On the Weakest Falloff Conditions in the Metric for an Isolated System

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ABSTRACT

The weakest asymptotic behavior in the metric for defining an asymptotically flat spacetime at spatial infinity is obtained. The technique of the field formulation of general relativity developed earlier in the Lagrangian description is used. The properties of the latter are similar to those of an ordinary gauge field theory in a fixed background spacetime. The role of the auxiliary background is plaid by Minkowski space. Integrals of motion are defined with the help of a stress-energy tensor of the gravitational field together with its sources and Killing vectors of the background spacetime. It is shown that the weakest asymptotics of gauge transformations, which conserve values of the integrals of motion, defines the weakest falloff conditions in the field gravitational potentials (the same, in the dynamic metric of general relativity if the ordinary geometrical formulation is used). The results are compared with the some known ones.

I. INTRODUCTION

Asymptotically flat spacetimes (AFSTs) in general relativity (GR) attract considerable interest of many researches over many years. The proof of the positive-energy theorem in [1, 2] has provoked a new additional interest in AFSTs. In particilar, these investigations shown that the standard asymptotic conditions at spatial infinity (spi) ($1/r$ falloff in metric) can be significantly relaxed [3 - 9]. In [3] it was proved that for given an asymptotically flat initial data set in a very weak sense there always exists a spinor field that satisfies Witten's equation [2] and that becomes constant at spi. Thus it was shown that Witten's arguments on nonnegativity of the total mass of an isolated system are also the case for this initial data set. The Witten spinorial approach and the weakest falloff conditions in the metric are also discussed in [4]. The works [5, 6] in more degree were devoted to a search for the weakest falloff conditions in the metric. In [5] the Regge-Teitelboim cocept [10] for a definition of integrals of motion for an AFST was developed. In [6] properties of weight Sobolev spaces and a statement of the boost theorem [11] was used. In [7] the weakest possible falloff conditions in the positive-energy theorem were discussed and two problems were considered. Namely: (a) Under what conditions can ADM mass, finite or infinite (stress this), be defined in a meaningful way? (b) Can this mass, even though defined at spi, change in time? In [8] the ADM mass expression was generalized to n-dimentions for $n \geq 3$. Then, appropriate decay conditions on the metric give a geometric invariant. For $n = 3$ these decay conditions are the known weakest falloff conditions for AFSTs in GR. In [9] an asymptotic symmetries theorem was proved under the weakest falloff conditions in the metric at spi. It implies that the ADM mass can be invariantly assigned to an asymptotically flat four dimensional end if the metric is a non-radiation metric or if the end is defined in terms of a collection of boost-type domains.

As a rule, studing AFSTs was carried out in the framework of the traditional geometrical formulation (GF) of GR. However, it is evident that AFSTs are more convenient to consider in the framework of a field theory. Many authors noted this fact (see, for example, [12]). In [13] we used the detaily developed technique of the field formulation (FF) of GR [14 - 16] to study AFSTs at spi. The FF of GR has the following properties of a field theory. The Einstein gravitational field and other physical fields, which are its sources, are considered on a background of a fixed auxiliary spacetime (flat or curved). Both the equations of motion and the total stress-energy tensor (not pseudotensor!) of all the dynamic fields (including the gravitational field) follow from an action by the ordinary variational procedure. All the equations and the expressions in the FF of GR are coordinate independent. Besides of that, the FF of GR has properties of a gauge theory. We stress that the latters are very important for the present investigation. We stress also that the FF of GR and the GF of GR are two different formalisms for a description of the same physical reality and they lead to the same physical conclusions (for the details see [13 - 16]).

In [13], like in [3 - 9], we obtained also the weakest falloff conditions in the metric at spi for defining AFSTs. Only we obtained them by the other way than that in [3 - 9]. However, we did not set off this result from other many results of [13] especialy, and in fact this result was not discussed and was not compared with the earlier above mentioned results. Nevetherless, we think that this matter is important sufficiently and deserves more attention. Thus, in this paper we will describe a way of obtaining the weakest falloff conditions in the framework of the FF of GR in detail and we will discuss it. On the one hand, this work developes and supplements [13]. On the other hand, this work is independent and can be read without reading [13].

