda}{2}a^2\right) \tag{5.6}$$

where K is the imaginary Bessel function. For large a, the dominant behavior is as $\exp(-\frac{\lambda}{2}a^2)$. This shows that ψ is vanishingly small for $Ra \gtrsim l$. Qualitatively similar results are obtained[7] for other values of n (for example, with $n = 2$, the normalizable solution is proportional to $K_0(\frac{\lambda}{2}a^2)$). This ψ is concentrated around 3-geometries with sizes of the order of Planck length.

It is hard to give any physical meaning to a universe which is devoid of matter and has planck dimensions. It is usually speculated that these structures have some connection with the 'space-time foam'. We shall not pause to interpret these solutions.

The situation is quite different if there exists some form of 'matter-clock' or internal time. In these cases we will be working with the time dependent Wheeler-Dewitt equation:

$$i\frac{\partial\psi}{\partial\tau} = H\psi = -\frac{l^2}{2\pi^2R^3}\left\{-\frac{1}{2a^n}\frac{\partial}{\partial a}\left(a^{n-1}\frac{\partial\psi}{\partial a}\right) + \frac{\lambda^2}{2}a\psi\right\} . \tag{5.7}$$

The solutions to this equation can be analyzed in terms of 'stationary states' in which ψ can be separated as:

$$\psi(a, \tau) = e^{+i\omega\tau}g_E(a) .$$

The functions $g_E(a)$ satisfy the equation

$$-\frac{1}{a^n}\left\{\frac{d}{da}\left(a^{n-1}\frac{dg}{da}\right)\right\} + \lambda^2 ag = Eg ; \qquad E = \frac{2\pi^2R^3}{l^2}\omega . \tag{5.8}$$

For the sake of simplicity, let us take $n = 1$. Then (5.9) can be transformed to

$$-\frac{d^2g}{dx^2} + \lambda^2x^2g = \frac{E^2}{4\lambda^2}g \tag{5.9}$$

where $x = (a - E/2\lambda^2)$. The properly normalized solutions to this harmonic oscillator equation are characterized by an integer n. (We now take both x and a in the range $(-\infty, +\infty)$.) In the nth quantum state we have:

$$E^2 = 8\lambda^3\left(n + \frac{1}{2}\right); \qquad \langle n|a|n\rangle = \sqrt{\frac{2}{\lambda}}\left(n + \frac{1}{2}\right)^{1/2}; \qquad \langle n|a^2|n\rangle = \frac{3}{\lambda}\left(n + \frac{1}{2}\right) .$$

The spacetime geometry corresponding to this quantum state is described by the 3-metric

$$\langle dl^2\rangle = \frac{3R^2}{\lambda}\left(n + \frac{1}{2}\right)\{d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2)\}$$

$$= \frac{3\sqrt{2}}{8\pi^2}l^2\left(n + \frac{1}{2}\right)\{d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2)\} . \tag{5.10}$$

The class of such geometries parametrized by the integer n may be called the 'quantum stationary geometries' (QSG) for this particular mini-superspace. Similar structures exist for all mini-superspace models and have been analyzed in detail in the literature.[37] Notice that the 'ground state' for this system has the r.m.s. size of (Planck length)^2.

We can easily construct the classical limit of our system by using either the Gaussian states or the WKB states. From (5.9) it follows that x will behave as an oscillator with frequency λ and energy $(E^2/8\lambda^2)$. The classical limit can be worked out from this. We easily find that the classical trajectory is given by

$$\dot{a}^2 + \frac{1}{R^2} = \frac{l^2}{\pi^2}\frac{\omega}{R^3}\frac{1}{a}$$

or, equivalently by,

$$\frac{a^2 + R^{-2}}{a^2} = 2l^2\left(\frac{\omega}{2\pi^2 R^3 a^3}\right). \tag{5.11}$$

This is the classical (Einstein's) equation for a dust filled, closed universe with energy ω. The geometry described by (5.11) is singular. (It has a 'big bang' and 'big crush'.) The QSG's described in (5.10) avoid this singularity due to the lower bound at planck length. This feature is even more transparent if we use a Gaussian state to describe the semi-classical limit. It can be shown that,[38] in such a Gaussian state,

$$\langle a^2(t)\rangle = a^2_{\text{classical}}(t) + \frac{3\sqrt{2}}{16\pi^2}\frac{l^2}{R^2} \tag{5.12}$$

where $a^2_{\text{class}}(t)$ is the singular solution to (5.11). This Gaussian state describes the space-time metric:

$$\langle dl^2\rangle = \left(R^2 a^2_{\text{class}}(t) + \frac{3\sqrt{2}}{16\pi^2}\,l^2\right)(d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2)) \tag{5.13}$$

which is also nonsingular.

The results discussed above turn out to be generic. QSG's exist for a wide variety of mini-superspace models. All such QSG's are singularity free and have a Planck size lower length cut-off. As a bonus, we see that the model in (5.13) is also devoid of cosmological horizons.

Similar conclusions are reached if we treat the overall conformal factor of the spacetime metric as a quantum variable,[39] that is, if we consider the class of all space-times described by the line element:

$$ds^2 = \Omega^2(t)(dt^2 - R^2(d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2)))$$

and quantize $\Omega(t)$. This procedure is equivalent to quantizing the 3-geometries described by (5.1) in the 'gauge' $N = a$. Since we expect the theory to be indifferent (except as regards factor ordering) to the choice of N, it is not surprizing that the results are similar.

The above connection also stresses the fact that Ω in (5.17) and a in (5.1) enjoy the same physical status. Each is as physical as the other, and differ only by a gauge choice. (Statements in literature, that Ω is not a 'physical' degree of freedom, are incorrect.)

The results (5.10) and (5.13) indicate that the Planck length sets a lower bound on the proper size of the quantum universe. This is indeed true: Planck length can be interpreted as a 'zero-point length' to space-time. The reader is referred to the original literature for discussion of this result and its consequences.[40]

5.2. *Mini-superspace of conformally constant three geometries*

In quantizing the Friedmann models described by (5.1) we are actually quantizing the overall scale of the 3-geometry described by a. In the above analysis, we have assumed this scale factor to be independent of the spatial coordinates and dependent only on time. It is possible to relax this assumption and consider space-dependent conformal factors.

Consider a class of 3-geometries which are conformally related to some constant, compact 3-manifold with positive curvature. The 3-metric for these geometries can be taken to be

$$\dot{g}_{\alpha\beta}(\underset{\sim}{x}, t) = \Omega^2(\underset{\sim}{x}, t) g_{\alpha\beta}$$

where Ω is the dynamical variable and $g_{\alpha\beta}$ is some, given, constant 3-metric on a compact 3-manifold. Note that we are allowing explicit spatial dependence for Ω, which is an improvement over conventional mini-superspace models.

The action which describes the dynamics of Ω can be easily computed from (2.47) or (2.53). We will need the formula

$$\dot{R} = \Omega^{-2}R - 2(n-1)\Omega^{-3}\Omega_{;\alpha\beta}g^{\alpha\beta} - (n-1)(n-4)\Omega^{-4}\Omega_{;\alpha}\Omega_{;\beta}g^{\alpha\beta}$$

which connects the Ricci scalars of $\dot{g}_{\alpha\beta}$ and $g_{\alpha\beta}$ in n-dimensions. Using this, and the fact that $\sqrt{\hat{g}} = \Omega^3\sqrt{g}$, we can write (2.47) as:

$$S = \frac{1}{16\pi G}\int dt d^3\underset{\sim}{x}\sqrt{g}N\Omega^3\left\{-\frac{6\dot{\Omega}^2}{\Omega^2N^2} - \frac{4}{\Omega^3}\Box\Omega + \frac{2}{\Omega^4}\Omega_{,\alpha}\Omega^{,\alpha} + \frac{R}{\Omega^2}\right\}.$$

For the sake of future convenience we have also included the 'lapse factor' $N(\underset{\sim}{x}, t)$, by replacing 'dt' in (2.47) by 'Ndt'. After some simple rearrangement and integration by parts, S can be written as

$$S = \frac{1}{16\pi G}\int dt\, d^3\underset{\sim}{x}\sqrt{g}\left\{-\frac{6\Omega\dot{\Omega}^2}{N} + 4\Omega^{,\alpha}N_{,\alpha} + 2\frac{N}{\Omega}\Omega^{,\alpha}\Omega_{,\alpha} + R\Omega N\right\}. \quad (5.14)$$

To proceed further we have to fix the gauge by choosing $N(\underset{\sim}{x}, t)$. The conventional choice is $N = 1$. Then S becomes

$$S = \frac{1}{l^2}\int dt\, d^3\underset{\sim}{x}\sqrt{g}\left\{-\frac{1}{2}\Omega\dot{\Omega}^2 + \frac{1}{6}\Omega^{-1}\Omega^{,\alpha}\Omega_{,\alpha} + \frac{1}{12}R\Omega\right\}; \qquad l^2 = \frac{4\pi G}{3}$$

leading to the Hamiltonian

$$H = \int d^3\underset{\sim}{x}\sqrt{g}\left\{-\frac{l^2}{2\Omega}\pi^2 - \frac{1}{l^2}\left(\frac{1}{6}\Omega^{-1}\Omega_{,\alpha}\Omega^{,\alpha} + \frac{1}{12}R\Omega\right)\right\}.$$

The Wheeler-DeWitt equation for the Ω-mode can be written, if we choose a factor ordering for the $\Omega^{-1}\pi^2$ term. We will make the choice that $(\Omega^{-1}\pi^2)$ should go over to $(\hat{\pi}^2\hat{\Omega}^{-1})_{\text{quantum}}$. Then Wheeler-DeWitt equation will be:

$$\int d^3\underset{\sim}{x}\sqrt{g}\left\{+\frac{l^2}{2}\frac{\delta^2}{\delta\Omega^2} - \frac{1}{l^2}\left(\frac{1}{6}\Omega_{,\alpha}\Omega^{,\alpha} + \frac{1}{12}R\Omega^2\right) + \Omega\hat{H}^{(q)}_{\text{matter}}\right\}\frac{\psi(\Omega, q)}{\Omega} = 0 \quad (5.15)$$

where H_{matter} is the matter Hamiltonian.

The crucial point to note here is that the gravitational part of Wheeler-DeWitt equation is a quadratic functional of Ω. Substituting $\psi\Omega^{-1} = F(\Omega)$ and setting $H_m = 0$, we get the pure-gravity part:

$$\int d^2x\left\{-\frac{l^2}{2}\frac{\delta^2}{\delta\Omega^2} + \frac{1}{l^2}\left(\frac{1}{6}\Omega_{,\alpha}\Omega^{,\alpha} + \frac{1}{12}R\Omega^2\right)\right\}F(\Omega) = 0\,. \quad (5.16)$$

Instead of Ω, let us use the variable ϕ defined as $\phi^2 = (\sqrt{3}l^2)^{-1}\Omega^2$. Then (5.16) becomes

$$\int d^3x\sqrt{g}\left\{-\frac{1}{2}\frac{\delta^2}{\delta\phi^2} + \frac{1}{2}\phi_{,\alpha}\phi^{,\alpha} + \frac{1}{4\sqrt{3}}R\phi^2\right\}F(\phi) = 0\,.$$

This is precisely the functional Schrödinger equation for a free scalar field with mass $(2\sqrt{3})^{-1/2}R^{1/2}$. We could have obtained the same equation starting with the Lagrangian density

$$L = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\phi_{,\alpha}\phi^{,\alpha} - \frac{R}{2\sqrt{3}}\phi^2\,.$$

Thus there is a formal similarity between a free massive scalar field and the conformal mode.