The paper is organized as follows. In section 2 the main notions of the FF of GR are given. Section 3 is the central section of the paper. In the framework of the FF of GR the definition of an AFST is given and integrals of motion are constructed. Then we find the permissible asymptotics for gauge transformations such that values of the integrals of motion are left unchanged under the gauge transformations. This leads to the weakest falloff conditions in the metric at spi. In section 4 the results obtained are compared with those in the works [5, 6] where the investigation of the problem was carried out in more detail and carefully.

II. ON THE FIELD APPROACH TO GENERAL RELATIVITY

In this section we briefly review only the results of [14 - 16] which present a necessary basis for investigating AFSTs. The action in the FF of GR is given in a background spacetime with the metric $\gamma_{\mu\nu}$:

$$S = -\frac{1}{2\kappa c}\int d^4x L^g + \frac{1}{c}\int d^4x L^m \equiv \frac{1}{c}\int d^4x L^{tot} \tag{2.1}$$

where $\kappa \equiv 8\pi G/c^4$ and the Lagrangian for the free gravitational field $h^{\mu\nu}$ has the form:

$$L^g = -h^{\mu\nu}{}_{;\alpha}(K^\alpha_{\mu\nu} - K^\beta_{\beta(\mu}\delta^\alpha_{\nu)}) + (\sqrt{-\gamma}\gamma^{\mu\nu} + h^{\mu\nu})(K^\alpha_{\mu\nu}K^\beta_{\beta\alpha} - K^\alpha_{\mu\beta}K^\beta_{\nu\alpha}). \tag{2.2}$$

Henceforth, Greek indeces are equal to 0, 1, 2, 3; $x^0 = ct$; Latin indeces numerate spatial coordinates; $h^{\mu\nu}$ is the symmetric tensor density of the weight +1; $(\alpha\beta)$ means the symmetrization in α and β. In (2.2) the tensor $K^\alpha_{\beta\gamma}$ is symmetric with respect to the lower indeces and obeys the equation

$$h^{\alpha\beta}{}_{;\gamma} - (\sqrt{-\gamma}\gamma^{\alpha\beta} + h^{\alpha\beta})K^\pi_{\pi\gamma} + (\sqrt{-\gamma}\gamma^{\alpha\pi} + h^{\alpha\pi})K^\beta_{\pi\gamma} + (\sqrt{-\gamma}\gamma^{\pi\beta} + h^{\pi\beta})K^\alpha_{\pi\gamma} = 0. \tag{2.3}$$

For simplicity we suppose that the matter sources are presented by a set of tensor densities and are denoted by the generalized field variable ϕ^A and that the source Lagrangian has the form:

$$L^m = L^m\left[\phi^A;\ \phi^A_{\ ,\alpha};\ \sqrt{-\gamma}\gamma^{\mu\nu} + h^{\mu\nu};\ (\sqrt{-\gamma}\gamma^{\mu\nu} + h^{\mu\nu})_{,\alpha}\right]. \qquad (2.4)$$

In (2.2) - (2.4) and below, $(,\alpha)$ and $(;\alpha)$ are the ordinary and the covariant (with respect to $\gamma_{\mu\nu}$) derivatives; $\gamma \equiv \det \gamma_{\mu\nu}$.

The main reason why AFSTs are suitable models for applying the methods of the FF of GR lies in the fact that a background spacetime is selected by a natural way [13]. Indeed, in the AFST case the existence of the physical flat spacetime at infinity defines the choice in favour of Mikowski space. Therefore, we will use the background equations [14] which can be written in the form of the simplest condition

$$R^{(0)}_{\mu\nu} = 0 \qquad (2.5)$$

for the background Ricci tensor. After varying the action (2.1) with respect to $h^{\mu\nu}$ and some algebraic operations the equations of the gravitational field are obtained,

$$G^L_{\mu\nu}(h) \equiv \frac{1}{2\sqrt{-\gamma}}(h_{\mu\nu}{}^{;\alpha}{}_{;\alpha} + \gamma_{\mu\nu}h^{\alpha\beta}{}_{;\alpha\beta} - h^{\alpha}{}_{\mu;\nu\alpha} - h^{\alpha}{}_{\nu;\mu\alpha}) = \kappa(t^g_{\mu\nu} + t^m_{\mu\nu}) \equiv \kappa t^{tot}_{\mu\nu}. \qquad (2.6)$$

Here, $t^g_{\mu\nu}$ is the stress-energy tensor (SET) of the free gravitational field, and $t^m_{\mu\nu}$ is the SET of the matter sources interacting with the gravitational field. They are obtained in the standard way from (2.1)

$$t^g_{\mu\nu} = -\frac{1}{\kappa\sqrt{-\gamma}}\frac{\delta L^g}{\delta\gamma^{\mu\nu}}, \qquad (2.7a)$$

$$t^m_{\mu\nu} = \frac{2}{\sqrt{-\gamma}}\frac{\delta L^m}{\delta\gamma^{\mu\nu}}. \qquad (2.7b)$$

The source equations of motion have the form:

$$\frac{\delta L^m}{\delta\phi^A} = 0. \qquad (2.8)$$

After taking into account the condition (2.5) from Eq. (2.6) the differential conservation law

$$t^{tot;\nu}_{\mu\nu} = 0 \qquad (2.9)$$

follows. Note that for the first time the Einstein equations in the closed form of Eq. (2.6) were constructed and explained in detail in the work [17].

After the identification

$$\sqrt{-g}g^{\mu\nu} \equiv \sqrt{-\gamma}\gamma^{\mu\nu} + h^{\mu\nu} \qquad (2.10)$$

it becomes evident that the FF of GR transfers to the ordinary GF of GR with the dynamic metric $g^{\mu\nu}$, $g \equiv \det g_{\mu\nu}$, see [14].

Stress some points for Eq. (2.6). 1) An auxiliary metric of Minkowski space was included; 2) the equations (2.6) are coincided with the Einstein equations in the ordinary GF of GR exactly (without approximations); 3) the source of the linear field of the spin 2 is the total metric SET of the system (of the gravitational field together with its matter sources).

Appart from evident covariance (see (2.1) - (2.9)), the FF of GR, unlike the GF of GR, has invariance properties under gauge transformations (GTs) which act only upon the dynamic variables and do not affect the coordinates and the background metric. The gauge invariance is expressed as follows.

The GTs for the dynamic variables can be written as

$$h'^{\mu\nu} = h^{\mu\nu} + \sum_{k=1}^{\infty} \frac{1}{k!}\, L^k_\xi(\sqrt{-\gamma}\gamma^{\mu\nu} + h^{\mu\nu}) \equiv h^{\mu\nu} + \Delta_\xi h^{\mu\nu}, \qquad (2.11a)$$

$$\phi'^A = \phi + \sum_{k=1}^{\infty} \frac{1}{k!}\, L^k_\xi \phi^A \equiv \phi^A + \Delta_\xi \phi^A \qquad (2.11b)$$

where L_ξ means the ordinary Lie-derivatives with respect to ξ^α [18]. Owing to the fact that L^{tot} is the scalar density of the weight $+1$ and after taking into account the background equation (2.5) one concludes that L^{tot} is invariant under GTs (2.11) up to a four-divergence. Then, it is clear that the action (2.1) is invariant under the GTs up to surface terms. The Eqs. (2.6) and (2.8) are invariant under the GTs on themselves. In the present work, the GTs for the total SET are more important, therefore we write out them in an explicit form. The substitution of GTs (2.11) into the total SET and using Eq. (2.6) give

$$t'^{tot}_{\mu\nu} = t^{tot}_{\mu\nu} + \frac{1}{k}\, G^L_{\mu\nu}(\Delta_\xi h) \equiv t^{tot}_{\mu\nu} + \Delta_\xi t^{tot}_{\mu\nu} \qquad (2.12)$$

where the gauge remainder is a covariant four-divergence.