A different gauge choice for N makes this conclusion more apparent. If we take $N = \Omega$, (5.14) becomes

$$S = \frac{1}{l^2}\int dt\, d^3\underset{\sim}{x}\sqrt{g}\left\{-\frac{1}{2}\dot{\Omega}^2 + \frac{1}{2}\Omega_{,\alpha}\Omega^{,\alpha} + \frac{1}{12}R\Omega^2\right\}. \quad (5.17)$$

With this choice, there is no factor ordering problem at all! (which is one definite reason to favor this gauge choice). Expression (5.17) is clearly the action for a free scalar field Ω with mass $[(1/6)R]^{1/2}$. The two gauge choices change the coefficients of Ω^2 term in S; but except for this difference, the structure of the theory remains the same.

The quantum gravitational models based on (5.17) are studied in detail in the literature,[41] treating the conformal factor as a perturbation around a background. (To

quantize $\Omega(\underset{\sim}{x}, t)$ exactly, we need to tackle the momentum constraints as well; these are either too difficult or uninterestingly trivial.) The results are in agreement with the conclusions reached earlier in Sec. 5.1.

5.3. *Semi-classical limit: examples and issues*

In Sec. 4, we discussed several approaches to semi-classical gravity. We shall now discuss the application of these concepts using a simple mini-superspace model. This will also give us an opportunity to bring some of the criticism raised against the semi-classical models in Sec. 4 into sharper focus.

The model we take up is that of a massless, homogeneous scalar field ϕ, in a closed Friedmann universe.[42] The total Hamiltonian for our system is

$$H = -\left\{\left(\frac{l^2}{2\pi^2R^3}\right)\frac{1}{2}\frac{p^2}{a} + \frac{2\pi^2R}{l^2}a\right\} + \frac{1}{(2\pi^2R^2)}\cdot\frac{1}{2}\frac{p_\phi^2}{a^3} \tag{5.18}$$

where p_ϕ is the momentum conjugate to the scalar field ϕ. The classical constraint equation, added to the Hamilton's equation for ϕ implies the trajectories $a = a(t)$, $\phi = \phi(t)$, where:

$$\dot{\phi}a^3 \equiv \frac{k}{l^2} = \text{constant}\,; \qquad \dot{a}^2 + R^{-2} = l^2a^2\dot{\phi}^2\,. \tag{5.19}$$

This is equivalent to a relation $\phi = \phi(a)$ between ϕ and a where:

$$l\left(\frac{d\phi}{da}\right) = \pm\frac{1}{a}\frac{k(R/l)}{\sqrt{k^2\left(\frac{R}{l}\right)^2 - a^4}}\,. \tag{5.20}$$

Integrating, we get:

$$l\phi = \pm\frac{1}{2}\cosh^{-1}\left(\frac{kR}{la^2}\right) \tag{5.21}$$

which connects the classical scalar field ϕ to the expansion factor of the universe.

The quantum physics of this system is based on the Wheeler-DeWitt equation:

$$\frac{l^2}{a^n}\frac{\partial}{\partial a}\left(a^{n-1}\frac{\partial\psi}{\partial a}\right) - \frac{1}{a^3}\frac{\partial^2\psi}{\partial\phi^2} = l^2\lambda^2a\psi\,. \tag{5.22}$$

Exact solutions to this equation can be written down for several choices of n. For example, let us consider the $n = 2$ case for which the Wheeler-DeWitt equation has an invariant ∇^2-operator. With this choice, the normalizable solutions to (5.22) are of the form:

$$\psi(\phi, a) = Ne^{-iq\phi} \cdot K_\mu\left(\frac{1}{2}\lambda a^2\right) \tag{5.23}$$

where $\mu = (iq/2l)$ and K_μ is the MacDonald function. (One can, of course, superpose solutions with different values for q). For large values of the argument, the MacDonald function K is real, and goes as:

$$K_{i\theta}(z) \approx \frac{1}{2}\sqrt{\frac{\pi}{2}} \exp\left(-\frac{\pi\theta}{2}\right)(\theta^2 - z^2)^{-1/4} \cos\left\{\theta \cosh^{-1}\left(\frac{\theta}{z}\right) - \sqrt{\theta^2 - z^2} - \frac{\pi}{4}\right\}. \tag{5.24}$$

Let us now look at the semi-classical limit to (5.22), obtained in the limit of ($l^2 \to 0$). We rewrite it (5.22)—with $n = 2$—as

$$\frac{1}{2} l^2 a \frac{\partial}{\partial a}\left(a \frac{\partial\psi}{\partial a}\right) - \frac{(2\pi^2 R^2)^2}{l^2} a^4\psi - \frac{1}{2}\frac{\partial^2\psi}{\partial\phi^2} = 0 \tag{5.25}$$

and try out the usual ansatz:

$$\psi(a, \phi) = \frac{F(a, \phi)}{\sqrt{S'}} \exp(il^{-2}S(a)) . \tag{5.26}$$

Equating the coefficients of equal powers of l^2 we get

$$-\frac{1}{2} l^{-2}a^2\left(\frac{dS}{da}\right)^2 - \frac{(2\pi^2 R^2)^2}{l^2} a^4 = 0 \tag{5.27}$$

$$-ia^2 S'(a)\frac{\partial F}{\partial a} = -\frac{1}{2}\frac{\partial^2 F}{\partial\phi^2} \equiv i\frac{\partial F}{\partial t} . \tag{5.28}$$

These equations offer valuable insights into the nature of our semi-classical approximation and the issue of back reaction.

To begin with, note that the ansatz in (5.26) has the effect of producing a complex Eq. (5.28) out of a real Eq. (5.25). This is required if we want to interpret (5.28) as "quantum field theory in curved space-time"; this point was discussed earlier in Sec. (4.6).

We can recast (5.27) into the form:

$$\frac{1}{2}\frac{p^2}{a} + \left(\frac{2\pi^2 R^2}{l^2}\right)^2 a = 0 \tag{5.29}$$

by defining the momentum p as the derivative of the WKB phase: $p(a) \equiv l^{-2}S'(a)$. This definition is in accordance with our rule that "the WKB phase shall be the classical action". Indeed, we see that (5.29) is the same as (5.4)—the constraint equation for the

closed empty universe. The trouble, of course, is that (5.29)—or (5.27)—has no real solutions for S or p. The reason for this difficulty is also obvious: Our model contains only a scalar field and gravity. The semi-classical prescription which we are using (viz. $G \to 0$) suppresses the action of ϕ on a. Thus, in the semi-classical limit, our universe becomes sourceless.

People have different views on how this situation should be interpreted. One possibility, of course is to go ahead and solve (5.27) obtaining an imaginary S:

$$l^{-2}S(a) = \frac{i}{2}\lambda a^2 \,. \tag{5.30}$$

This will give a WKB wave function for a which is a decreasing exponential. It is usual to consider this solution as indicating the "origin of universe by tunnelling" from "nothing".[43] With this solution for S, the left-hand side of (5.28) loses its complex character, or equivalently, "time" becomes imaginary. There have been attempts in literature to treat this as origin of "temperature".[44] At present it is difficult to judge whether these attempts are mere mathematical curiosities or contain some profound physical principle.

The semi-classical limit in (5.26) is tailor made to produce a situation in which gravity is classical and ϕ is quantum mechanical. Instead of taking this limit ($\Omega^2 \to 0$, $\hbar$ = constant) we could have taken the classical limit of the full Eq. (5.22). This will lead to the WKB wavefunction in both a and ϕ of the form

$$\psi(a, \phi) = \frac{N}{\sqrt{S_0'}} \exp\left\{\frac{i}{\hbar} S_0(a, \phi)\right\} \tag{5.31}$$

where $S_0(a, \phi)$ satisfies the equation

$$-\frac{1}{2} l^2 a^2 \left(\frac{\partial S_0}{\partial a}\right)^2 - \frac{(2\pi^2 R^2)^2}{l^2} a^4 = -\frac{1}{2}\left(\frac{\partial S_0}{\partial \phi}\right)^2 . \tag{5.32}$$

On identifying $(\partial S_0/\partial a)$ and $(\partial S_0/\partial \phi)$ with p and p_ϕ we find that this equation is identical to "$H = 0$", where H is given by (5.18). The same WKB limit can also be obtained by using (5.24) in (5.23) and identifying the phase. Thus the full classical limit ($\hbar \to 0$) poses no difficulties at all. Solutions with scalar field as the source exist.

The above discussion stresses, in an explicit manner, some of the observations made in Sec. 4. If we treat the quantum nature of both ϕ and a at the same footing, then we obtain the classical limit without any difficulty. Troubles arise only when we take ($l^2 \to 0$) limit and treat gravity as classical, keeping ϕ quantum mechanical.

Lastly, we can use this model to study the issue of "backreaction" as well. To do this we have to look for a phase factor in F which can be combined with the WKB phase of (5.26). Fortunately, Eq. (5.28) can be solved exactly. We get the general solution as the superposition

$$F(\phi, a) = \int_{-\infty}^{+\infty} dk A(k) \exp\left\{ik\phi + \frac{ik^2}{2}\int \frac{da}{a^2 S'(a)}\right\} . \tag{5.33}$$

To understand the effects of $\mathbb{O}(1)$ "backreaction" we will consider two extreme kinds of superpositions in (5.33). As a first example, let us consider a plane wave with $A(k) = \delta(k - k_0)$. Then the complete wavefunction is:

$$\psi(a, \phi) = \exp i\left[l^{-2}S(a) + \frac{k_0^2}{2}\int \frac{da}{a^2 S'}\right] \cdot e^{ik_0\phi} . \tag{5.34}$$

The total, ϕ-independent phase of the wavefunction—to $\mathbb{O}(1)$, is

$$l^{-2}S_t \equiv l^{-2}S(a) + \frac{k_0^2}{2}\int \frac{da}{a^2 S'} .$$

According to our "rule", we will identify S_t as the semi-classical action. Since we know that $S(a)$ satisfies Eq. (5.27), it is easy to derive the equation satisfied by S_t. We have

$$l^{-4}S'^2(a) = \left[l^{-2}S_t' - \frac{k_0^2}{2}\frac{1}{a^2 S'}\right]^2$$

$$= l^{-4}S_t'^2 - l^{-2}k_0^2 \frac{S_t'}{a^2 S'} + \mathbb{O}(1) .$$

In the second term, we can replace $a^2 S'$ by $a^2 S_t'$ to the same order of accuracy. Doing this and substituting into (5.27) we get

$$-\frac{1}{2} l^{-2}a^2\left\{S_t'^2 - \frac{l^2 k_0^2}{a^2}\right\} - \frac{(2\pi^2 R^2)^2}{l^2} a^4 = 0 . \tag{5.35}$$

Or

$$l^{-4}(S_t')^2 + \lambda^2 a^2 = k_0^2 l^{-2}\frac{1}{a^2} . \tag{5.36}$$

This is the 'Hamilton-Jacobi equation' for S_t. This equation implies a source term to the Einstein's equation arising from the right-hand side. In fact, (5.36) in equivalent to the equation

$$\left\{\frac{2\pi^2 R^3}{l^2} a\dot{a}\right\}^2 + \left(\frac{2\pi^2 R^2}{l^2}\right)^2 a^2 = \frac{k_0^2}{l^2}\frac{1}{a^2} \tag{5.37}$$

which is the same as,

$$\frac{\dot{a}^2 + R^{-2}}{a^2} = \frac{l^2}{(2\pi^2 R^3)^2}\frac{k_0^2}{a^6} = \frac{8\pi G}{3}\left\langle \frac{1}{2}\dot{\phi}^2 \right\rangle \tag{5.38}$$

where we have used the fact $k_0^2 = \langle p_\phi^2 \rangle = (2\pi^2 R^3)^2 a^6 \langle \dot{\phi}^2 \rangle$ and the relation $l^2 = (4\pi G/3)$. The dynamics is completely determined by solving Eq. (5.38), coupled with the Schrödinger Eq. (5.28). We get a "backreaction".

As a second example, we will consider the opposite extreme: a sharply localized Gaussian state for $F(\phi, a)$. We take

$$F(\phi, a) \equiv F(\phi, t) = N(t) \exp[-B(t)(\phi - f(t))^2 + ip(t)\phi + i\varepsilon(t)] \tag{5.39}$$

where we have used the "time", t, defined via (5.28). We assume that N, f, p and ε are real and B complex. Substituting (5.39) into (5.28) and equating coefficients of powers of ϕ we get:

$$i\dot{B} = 2B^2\,; \quad \dot{p} = 0\,; \quad p = \dot{f}\,; \quad \dot{\varepsilon} = -\left(\frac{1}{2}p^2 + B_R\right)\,; \quad i\frac{\dot{N}}{N} = B_I \tag{5.40}$$

where $B = B_R + iB_I$. The B equation can be solved to give

$$B(t) = \frac{B(0)}{1 + 2iB(0)t} \equiv \frac{B_0}{1 + 2iB_0 t}\,. \tag{5.41}$$

The wavefunction is now

$$\psi(\phi, a) \equiv \psi(\phi, t) = \exp i[l^{-2}S(a) + \theta(a)]$$

$$\times N \exp(-B_R(\phi - f)^2 + i\xi)$$

where $\theta = \varepsilon - f^2 B_I$, and $\xi = -B_I\phi^2 + (p + 2B_I f)\phi$. The equation satisfied by the ϕ-independent part of the total phase, $l^{-2}S_t = l^{-2}S(a) + \theta$, can be derived exactly as before. We get:

$$-\frac{1}{2}a^2\left\{l^{-2}S_t' - \frac{\dot{\theta}}{\dot{a}}\right\}^2 - \frac{(2\pi^2 R^2)^2}{l^4}a^4 = 0\,. \tag{5.42}$$

Using $\dot{a} = -a^2 S' \approx -a^2 S_t'$ (since the difference is $\mathcal{O}(l^2)$) and retaining the leading terms we get:

$$(l^{-2}S_t')^2 + \lambda^2 a^2 = \frac{2}{a^2 l^2}(-\dot{\theta})\,. \tag{5.43}$$

This is equivalent to the dynamical equation:

$$\frac{\dot{a}^2 + R^{-2}}{a^2} = \frac{8\pi G}{3} \frac{(-\dot{\theta})}{(2\pi^2 R^3)^2 a^6} . \tag{5.44}$$

In the right-hand side $(-\dot{\theta})$ has the dominant contribution from the $\frac{1}{2}p^2$ term of $\dot{\varepsilon}$; the less dominant terms are the quantum corrections to this classical energy. Explicitly:

$$\frac{\dot{a}^2 + R^{-2}}{a^2} = \frac{8\pi G}{3} \cdot \frac{1}{(2\pi^2 R^3)^2 a^6} \cdot \left[\frac{1}{2} p^2 + B_R + 2f B_I p - 2f^2 (B_I^2 - B_R^2) \right] \tag{5.45}$$

where $B = B_R + iB_I$ and f and p are governed by (5.40).

We see that nontrivial solutions can be obtained to the scalar field—gravity system when the $\mathcal{O}(1)$ corrections to the phase of WKB function are included. The source term is just the expectation value of energy of the scalar field, if the field is in an energy eigenstate. It not, the effects of quantum fluctuations complicate the nature of the semi-classical source term.

If we take the $(\hbar \to 0)$ limit of (5.45), we again recover the classical equations with scalar field as source. Thus the sequence, "full quantum theory $\to$ semi-classical limit, with terms up to $\theta(1)$ retained $\to$ classical limit via $\hbar \to 0$" seems to be completely meaningful. On the contrary, if we try, "full theory $\to$ semi-classical limit up to $\mathcal{O}(l^{-2}) \to$ classical limit", we run into an empty universe at the second stage. [See (5.27) to (5.29)]. This fact suggests that the $\mathcal{O}(l^{-2})$ approximation has to be used with caution.

We conclude with a comment on the semi-classical limit encountered in this section and the adiabatic approximation discussed in Sec. 4.3. The model considered here allows an exact separation of the matter degree of freedom [see Eq. (5.23)]. In this case, adiabaticity becomes exact and it is no wonder that a backreaction is obtained when the matter field is in an energy eigenstate. The general situation can be summarized as follows: The exact Wheeler-DeWitt equation in the presence of matter has the form

$$\left[-\frac{1}{2M} \nabla^2 + MV(g) + \hat{H}_{\mathrm{m}}(g, q) \right] \psi(g, q) = 0 . \tag{5.46}$$

Suppose we can find approximate energy eigenstates $f(g, q)$ for matter variable which satisfy the relation:

$$\hat{H}_m(g, q) f(q, g) \approx E(g) f(q, g) .$$

Then the solution to (5.45) can be taken in the form $\psi(g, q) = W(g) f(g, q)$ where W will satisfy the approximate equation

$$-\frac{1}{2M} \nabla^2 W + MV(g) W \approx EW . \tag{5.47}$$

In arriving at (5.47) we have ignored terms like $\nabla^2 f$, ∇f etc. on the assumption that the g-dependence of f is weak. ("adiabatic limit"). The term $E(g)$ in (5.47) leads to the

"backreaction" in this limit. To be consistent, one should solve (5.47) in WKB approximation and make sure that ∇f etc. are indeed negligible.

5.4. *Initial conditions*[45]

The Wheeler-DeWitt equation, like any other differential equation requires some boundary condition for its solution. Since we have not identified an intrinsic time coordinate in the superspace, it is not easy to give this boundary condition as a well-defined initial condition. This poses yet another problem in the interpretation of the quantum gravity, regarding which several suggestions have been made in the literature.

The simplest way out is to explore all possible solutions to the Wheeler-DeWitt equation and obtain their general properties. This could eliminate the need to impose any initial condition. This can be done in the simple mini-superspace models by studying the complete set of QSG's.[37] Any, arbitrary, quantum state can be obtained by suitable superposition of these states.

The second possibility is to work backwards.[29] we know that the quantum state of the universe must be something which allows classical description for large expansion factors. One can put this information by hand and construct very special, semi-classical, solutions which mimic the classical behavior at "late times".

Lastly, it is possible to by-pass the issue of the boundary conditions by defining the wavefunctions directly in terms of a path integral prescription in the Euclidean space.[46] The reader is referred to original literature for detailed discussion of these issues.[46,47]

Unfortunately, none of the above proposals lead us to any deeper insight in quantum gravity. This is essentially due to the fact that all the above proposals are "cooked up" so as to produce some desirable features—like classical behavior at "late times". In this connection it should be noted that while quantizing a hydrogen atom one never worries about initial conditions (even though Schrödinger equation for a hydrogen atom is a second order equation). We merely solve the equation for a complete set of states. In which quantum state, a particular hydrogen atom finds itself at a particular instant, depends on the measurements performed on the system previously. It is possible that, to find out that in what state our present universe is in, we need to know much more about the measurement theory of the quantum gravity.

5.5. *Mach's principle, time and quantum cosmology*

There seems to be a way of interpreting Mach's principle in terms of the choice of wavefunction of our universe. This interpretation also throws light on the role of time in Mach's principle.

Consider the conventional Newtonian mechanics in which we use a "frame of fixed stars", S. This is the frame in which all the distance matter in the universe (which we shall call, loosely, as "stars") is at rest. Newton's laws state that a particle shielded from external influences will follow an unaccelerated trajectory in this frame. Let $\underline{x}_1(t)$, $\underline{x}_2(t) \cdots \underline{x}_N(t)$ be the position vectors of N stars and $\underline{x}(t)$ be the position of a free test particle. In a reference frame in which $\underline{x}_i(t) = \underline{x}_i(0)$ ('distance stars are fixed'), $\underline{x}(t)$ satisfies the equation

$$\frac{d^2\mathbf{x}}{dt^2} = 0 \,. \tag{5.48}$$

Thus, by connecting up the local behavior of the test particles with the state of motion of distant matter, we have brought in 'the Mach's principle'.[48] Consider another frame S' in which the distant stars are not at rest but move according to the law:

$$\underset{\sim}{x}_i(t) = \frac{1}{2}\underset{\sim}{g}t^2 \,. \tag{5.49}$$

In this frame—in which the distant stars are moving—we cannot use (5.48). However, it is easy to transform the coordinates bringing these stars to rest; we can, then, use (5.48) in such a frame. Transforming back we can find the equation of motion for the free test particle in our original frame S'. We will find that $\underset{\sim}{x}'$ satisfies the equation:

$$\frac{d^2\underset{\sim}{x}'}{dt^2} = -\underset{\sim}{g} \,. \tag{5.50}$$

It is usual to call S as 'inertial frame' and S' as 'noninertial frame'; and the acceleration $\underset{\sim}{g}$ (in (5.50)), experienced by a test particle in S' as due to a 'pseudo-force'. In its true form, Mach's principle does not distinguish between the coherent motion of all the distant matter in the universe and a local transformation to a noninertial frame.

Consider now a different frame in which the following situation occurs[49]:

(a) For all stars, $\underset{\sim}{x}_i(t) = \underset{\sim}{x}_i(0)$. That is, all distant stars are fixed.

(b) All test particles shielded from external influences follow the trajectory $\underset{\sim}{x}(t)$ such that

$$\frac{d^2\underset{\sim}{x}'}{dt^2} = -\underset{\sim}{g} \,. \tag{5.50}$$

What can we say about this situation? Since the distant stars are fixed, we should assume that the frame is inertial and that (5.51) implies a velocity dependent drag force on the nearby test particles. This conclusion, however, is premature. A transformation of the time coordinate from t to T such that

$$T = \int^t dt \exp\left(-\int \alpha(t)\, dt\right) \tag{5.52}$$

will reduce (5.51) to the inertial trajectory:

$$\frac{d^2\underset{\sim}{x}}{dT^2} = 0 \,. \tag{5.53}$$

We have two coordinate systems for the space-time: $(\underset{\sim}{x}, t)$ and $(\underset{\sim}{x}, T)$ with t and T related by (5.52). The fixed stars remain fixed in both these frames but Newton's law picks up a pseudo-force in one of these frames.

In the usual discussions of Mach's principle, one never bothers about the time transformations like (5.52). This is because in Newtonian physics, there is an 'absolute time' which 'flows uniformly'. Newtonian physics permits arbitrary transformations of the space coordinates x but forbids transformations of time (except for scaling and translation, $t' = at + b$, under which (5.48) is invariant). By prescribing an absolute time in which (5.48) is valid and forbidding transformations of t, Newtonian physics has effectively bypassed this question. Under such circumstances, the frame of fixed stars defines for us a useful inertial frame. If, however, we have no information about the time coordinate which is used then we cannot exclude pseudo-forces even in a frame of the fixed stars.

This result is very surprising. We know that the Newtonian description is only an approximation to the fully general relativistic, quantum mechanical description of the universe. But that exact description is reparametrization invariant in the time coordinate. In fact, the quantum gravitational models do not give us any time coordinate—let alone a preferred one! Then how do we pick up a preferred time coordinate in the Newtonian limit?

The Newtonian limit is obtained by taking two separate limits in the exact description: the limit of weak gravitational field ($G \to 0$) and the limit of classical physics ($\hbar \to 0$). Let us begin by reversing this process and proceed from (5.48) to the more exact descriptions. A classical free particle obeying (5.48) has its quantum mechanical equivalent, described by the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla_x^2 \psi \,. \tag{5.54}$$

Arbitrary transformation of the time coordinate is still forbidden. The Schrödinger equation retains the above form only when the 'sacred time coordinate' is used. The transition from (5.54) to (5.48) is via the expectation value

$$\langle \underset{\sim}{x} \rangle \equiv \int d^3\underset{\sim}{x} \psi^*(\underset{\sim}{x}, t) \hat{\underset{\sim}{x}} \psi(\underset{\sim}{x}, t) \tag{5.55}$$

and Ehrenfest's theorem or via the WKB states. There are no surprises here.

In the next stage—that of the quantum field theory—the situation becomes trickier. Let us suppose we are working in the flat spacetime and that our particle is described by a scalar field obeying the Klein-Gordon equation:

$$\left(\Box + \frac{m^2c^2}{\hbar^2}\right)\hat{\phi} = \left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \underset{\sim}{\nabla}^2 + \frac{m^2c^2}{\hbar^2}\right]\hat{\phi} = 0 \,. \tag{5.56}$$

How do we make the transition from (5.56) to (5.54)? Note that (5.56) is an equation in Heisenberg picture for the operator $\hat{\phi}$ while (5.54) is an equation in Schrödinger picture for a c-number state function ψ. To make the proper identification we have to define the Fock basis corresponding to $\hat{\phi}$ in (5.56). Let $|0\rangle$ and $|1_{\underset{\sim}{k}}\rangle$ be the vacuum and one-particle

states of the quantum field theory described by $\hat{\phi}(t, \underline{x})$. The correct identification between (5.56) and (5.54) can be made most easily by using the transition element $\langle 0|\phi|l_{\underline{k}}\rangle$:

$$\langle 0|\hat{\phi}(t, \underline{x})|l_k\rangle = \frac{1}{\sqrt{2\omega_{\underline{k}}}} \exp(-i\omega_{\underline{k}} t + i\underline{k} \cdot \underline{x}) ;$$

$$\hbar\omega_k = (\hbar^2 k^2 c^2 + m^2 c^4)^{1/2} . \tag{5.57}$$

In the nonrelativistic limit of $c \to \infty$, we should identify $\langle 0|\phi|l_k\rangle e^{imc^2 t/\hbar}$ with the Schrödinger wave function $\psi(\underline{x}, t)$. This is easily seen from noting that $\langle 0|\phi|l_{\underline{k}}\rangle e^{imc^2 t/\hbar}$ has the limiting form

$$\langle 0|\hat{\phi}|l_{\underline{k}}\rangle e^{imc^2 t/\hbar} \simeq \exp\left(i\underline{k} \cdot \underline{x} - \frac{i\underline{k}^2}{2m} t\right) \tag{5.58}$$

which is just the free particle wave function with momentum $\underline{k}$. In other words, the correct limiting form for free particle is obtained only after we have defined the one-particle Fock state $|l_k\rangle$. As long as we work within the framework of the special relativity and Lorentz transformations, the Fock basis is unique. Though time can be mixed with space in Lorentz transformations, inertial frames and particles retain their identity.

The transition from (5.56) to (5.54) through (5.58) may appear somewhat unconventional. We stress that this is the usual prescription for taking the nonrelativistic limit. The method appears unconventional only because we usually use Heisenberg picture in the field theory and Schrödinger picture in the quantum mechanics. We could have achieved uniformity by replacing (5.56) by a functional Schrödinger equation. The wave functions for a one-particle state in Schrödinger picture field theory will lead to exactly the same conclusion.

The first place where we lose uniqueness of time coordinate is when we allow arbitrary coordinate transformation. Consider, for example, the Rindler frame in which the line element has the form

$$ds^2 = \left(1 + \frac{gx}{c^2}\right)^2 c^2 d\tau^2 - dx^2 - dy^2 - dz^2 . \tag{5.59}$$

As is well known, the Fock basis defined using τ is not the same as the one defined using the inertial time t.[50] The transition element constructed from the Rindler states has the form:

$${}_R\langle 0|\phi|l_{\underline{k}}\rangle_R = \frac{(\sinh \omega\pi)^{1/2}}{2\pi^2} (e^{-i\omega\tau + i\underline{k}\cdot\underline{x}}) K_{i\nu}(\omega(1 + gx)) \tag{5.60}$$

(where we have temporarily set $c = \hbar = 1$; $K_{i\nu}$ is the modified Bessel function). In the nonrelativistic limit, $(c \to \infty)$ the function ${}_R\langle 0|\phi(\underline{x}, \tau)|l_k\rangle_R \, e^{+imc^2\tau/\hbar}$ satisfies the Schrödinger equation for a uniformly accelerated particle:

$$i\hbar \frac{\partial \psi}{\partial \tau} = -\frac{\hbar^2}{2m} \nabla^2 \psi + mgx\psi \,. \tag{5.61}$$

In the corresponding Newtonian limit, these particles obey equation (5.5)—with a pseudo-force. Suppose we had described our flat spacetime using χ and τ. Then, in the appropriate nonrelativistic limit all our test 'particles' (defined using the mode functions in (5.60)) will experience an acceleration. In the flat space-time, of course, this difficulty is easy to cure; we use the Minkowski frame, rather than (5.59) to define the particles. But this example illustrates how a particular time coordinate achieves preference in the nonrelativistic limit. It is the definition of 'particle' which breaks the invariance under time transformations. We choose the quantum vacuum state in such a way that "particles" in the classical limit experience no pseudo-force.

The real universe, of course, is not flat. To define particles in the real universe, we have to quantize the fields in a given background. Since quantum state is a global concept—defined on a spacelike hypersurface cutting through the universe—we now have to introduce 'distant matter' in the discussion. We must demand that: "One particle state should be defined in our universe in such a way that, in the nonrelativistic limit, these particles must be unaccelerated with respect to fixed stars".

It is clear that this requirement must impose constraints on the possible wave functions for the universe. Any prescription which assigns a quantum state to the universe also contains information about the semi-classical time coordinate and the quantum state of the matter fields. To obtain unaccelerated particles in the Newtonian limit, this quantum state for the universe must be chosen with care.

In simple models, the time coordinate defined by the overall volume of the three geometry leads to the correct Newtonian limit. It has also been shown that the vacuum state for the quantum fields turns out to be the "natural" one if we use the Euclidean "no boundary" prescription for the wave function of the Universe.[51] Not much more is known about this issue. It is, for example, not clear what precisely mathematical restrictions on the solutions of the Wheeler-DeWitt equations will ensure the validity of the above condition.

6. Quantum Fluctuations and Gravity

In the material covered so far, the emphasis has been mostly on the technical side of the quantum gravity. We assumed that gravity may be quantized "just like" any other system. We probably made some progress; but the resulting structures have not given us any fresh insight into the mystical world of quantum gravity. In this last part, we will explore certain basic issues which may hold the key to a deeper understanding of quantum gravity.

6.1. *Zero point energy and gravity*

It is often stated that combining special relativity with quantum theory opened the door to the world of antiparticles. The existence of antiparticles is intimately linked to the fact that the concept of vacuum state in quantum theory is fairly nontrivial. It is usual to

describe the quantum free fields by quadratic Lagrangians which can be decomposed into set of harmonic oscillators. The quantum state of the field can be specified by specifying the quantum state of each of the harmonic oscillator—that is, by giving a set of integers, one for each oscillator. These integers are suggestively identified as the number of "particles" present in that particular quantum state. The state with no particles corresponds to the ground state or "vacuum". In the vacuum state, all the oscillators are in their respective ground states.

This ground state has two peculiar features. Firstly, the ground state is a Gaussian state; the fluctuations of the field amplitudes are governed by a Gaussian stochastic process. Secondly, the ground state of a harmonic oscillator has nonzero energy. Since there are infinite number of oscillators the ground state energy of any field becomes infinite.

Both the above facts create difficulties when gravity is introduced into the picture. The first fact, namely that there are field fluctuations in the vacuum implies that the expressions quadratic in the field variables will exhibit Gaussian fluctuations. If these fluctuations are observable then it is necessary to take into account their gravitational effects. For example, one would have liked to include the quantum fluctuations in the stress tensor T_{ik} in the source term for gravity in some (as yet unknown) manner. The second fact creates the difficulty precisely at this point. Any naive (that is, honest) calculation will lead to infinite values for these fluctuations and hence to infinite gravitational effects. It is necessary to regularize (that is, throw away the divergent terms in a systematic, though *adhoc*, manner) such infinite expressions before their gravitational effects are computed.

There are several ways of regularizing such expressions and not all of them will lead to the same result. It was realized by several people that if these regularized expressions are going to be used as a source for gravity, they better be generally convariant.[20] In other words, one would like to use regularization procedures in which the following feature is built in: The regularized expectation value of tensorial quantities like T_{ik} should transform tensorially. In particular if they vanish in one frame they should vanish in all frames.

The above dictum disallows several straightforward, intuitively obvious, regularization schemes popular in special relativistic theory. The simplest among them, for example, is "normal ordering". Normal ordering involves identification of creation and annihilation operators, which in turn involves definition of the positive and negative frequency modes. This is clearly not a covariant procedure. In fact, any consistent covariant regularization procedure will make $\langle T_{ik} \rangle_R$ zero in the flat space-time in all coordinate systems.

It seems eminently reasonable that the flat space-time should have zero energy. It is, probably, required for consistency if we are going to include the gravitational effects. But the "reasonableness" of any procedure in physics needs to be verified by explicit experimentation. In particular such a procedure should give consistent answers in simple thought experiments.

Here is where trouble begins. It turns out that the vacuum functional for a quantum field will have different forms when expressed in different coordinate systems. This, by itself, is no surprise because vacuum functionals are constructs in the Schrödinger picture which does not respect manifest covariance. However, even the fluctuation pattern of the vacuum—by which we mean the behavior of rms fluctuations of the field variables—

change under coordinate transformation. Unfortunately, it is possible to construct detectors which will actually measure these fluctuations. This brings into question the operational validity of general covariance in quantum theory.

We will discuss this effect in detail in the next section. However, before we do that, it is probably worthwhile to look at a simpler and better understood phenomena in the inertial frame itself. This is the experimentally verified result[52] called the *Casimir effect*. It turns out that there is striking mathematical similarity between the Casimir effect and the issue of vacuum fluctuations in noninertial frames of reference.

Consider two plane, parallel, conducting plates located with a separation L in between. It is experimentally observed that they attract each other with a force given by

$$F = -\frac{\pi^2}{240}\left(\frac{\hbar c}{L^4}\right). \tag{6.1}$$

It is possible to derive this result in several equivalent ways. For our purpose, it is best to consider it as arising from the fact that the insertion of the capacitor plates modifies the vacuum functional.[53] The action for the electromagnetic field can be decomposed into harmonic oscillators by using some suitable set of mode functions. In the flat space-time with R^4 topology it is usual to take these model functions to be plane waves of the form $\exp(ikx)$. This will lead to a vacuum functional of the usual form. Suppose we now introduce two conducting plates in the space-time. We can no longer use the mode functions $\exp(ikx)$ because they do not vanish on the location of the conductors. Instead we use some other suitable set of functions (like, say, $\sin(kx)$ etc.) ensuring the vanishing of mode functions on the plates. The vacuum functional now calculated will be quite different. In other words, the vacuum state depends on the choice of mode functions—or, equivalently on the choice of the 'harmonic oscillators'. These choices, in turn, are dictated by the boundary conditions of the problem.

The mathematical description of this phenomenon is quite straightforward. Since, in the later sections, we will be using the example of a scalar field in $(1 + 1)$ dimensions, we will briefly illustrate the above phenomenon in this context. The action for a scalar field ϕ in $(1 + 1)$ dimension will be

$$A = \frac{1}{2}\int dT dX\left[\left(\frac{\partial\phi}{\partial T}\right)^2 - \left(\frac{\partial\phi}{\partial X}\right)^2\right]. \tag{6.2}$$

If we expand the field in terms of the plane waves $\exp(ikx)$ then the action reduces to that for a sum of harmonic oscillators;

$$A = \frac{1}{2}\int \frac{dK}{(2\pi)}\int dT(|\dot{q}_K|^2 - \omega_K^2|q_K|^2) \tag{6.3}$$

where

$$\phi(X, T) = \int \frac{dK}{(2\pi)} q_K(T)e^{iKX}. \tag{6.4}$$

The ground state wavefunctional can be expressed in terms of the field in the form

$$\Psi[\phi(X),\,0] = N\exp\left(-\frac{1}{2}\int_0^\infty \frac{dK}{(2\pi)}\,|K||q_K|^2\right)$$

$$= N\exp\left[-\int dXdY\nabla\phi(X)\nabla\phi(Y)\mathcal{G}(X,\,Y)\right] \tag{6.5}$$

where $\mathcal{G}$ is the green's function,

$$\mathcal{G}(X,\,Y) = \frac{1}{2}\int \frac{dK}{(2\pi)}\,\frac{\exp iK(X-Y)}{|K|}\,. \tag{6.6}$$

If there are two conductors at $x = 0$ and at $x = L$, then we will expand the field in terms of the mode functions $\sin(n\pi x/L)$ where n is an integer. The wave functional can again be calculated in terms of the field. It can be expressed in exactly the form as (6.5) but with a different Green's function. We find:

$$\mathcal{G}'(X,\,Y) = \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{L}{n\pi}\right)\exp\left[\left(\frac{in\pi}{L}\right)(X-Y)\right]. \tag{6.7}$$

Since the vacuum functionals are different, so are the fluctuation patterns and the vacuum expectation values of operators. In particular, calculations show that the $\langle T_{ik}\rangle$ calculated from (6.6) and from (6.7) are different. From (6.6) we get the result

$$E = \frac{1}{2}\int_{-\infty}^{+\infty}\frac{LdK}{(2\pi)}\,|K| = \frac{1}{2}\int_0^\infty dn\left(\frac{n\pi}{L}\right) \tag{6.8}$$

while from (6.6) we get

$$E' = \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{n\pi}{L}\right). \tag{6.9}$$

These two quantities differ by a finite amount. If we decide to regularize the $\langle T_{ik}\rangle$ in the presence of the plates—given by (6.9)—by merely subtracting the value in the absence of the plates, then we get the finite remainder:

$$\Delta E = -\left(\frac{\pi}{4L}\right) \tag{6.10}$$

(To arrive at the above expression we have used the following simple regularization procedure. In defining the difference between (6.8) and (6.9) we have used the limit

$$\lim_{\lambda \to 0} \left(\int_0^\infty n e^{-\lambda n} dn - \sum_{n=0}^{\infty} n e^{-\lambda n} \right) = \frac{1}{2}$$

which can be derived by straightforward calculation.). Similar calculation can be performed for electromagnetic field in (3 + 1) dimensions. In that case the corresponding result is:

$$\left(\frac{\Delta E}{\text{area}} \right) = -\left(\frac{\pi^2}{720} \right) \left(\frac{\hbar c}{a^3} \right) .$$

This finite remainder can exactly account for the observed Casimir effect. In other words, it seems very reasonable—at least in this context—to compute $\langle T_{ik} \rangle$ using two different vacuum functionals made of different mode functions and subtract one from the other to obtain a finite remainder.

It is also important to realize that this finite residual quantity must contribute to the gravitational attraction. To see this, consider the following thought experiment: Two parallel capacitor plates A and B are kept at a distance L apart. A mass M is attached to each plate through the pulley arrangement. As usual we will assume the plates to be weightless etc. (These details are given only for the sake of visualization; our argument is independent of these details.) The plates were kept in the position by external forces for $t < 0$ and were let go at $t = 0$. Because of the Casimir effect, the plates will feel mutual attractive force and will start moving towards each other. At a short time later, the masses would have acquired some kinetic energy. This kinetic energy comes from the change in the Casimir energy between the plates. (The Casimir energy depends on the plate separation which changes when the plates come closer.) If $\langle 0 | T_{ik} | 0 \rangle_C$ denotes the Casimir energy of the vacuum and t_{ik}^M is the stress-energy of the masses, then we have, from energy conservation:

$$[\langle 0 | T_k^i | 0 \rangle + t_k^i(M)]_{;t} = 0 . \tag{6.11}$$

However, we know that,

$$t_k^i(M)_{;t} \neq 0 \tag{6.12}$$

since the masses are acted upon by external forces.

Now consider a region X far away from our Casimir plates. The gravitational field due to our setup is measured at X. If the Casimir energy $\langle 0 | T_{ik} | 0 \rangle_C$ does not contribute to gravity, then the only source of gravity in our arrangement is M. Then Einstein's equations will read as:

$$G_{ik} = R_{ik} - \frac{1}{2} g_{ik} R = -8\pi G t_{ik}(M) . \tag{6.13}$$

This is blatantly wrong because G_{ik} has zero divergence while $t_{ik}(M)$ does not. We are forced to conclude that $\langle 0|T_{ik}|0\rangle_C$ does contribute to gravity and that (6.13) must be modified to

$$G_{ik} = -8\pi G[t_{ik}(M) + \langle 0|T_{ik}|0\rangle_C] \,. \tag{6.14}$$

It therefore follows that, in the absence of M, that is, if we take the limit of $(M \to 0)$, $\langle 0|T_{ik}|0\rangle_C$ should lead to observable gravitational effects.

The above example is extremely important and should be taken seriously. It shows that the finite parts of the expectation value T_{ik}, computed by subtracting one divergent quantity from another, must act as a source of gravity. If it does not, we run into contradictions and even perpetual motion. In this particular case the two divergent values for $\langle T_{ik}\rangle$ are obtained by using two different sets of mode functions in the flat space-time. We shall now consider similar effects which arise in the noninertial frames.

6.2. *Vacuum fluctuations in the Rindler frame*

To illustrate the concepts involved, we will use a scalar field in a (1 + 1)-dimensional spacetime and indicate the generalizations to four dimensions whenever needed.

Consider the two-dimensional Minkowski space-time with the line element:

$$ds^2 = dT^2 - dX^2 \,. \tag{6.15}$$

The complete manifold is covered by the range $(-\infty < T < +\infty, -\infty < X < +\infty)$. We now introduce two sets of coordinate patches, (x, t) and (x', t') on the regions $X > T$ ("Right", R) and $-X > T$ ("Left", L) by the transformations:

$$X = g^{-1}e^{gx}\cosh gt \,; \qquad T = g^{-1}e^{gx}\sinh gt \qquad (\text{in } R) \tag{6.16}$$

$$X = -g^{-1}e^{gx'}\cosh gt' \,; \qquad T = -g^{-1}e^{gx'}\sinh hgt' \qquad (\text{in } L) \,. \tag{6.17}$$

All the coordinates (x, t), (x', t') vary from $(-\infty, +\infty)$. The metric (6.15) in terms of (x, t) [or (x', t')] has the form

$$ds^2 = e^{2gx}(dt^2 - dx^2) \,. \tag{6.18}$$

It can be shown that (x, t) corresponds to the proper coordinate system of a uniformly accelerated observer in the region R, t being the proper time of his clocks. For an observer, completely confined to R, the surfaces $t =$ constant provide a set of spacelike hypersurfaces. In the region R, one can introduce two Killing vector fields:

$$\xi^i_R = e^{-gx}(1, 0)$$

and

$$\xi^i_M = e^{-gx}(\cosh gt, \sinh gt) \,.$$

The ξ^i_R leads to translation in the t-coordinate while ξ^i_M corresponds to translations in T-coordinate ; the latter has components (1, 0) in the inertial frame.

Consider a scalar field $\phi(X, T) = \phi'(x', t')$, described by the generally convariant action

$$A = \frac{1}{2}\int d^2x\sqrt{-g}g^{ik}\partial_i\phi\partial_k\phi$$

$$= \frac{1}{2}\int_{-\infty}^{+\infty} dT \int_{-\infty}^{+\infty} dX\left[\left(\frac{\partial\phi}{\partial T}\right)^2 - \left(\frac{\partial\phi}{\partial X}\right)^2\right]. \tag{6.19}$$

In the region R, the action can be written in terms of x and t in the same form. (This is because in 2-dimension, $\sqrt{-g}g^{ik} = \eta^{ik}$ for (6.18)):

$$A_R = \frac{1}{2}\int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} dx\left[\left(\frac{\partial\phi}{\partial t}\right)^2 - \left(\frac{\partial\phi}{\partial x}\right)^2\right]. \tag{6.20}$$

We shall quantize this field in the Schrödinger picture by decomposing the field into harmonic oscillator modes. Consider a field configuration $\phi(X)$ on the $T = 0$ hypersurface, expanded as,

$$\phi(X) = \int_{-\infty}^{+\infty} \frac{dK}{2\pi} q_K e^{iKX} . \tag{6.21}$$

The hypersurface $T = 0$ is covered completely by $(t = 0, x)$ for $X > 0$ and $(t' = 0, x')$ for $X < 0$. In R and L the field can be decomposed as,

$$\phi(X > 0) = \phi(x) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} a_k e^{ikx} \tag{6.22}$$

$$\phi(X < 0) = \phi(x') = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} b_k e^{ikx'} . \tag{6.23}$$

Using the relations (6.21), (6.22) and (6.23), we can express q_K in terms of a_k and b_k. After some simple algebra, we get

$$q_K = \int_{-\infty}^{+\infty} \phi(X)e^{-iKX}dX = \int_{-\infty}^{+\infty} \frac{dk}{2\pi}\left[a_k f(k, -K) + b_k f(k, +K)\right] \tag{6.24}$$

where the function $f(k, K)$ is defined by

$$f(k, K) = \int_{-\infty}^{+\infty} dx \exp\left[gx + i\left(kx + \left(\frac{K}{g}\right)e^{gx}\right)\right]. \tag{6.25}$$

This function can be expressed in terms of the gamma function of imaginary argument; but we will not need its explicit form.

The vacuum state in the Minkowski frame is described by the functional

$$\Psi[\phi(X), 0] = \Psi[(q_K)] = \prod_K \left(\frac{\omega_K}{\pi}\right)^{1/4} \exp\left(-\frac{1}{2}\,\omega_K |q_K|^2\right)$$

$$= N \exp\left[-\frac{1}{2}\int_0^\infty \frac{dK}{2\pi}\,|K||q_K|^2\right]. \tag{6.26}$$

Using (6.24) we can express this wave functional in terms of a_k's and b_k's. A detailed but straightforward calculation gives,

$$\Psi[a_k, b_k] = N \exp\left[-\frac{1}{2}\int_0^\infty \frac{dk}{2\pi}\,(P(k)(|a_k|^2 + |b_k|^2) - Q(k)(a_k^* b_k + a_k b_k^*))\right] \tag{6.27}$$

with

$$P(k) = \coth\left(\frac{\pi\omega_k}{g}\right), \qquad Q(k) = \operatorname{cosech}\left(\frac{\pi\omega_k}{g}\right); \qquad \omega_k = |k|. \tag{6.28}$$

An observer confined to R will have his observables made out of a_k's. Let $\mathcal{O}(a_k)$ be any such observable. The expectation value of $\mathcal{O}$ (at $T = t = 0$) in the state Ψ is given by

$$\langle\mathcal{O}\rangle = \int \prod_k da_k \int \prod_k db_k \Psi^*(a_k, b_k)\mathcal{O}\Psi(a_k, b_k)$$

$$= \int \prod_k da_k \rho(a_k, a_k)\mathcal{O}(a_k) = \mathrm{Tr}(\rho\mathcal{O}) \tag{6.29}$$

where

$$\rho(a_k', a_k) \equiv \int \prod_k db_k \Psi^*(a_k', b_k)\Psi(a_k, b_k)$$

$$= N \exp\left[-\frac{1}{2}\int_{-\infty}^{+\infty} \frac{dk}{2\pi}\,\frac{\omega_k}{4}\left((a_k - a_k')^2 \coth\left(\frac{\omega_k}{2T}\right) + (a_k + a_k')^2 \tanh\left(\frac{\omega_k}{2T}\right)\right)\right] \tag{6.30}$$

is a thermal density matrix corresponding to the temperature $T = (g/2\pi)$. This is a well known result, showing the thermal nature of the vacuum state (6.26) when expressed in terms of the Rindler mode functions.[54]

Identical results hold in 4-dimensions. We are interested in the r.m.s. fluctuations of physical quantities evaluated using the two wave functionals given by (6.26) and (6.27).

For example, consider the quantity $\langle 0|\phi^2|0\rangle$ calculated using the two wave functionals. Since both the wave functionals are supposed to represent the same vacuum state and since the quantity $\langle 0|\phi^2|0\rangle$ is a generally covariant scalar we expect to get the same answer, when we use (6.26) or (6.27). The straightforward computation leads to the following result. On using (6.26) we get

$$\langle 0|\phi^2(X)|0\rangle = \int \prod_K dq_K \Psi^*[q_K]\left(\sum_{L,M} q_L q_M^* e^{i(L-M)X}\right)\Psi[q_K]$$

$$= \sum_P \left(\frac{1}{2\omega_P}\right) \tag{6.31}$$

while on using (6.27) we get

$$\langle 0|\phi^2(x)|0\rangle = \int \prod_k da_k db_k \Psi^*(a, b) \sum_{p,s} a_p a_s^* e^{i(p-s)x}\Psi(a, b)$$

$$= \sum_p \left[\frac{1}{2\omega_p} + \frac{1}{\omega_p(e^{(2\pi\omega_p/g)} - 1)}\right]. \tag{6.32}$$

These expressions can be cast into a more familiar form in (3+1) dimensions, which will facilitate our discussion. Replacing the summation by an integration,

$$\sum_{\mathbf{p}} \to \int \frac{d^3\mathbf{p}}{(2\pi)^3}$$

we get, in terms of the inertial frame variables,

$$\langle 0|\phi^2(\mathbf{X})|0\rangle = \int \frac{d^3\mathbf{P}}{(2\pi)^3}\left(\frac{1}{2|\mathbf{P}|}\right) = \lim_{\mathbf{R}\to 0} \int \frac{d^3\mathbf{P}}{(2\pi)^3}\left(\frac{1}{2|\mathbf{P}|}\right) e^{i\mathbf{P}\mathbf{R}}$$

$$= \lim_{\mathbf{R}\to 0}\left(\frac{1}{4\pi^2 R^2}\right) = \lim_{\mathbf{R}\to 0} \langle 0|\phi(\mathbf{R})\phi(0)|0\rangle \tag{6.33}$$

indicating the divergence due to the coincidence limit behavior of the Green's function $0|\phi(X)\phi(Y)|0\rangle$. On the other hand, in terms of the accelerated frame variables we get:

$$\langle 0|\phi^2(\mathbf{x})|0\rangle = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\left(\frac{1}{2|\mathbf{p}|}\right)\left[1 + \frac{2}{e^{\beta|\mathbf{p}|} - 1}\right]$$

$$= \langle 0|\phi^2(\mathbf{x})|0\rangle_{g=0} + \mathcal{R}^2_{\text{thermal}} \tag{6.34}$$

where the second term is a finite contribution coming from a thermal bath at temperature $g/2\pi$). How should we interpret the expressions (6.33) and (6.34)?

We note that, both the expressions are divergent. Thus $\langle 0|\phi^2|0\rangle$ is generally covariant in a trivial (and useless) manner. We can, of course, use some covariant regularization technique with (6.33) and (6.34). Any such regularization procedure will make both the expressions vanish.

On the other hand, we notice that the difference between the two terms is finite. It even has a simple physical interpretation in terms of the r.m.s. field fluctuations in thermal radiation at the temperature $(g/2\pi)$. This suggests that we subtract the background divergent term (6.33) from (6.34) and retain the finite $\mathcal{R}^2_{\text{thermal}}$ term. The moment we do that, we run into deep trouble with general covariance. In fact, the situation is more serious with $\langle 0|T_{ik}|0\rangle$. Similar calculation will give the following two divergent expressions for $\langle 0|T_{ik}|0\rangle$:

$$\langle 0|T_{00}|0\rangle_{\text{inertial}} = \int \frac{d^3\mathbf{K}}{(2\pi)^3}\frac{1}{2}|\mathbf{K}| \tag{6.35}$$

$$\langle 0|T_{00}|0\rangle_{\text{Rindler}} = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\left[\frac{1}{2}|\mathbf{k}| + \frac{|\mathbf{k}|}{(e^{\beta|\mathbf{k}|}-1)}\right]. \tag{6.36}$$

We can either make both of them vanish by a covariant regularization or subtract (6.35) from (6.36) and retain the finite difference as "physical" and due to thermal radiation. If we decide to retain the finite parts then we are also forced to use it as a source of gravity. The situation is, in principle, identical to that encountered in the case of Casimir effect. In both cases, we have two vacuum functionals constructed from two different set of mode functions. These vacuum functionals lead to divergent expectation values for $\langle 0|T_{ik}|0\rangle$. However, the difference between the two divergent expressions turns out to be finite. In the case of Casimir effect, this difference—as we saw—must act as a source of gravity. It seems reasonable that the same criterion should apply here. But if we retain the finite part we can bid farewell to the general covariance and even to an invariant description of the flat space.

This strongly suggests that we must ruthlessly regularize away such expressions and retain manifest covariance. This is in fact what is usually done. Unfortunately, there are several conceptual difficulties in adopting this procedure which we shall now discuss.

6.3. *Observability of gravitational effects of vacuum*

The major trouble in regularizing away (6.36) is that the finite part may actually lead to the gravitational effects which are observable—at least in principle. This is precisely what happens in the case of Casimir effect. There is a widespread belief that the Casimir effect and thermal effects in the Rindler frame are very different things. The conventional wisdom accepts the retention of the finite part of $\langle T_{ik}\rangle$ in the case of Casimir effect but insists on regularizing $\langle T_{ik}\rangle$ to zero in the Rindler frame. This certainly can be done as long as we are in the flat space-time with a preferred set of positive frequency modes. However, in an arbitrary curved space-time, one cannot make an invariant distinction between "Casimir-like phenomena" and "Rindler-like phenomena". The mathematical

formalism recognizes only the mode functions and vacuum functionals. When different kind of mode functions are chosen, the vacuum functionals (in general any quantum state functional) become different. The pattern of the vacuum fluctuation changes. There is no invariant prescription for deciding how much of the vacuum fluctuations will contribute to gravity. This is probably the most serious conceptual difficulty in reconciling the divergent quantities with general covariance and gravity.

It is also possible to argue that the finite thermal parts encountered in (6.34) and (6.36) are, in principle, observable. We will give two different arguments in this context. The first is based on an analogy with real blackbody cavities while the second is based on a detailed analysis of the response of an accelerated detector designed to measure energy densities.

We will present the first argument in the language of electromagnetic fields since they are more easily visualized than the scalar fields. This argument proceeds along the following steps:

(i) The expressions for $\langle\phi^2\rangle$, $\langle E^2\rangle \cdots$ etc. in the Rindler frame and in a blackbody cavity are identical. Both contain infinite zero point contributions.

(ii) A charged particle kept inside a blackbody cavity must respond to the finite part of the r.m.s. fluctuation of the electromagnetic field. In fact the equation of motion for a charged particle kept inside a blackbody cavity is usually taken to be

$$m\frac{d\mathbf{v}}{dt} = -\alpha\mathbf{v} + q\mathbf{E}(t)_{\text{random}} \,. \tag{6.37}$$

Here $\mathbf{E}(t)$ is the random, fluctuating electric field (corresponding to the thermal photons in the cavity) and $(-\alpha\mathbf{v})$ is a damping force, arizing due to the fact that a moving particle is "hit" by more photons in the 'front' than in the 'back'. This equation, for example, was used crucially by Einstein[55] in his derivation of the blackbody spectrum.

(iii) It is well-known that the velocity of such particles will soon reach a thermal equilibrium distribution. The random Brownian motion of this charged particle, therefore, contains information about the finite part of $\langle E^2\rangle$. In other words, the quantities $\langle E^2\rangle$, $\langle\phi^2\rangle$ etc. can, in principle, lead to observable effects.

The argument given above is a semi-classical analogue of the detection of "particles" in Rindler frame. A charged particle will detect a thermal bath of photons in the Rindler frame and will respond to it just as in any other thermal bath. It is incorrect to affirm the existence of photons but deny the existence of vacuum fluctuations in the Rindler frame.

A more direct argument for the observability of $\langle T_{ik}\rangle$ can be given in terms of detector models. Consider a detector coupled to the components of energy momentum tensor T_{ik} linearly and locally by the Hamiltonian

$$H_{\text{int}} = \int d\tau \mu_k^i(\tau) T_i^k[x(\tau)] \tag{6.38}$$

where μ_k^i is a detector variable and $x(\tau)$ is the trajectory of the detector.

Classically, such a detector measures the presence of T_{ik}. We ask the question: Will

this 'energy-coupled detector' click in the Rindler frame? That is, will it click if it is accelerated through the inertial vacuum? One may be tempted to answer 'no', based on the conventional wisdom that $\langle T_{ik} \rangle$ is a covariant object. Therefore, if regularized $\langle T_{ik} \rangle$ vanishes in one frame (in the Minkowski frame, say), then it must vanish in all frames and especially in the Rindler frame. Therefore, the detector will not click.

This argument, however, is fallacious. It is perfectly true that $\langle T_{ik} \rangle$ is a covariant object—if it is regularized covariantly, that is—and that it vanishes both in the inertial frame and in the accelerated frame. But the response of our detector has nothing to do with the $\langle T_{ik} \rangle$! What the detector will respond to is the power spectrum of the vacuum fluctuations in T_{ik}, i.e. to

$$\mathcal{G}^{ij}_{kl}(\nu) = \int_{-\infty}^{+\infty} d\tau e^{-i\nu\tau} \langle 0 | T^i_k(x(s+\tau)) T^j_l(x(s)) | 0 \rangle \ . \tag{6.39}$$

There is no reason for $\langle T_{ik} T_{jl} \rangle$ to vanish even though $\langle T_{ik} \rangle$ may vanish. Thus there is no *a priori* reason for the detector coupled to T_{ik} and not to click when accelerated.

A detailed calculation confirms[56] the suspicion that this detector will click when accelerated. For example, a detector coupled to the trace of T_{ik} will click at the rate proportional to

$$R = \frac{(\omega)}{(e^{2\pi\omega/g} - 1)} \left(1 + \frac{\omega^2}{g^2}\right)\left(1 + \frac{\omega^2}{4g^2}\right)\left(1 + \frac{\omega^2}{9g^2}\right) \tag{6.40}$$

where g is the acceleration and ω is the energy difference between the excited and ground states of the detector. The thermal contribution in (6.41) is clearly noticeable. Similar results hold for several other kinds of detectors.

This result raises serious questions regarding the concept of general covariance in the quantum theory. It is necessary to distinguish between the 'formal covariance' and 'operational covariance' of an observable.[56] The energy-momentum tensor T_{ik} or some scalar functional $A(\phi)$ of the field ϕ are formally covariant objects. These operators as well as their expectation values $\langle T_{ik} \rangle$ transform in a systematic tensorial manner. It has always been assumed that if T_{ik} and T'_{ik} are obtained from one another by a tensorial coordinate transformation, then the observers using the corresponding coordinates x^i and $x^{i\prime}$ will actually measure the values as T_{ik} and T'_{ik}. This assumption (valid classically) is not valid in quantum theory. Any operational procedure devised to measure T_{ik}—say, by a detector coupled to T_{ik}—will go about performing this measurement by the emission and absorption of quanta and hence will respond differently in inertial and accelerated trajectories. In other words, even though the formal covariance is assured by the relation

$$\langle 0 | T_{ik} | 0 \rangle_{\text{reg}} = 0 = \langle 0 | T'_{ik} | 0 \rangle_{\text{reg}} \tag{6.41}$$

the operational covariance is completely lost in quantum theory. The objects like $\langle 0 | T'_{ik} | 0 \rangle$ have no operational significance.

We stress that this result follows from three well accepted facts in the quantum domain:

(i) In states with well-defined number of particles, for example in the vacuum state, the field variables will not be well-defined.

(ii) Objects like field intensity, energy, (ϕ, T_{ik}) etc. are not directly measurable in any state. Any detector used to measure these objects can couple to such variables only via the exchange of the field quanta. Thus only the quanta of the field are directly observable.

(ii) The field quanta are not generally covariant objects. They are defined through the choice of positive frequency components for the mode functions. Granted these facts, there is no escape from our conclusions.

When the gravitational field is introduced, we run into fresh problems. An observer can try to measure both T_{ik} and the curvature of space-time independently and try to connect up these two via Einstein's equations. In an accelerated frame, his detector—designed to measure T_{ik}—will click signalling nonzero energy density. However, he (probably) will not see any curvature of space-time!

It is not clear how this question can be resolved within the existing framework. The only possible solution seems to be the following: postulate that in the accelerated frame, the observer will see an energy density and the space-time curvature. This curvature will be interpreted by him as due to the energy density of the Rindler particles. An inertial observer, of course, will be hardpressed to understand this result. He has to either claim that the curvature of space-time is not a generally covariant concept or interpret the curvature seen by his Rindler frame friend as due to the agency which is accelerating the detector. Both the interpretations run into difficulties. In the first one, we have to completely give up any semblance of the general covariance and it is not clear how we can do meaningful physics. The second interpretation seems too incredible to be true. The same acceleration can be imparted to a detector by different agencies. The response of a detector will be the same in all these cases but the different agencies will produce different space-time curvatures.

In a wider context, we expect the semi-classical limit of the quantum gravity to provide us with information on 'backreaction'. We now see that semi-classical gravity has to somehow manage to produce a covariant description of this backreaction as well. It is possible that all these issues—emergence of a semi-classical time coordinate, backreaction, general covariance etc.—are deeply related. Unfortunately, we do not yet know how exactly they are related.

One exciting possibility which has been suggested recently[57] is the following: The very fact that we observe a classical space-time may be due to the existence of vacuum fluctuations! There are broadly two possible points of view which one can take regarding the classical limit of a quantum theory. We may say that certain quantum states—like WKB states, Gaussian states etc.—exhibit near the classical behavior and that several other states do not. In this point of view a system is classical because of the way it is prepared. There is something very disturbing about this point of view. Somehow it does not seem to explain why most systems which we see—i.e., "large" systems—are classical. (For example, why is it that tables and chairs are never found in quantum states which are superpositions of different position eigenstates?) Several people have investigated an alternative approach to classical limit. They claim that the quantum behavior is suppressed automatically in systems with large degrees of freedom because of

interactions. In fact, a quantum system which comes into contact with an 'environment' containing large number of degrees of freedom will be forced to behave almost classically. The gravitational field is a very good example of such a system. It is constantly in interaction with other fields. Even in the absence of classical field distributions, gravity does interact with the vacuum fluctuations of the fields. It is possible to show that—under certain conditions—this interaction is capable of forcing the gravity into behaving classically. At present, these results have been proved only in the context of simple mini-superspace models.[33,57] But it is very likely that these results are quite general. If so, this could provide yet another connection between the quantum gravity, vacuum fluctuations and the semi-classical limit.

References

1. J. A. Wheeler, in *Relativity, Groups and Topology*, eds. B. S. DeWitt and C. DeWitt (Gordon and Breach, New York, 1964); K. Kuchar, in *Relativity, Astrophysics and Cosmology*, ed. W. Israel (Reidel, Dordrecht, 1973); C. W. Mišner *et al.*, *Gravitation* (Freeman, San Francisco, 1973), Chapter 21; C. Teitelboim, *Phys. Rev.* **D25** (1982) 3159.
2. See e.g. Chapter 21 of *Gravitation*, C. W. Misner *et al. op. cit.* (Ref. 1).
3. L. D. Landau and E. M. Lifshitz, *Classical Theory of Fields* (Pergamon, New York, 1975), p. 278–279.
4. R. Arnowitt, S. Deser and C. Misner, in *Gravitation: An Introduction to Current Research*, ed. L. Witten (Wiley, New York, 1962).
5. T. Padmanabhan TIFR-TAP preprint (1988).
6. In addition to references in Ref. 1, see J. A. Wheeler, in *Battelle Rencontres*, eds. C. DeWitt and J. A. Wheeler (Benjamin, New York, 1968).
7. B. S. DeWitt, *Phys. Rev.* **160** (1967) 1113; C. W. Misner in *Magic Without Magic*, ed. J. Klauder (Freeman, San Francisco, 1972), p. 442–473.
8. M. Ryan, *Hamiltonian Cosmology*, (Springer, New York, 1972); M. A. H. MacCallum, in *General Relativity—An Einstein Centenary Survey* (Cambridge University Press, Cambridge, 1979).
9. M. P. Ryan and L. C. Shapeley, *Homogeneous Relativistic Cosmologies* (Princeton University Press, Princeton, 1975); also see p. 381–386 of Ref. 3.
10. G. t'Hooft, *Nucl. Phys.* **B62**, (1973) 444; S. Deser and P. Van Nieuwenhuizen, *Phys. Rev.* **D10** (1974) 401 and 411.
11. C. W. Misner, *Phys. Rev.* **186** (1969) 1319 and 1328; *Phys. Rev.* **D8** (1973) 3271; B. S. DeWitt, *Phys. Rev.* **160** (1967) 1113; W. Blyth and C. J. Isham, *Phys. Rev.* **D11** (1975) 768.
12. K. Kuchar, Lectures on relativity at Princeton (1971) (unpublished); J. W. York, *Phys. Rev. Lett.* **26** (1971) 1656; *Phys. Rev. Lett.* **28** (1972) 1082; also see Ref. 2.
13. C. J. Isham and K. Kuchar, *Ann. Phys.* **164** (1985) 316; also see Ref. 7.
14. J. V. Narlikar, *MNRAS* **183** (1979) 159; *Gen. Rel. Grav.* **10** (1979) 883; J. V. Narlikar and T. Padmanabhan, *Gravity, Gauge Theories and Quantum Cosmology* (Reidel, Dordrecht, 1986); T. Padmanabhan, in *Gravitation, Gauge Theories and the Early Universe*, eds. B. R. Iyer *et al.* (Kluwer, Dordrecht, 1989); W. Unruh, in *Preceedings of the 4th Moscow Quantum Gravity Seminar* (1988) (to be published).
15. J. B. Hartle, in *Gravitation in Astrophysics*, eds. J. B. Hartle and B. Carter (Plenum Press, New York, 1986); T. Padmanabhan, *Class. Quant. Grav.* **4** (1987) 107; T. Banks, *Nucl. Phys.* **B262** (1985) 159.
16. J. B. Hartle, *Phys. Rev.* **D37** (1988) 2818; and 'Quantum Kinematics of Space-time, II, III' *Phys. Rev.* **D** (in press).

17. M. Duff, in *Quant. Grav. 2*, eds. C. J. Isham *et al.* (Oxford University Press, Oxford, 1981); also see the article by G. Horowitz in the same volume.
18. V. G. Lapchinsky and V. A. Rubakov, *Acta Phys. Pol.* **B10** (1979) 1041.
19. U. K. Gerlach, *Phys. Rev.* **177** (1969) 1929.
20. See e.g. N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, 1984).
21. J. B. Hartle, *op. cit* Ref. 15.
22. T. Padmanabhan, *Class. Quant. Grav.* **6** (1989) 533.
23. A. Guth, *Phys. Rev.* **D23** (1981) 347; A. D. Linde, *Phys. Lett.* **108B** (1982) 389; A. Albrecht and P. J. Steinhardt, *Phys. Rev. Lett.* **48** (1982) 1220.
24. A. Guth and S. Y. Pi, *Phys. Rev. Lett.* **49** (1982) 1110; S. W. Hawking, *Phys. Lett.* **115B** (1982) 295; A. Starobinsky, *Phys. Lett.* **117B** (1982) 117; J. Bardeen *et al.*, *Phys. Rev.* **D28** (1983) 679.
25. E.g. L. D. Landau and E. M. Lifshitz, *Quantum Mechanics* (Pergamon, New York, 1976) Chap. 7.
26. E. Wigner, *Phys. Rev.* **40** (1932) 749.
27. J. J. Halliwell, *Phys. Rev.* **D36** (1987) 3626; also see J. Hartle (1986) *op. cit.* Ref. 15.
28. See e.g. L. O'Raifeartough and A. Wipf, *Found. Phys.* **18** (1987) 307.
29. J. V. Narlikar, *MNRAS* **183** (1979) 159; *Gen. Relativ. Gravit.* **10**, 883; *Found. Phys.* **11** (1981) 473; *Found. Phys.* **14** (1984) 443; T. Padmanabhan and J. V. Narlikar, *Nature* **295** (1982) 677.
30. L. I. Schiff, *Quantum Mechanics* (McGraw Hill, Kogakusha, 1968).
31. T. Padmanabhan and T. P. Singh, TIFR-TAP-4 preprint (to be published) (1988).
32. V. R. Rubakov, *Phys. Lett.* **148B** (1984) 280; G. V. Lavrelashvili, *Phys. Lett.* **161B** (1985) 280; K. Kuchar and M. Ryan (Jr.), in *Gravitational Collapse and Relativity*, eds. H. Sato and T. Nakumara (World Scientific, Singapore, 1986); R. Brout *et al.*, *Phys. Lett.* **B192** (1987) 318; P. D'Eath and Halliwell, *Phys. Rev.* **D35** (1987) 1110; R. Laflamme, *Phys Lett.* **B198** (1987) 156; A. Vilenkin, *Phys. Rev.* **D37** (1988) 888.
33. C. Kiefer, *Class. Quantum. Grav.* **4** (1987) 1369; *Phys. Rev.* **D38** (1988) 1761; preprint HD-TVP-88-10 (1988).
34. T. Padmanabhan and T. P. Singh, TIFR-TAP preprint, (1988).
35. B. R. Holstein and A. R. Swift, *Am. J. Phys.* **50** (1982) 829.
36. S. Coleman, in *The Laws of Hadronic Matter*, ed. A. Zichichi (Academic Press, New York, 1975); M. R. Brown, in *Quantum Theory of Gravity*, ed. S. M. Christenson (Adam Hilger, Bristol, 1984).
37. T. Padmanabhan, *Phys. Rev.* **D28** (1983) 745; *Gen. Relativ. Gravit.* **14** (1982) 549; *Class. Quant. Grav.* **1** (1984) 149; X. Hu. and Y. L. Wu, *Phys. Lett.* **A125** (1987) 362.
38. A. Maheshwari, *Phys. Lett.* **73A** (1979) 295; T. Padmanabhan, *Phys. Lett.* **87A** (1982) 226.
39. J. V. Narlikar, *op. cit.* [see Ref. 29].
40. B. S. DeWitt, *Phys. Rev. Lett.* **13** (1964) 114; C. J. Isham *et al.*, *Phys. Rev.* **D3** (1971) 1805; T. Padmanabhan, *Ann. Phys.* **164** (1985) 38; *Class. Quant. Grav.* **4** (1987) 107; *Phys. Rev. Lett.* **60** (1988) 2229; T. Padmanabhan *et al.*, *Phys. Rev.* **D39** (1989) 2100.
41. J. V. Narlikar, *Found. Phys.* **11** (1981) 473; *Found. Phys.* **14** (1984) 443; J. V. Narlikar and T. Padmanabhan, *Phys. Reports* **100** (1983) 152; P. S. Joshi and J. V. Narlikar, *Int. J. Mod. Phys.* **A1** (1986) 243.
42. D. J. Kaup and A. P. Vitello, *Phys. Rev.* **D9** (1974) 1648; W. F. Blyth and C. J. Isham, (1975) *op. cit.* [Ref. 11]; C. Keifer, (1988) *op. cit.* [Ref. 33]; T. Padmanabhan, TIFR-TAP preprint (1988).
43. R. Brout *et al.*, *Nucl. Phys.* **170** (1980) 228; L. Lindley, *Nature* **291** (1981) 391; T. Padmanabhan, *Phys. Lett.* **93A** (1983) 116, A. Vilenkin, *Phys. Rev.* **D27** (1983) 2848.
44. R. Brout, *Found. Phys.* **17** (1987) 603; R. Brout *et al.*, (1987) *op. cit.* [Ref. 32].
45. J. Hartle, in *High Energy Physics 1985—Proceedings of the Yale Summer School*, eds. M. J. Bowick and F. Gursey (World Scientific, Singapore, 1985).

46. J. Hartle and S. Hawking, *Phys. Rev.* **D28** (1983) 2960; S. Hawking, *Nucl. Phys.* **B239** (1984) 257; J. Hartle and D. M. Witt, *Phys. Rev.* **D37** (1988) 2833; A. Vilenkin, *Phys. Rev.* **D33** (1986) 3560; *Phys. Rev.* **D37** (1988) 888.
47. L. Z. Fang and Z. C. Wu, *Int. J. Mod. Phys.* **A1** (1986) 887; G. W. Gibbons *et al.*, *Nucl. Phys.* **B281** (1989) 736; J. J. Halliwell and S. W. Hawking, *Phys. Rev.* **D31** (1985) 1777; Y. Kazama and R. Nakayama, *Phys. Rev.* **D32** (1985) 2500; I. Moss and W. Wright, *Phys. Rev.* **D29** (1984) 1067; D. N. Page, *Phys. Rev.* **D34** (1986) 2267; For a bibliography see J. J. Halliwell NSF-ITP-88-132 (1988).
48. E. Mach, *Die Mechanik in Ihrer Entwicklung Historisch Kritisch Dargestellt* (Broachans, Leipzig, 1912); see also J. V. Narlikar, in *Relativity, Quanta and Cosmology*, eds. M. Pantaleo and F. de Finis (Johnson, New York, 1979); J. Isenberg and J. A. Wheeler, in the same volume.
49. T. Padmanabhan, Gravity essay; TIFR-TAP preprint (1988).
50. S. Fulling, *Phys. Rev.* **D7** (1973) 2850.
51. R. Laflamme, *Phys. Lett.* **B198** (1987) 156; S. Wada, *Phys. Rev. Lett.* **59** (1987) 2375; H. D. Zeh, *Phys. Lett.* **A116** (1986) 9; *Phys. Lett.* **A126** (1988) 311.
52. H. Casimir, *Proc. Kon. Ned. Akad. Wetenschap.* **51** (1948) 793; M. J. Sparnaay, *Physica* **24** (1958) 751; D. Tabor and R. Winterton, *Nature* **219** (1968) 1120; also see C. Itzykson and Jean-Bernard Zuber, in *Quantum Field Theory* (McGraw-Hill, Singapore, 1985).
53. T. Padmanabhan, TIFR-TAP preprint (1988).
54. P. C. W. Davies, *J. Phys.* **A8** (1975) 365; W. Unruh, *Phys. Rev.* **D14** (1976) 870; T. D. Lee, *Nucl. Phys.* **B264** (1986) 437; K. Freese *et al.*, *Nucl. Phys.* **B255** (1985) 693.
55. A. Einstein, *Phys. Z.* **18** (1917) 121 (English translation available in *The Old Quantum Theory* by D. ter Haar, (Pergamon Press, 1967) p. 167.
56. T. Padmanabhan and T. P. Singh, *Class. Quant. Grav.* **4** (1987) 1397; T. Padmanabhan, *Mod. Phys. Lett.* **A3** (1989) 367.
57. E. Joos and H. D. Zeh, *Z. Phys.* **B59** (1985) 223; E. Joos, *Phys. Lett.* **A116** (1986) 6; S. Wada, *Prog. Theor. Phys.* **75** (1986) 1365; H. D. Zeh, *op. cit.* [Ref. 51].
58. See e.g. S. W. Hawking, in *General Relativity—An Einstein Centenary Survey* (Cambridge University Press, Cambridge, 1979); in *Relativity, Groups and Topology*, eds. B. S. DeWitt and R. Stora (North Holland, Amsterdam, 1984).
59. S. Coleman, Harvard preprint HUTP-88/A022, HUTP-88/A008 (1988); S. Weinberg, Morris Loeb lectures in physics; UTTG preprint 12-88 (1988).
60. See e.g. M. B. Green *et al.*, *Superstring Theory* (Cambridge University Press, Cambridge, 1987).
61. A. Ashtekar, *New Perspectives in Canonical Gravity* (Bibliopolis, Naples, 1988).