All the gauge invariance properties in the FF of GR follow from the invariance properties of the GF of GR under the Lie displacement transformations which are directly connected with its general covariance [15, 16]. Let us show the way in which the coordinate transformations in the GF of GR are connected with GTs (2.11). Decompose the components $\sqrt{-g}g^{\mu\nu}$ into the sum of the background and the dynamic parts, like in (2.10),

$$(\sqrt{-g}g^{\mu\nu})(x) = (\sqrt{-\gamma}\gamma^{\mu\nu})(x) + h^{\mu\nu}(x). \qquad (2.13)$$

Make an arbitrary coordinate transformation

$$x'^\alpha = x'^\alpha(x) \qquad (2.14)$$

which has an inverse one. Then make the new decomposition

$$(\sqrt{-g}g^{\mu\nu})'(x') = (\sqrt{-\gamma}\gamma^{\mu\nu})(x') + h'^{\mu\nu}(x') \tag{2.15}$$

where the form of the background part $(\sqrt{-\gamma}\gamma^{\mu\nu})(x')$ is the same, like in (2.13). Now, pass from point x'^{α} to point x^{α} within frame $\{x'^{\alpha}\}$. After that, from (2.13) and (2.15) one obtains that $h'^{\mu\nu}(x)$ and $h^{\mu\nu}(x)$ are connected by GTs (2.11a) in the case if the coordinate transformation (2.14) is given in the form

$$x'^{\alpha} = x^{\alpha} + \xi^{\alpha}(x) + \frac{1}{2!}\,\xi^{\beta}\xi^{\alpha}{}_{,\beta} + \frac{1}{3!}\,\xi^{\pi}(\xi^{\beta}\xi^{\alpha}{}_{,\beta})_{,\pi} + \ldots \tag{2.16}$$

Below, we will show that an investigation of GTs gives a possibility to find the weakest falloff conditions defining AFSTs. This is a reason why in (2.11) - (2.16) the description of the GTs is given in detail.

III. THE WEAKEST FALLOFF CONDITIONS AT SPATIAL INFINITY

In this section we get the weakest asymptotic conditions for defining AFSTs at spi. The methods of the FF of GR are used. We name a spacetime as an AFST if it corresponds to a real isolated system. For these systems all the physical fields are effectively concentrated in a confined space at finite time intervals. As an initial definition of an AFST we use the following naive one in the framework of the GF of GR. A spacetime is called as an AFST if the following points 1) - 4) are satisfied:

1) At spi world points can be paramitrized one-to-one by some four coordinates x^{α} which satisfy $-\infty < x^{\alpha} < \infty$;

2) In the one of these frames the metric has the behavior

$$g_{\mu\nu} = \eta_{\mu\nu} + O(r^{-1}), \qquad g_{\mu\nu,\alpha} = O(r^{-2}) \tag{3.1}$$

at $r \to \infty$ where $\eta_{\mu\nu} \equiv diag\,(-1,\ 1,\ 1,\ 1)$ and $r^2 \equiv {x^1}^2 + {x^2}^2 + {x^3}^2$. Henceforth $O(r^{\omega})$ means the asymptotic behavior at $r \to \infty$ only;

3) No a coordinate system exists where the gravitational potentials fall off faster than in (3.1);

4) The matter SET in the coordinate system (3.1) has the asymptotics

$$T_{\mu\nu} = O(r^{-3-\alpha}), \quad \alpha > 0. \tag{3.2}$$

This definition of an AFST is very simple and similar to that in [19].

In order to evaluate a field configuration for an AFST in the framework of the FF of GR one has to select a background spacetime. We select Minkowski space (see Sec. II) with the metric $\eta_{\mu\nu}$ from (3.1). The Lorentz coordinates are used as matter of convenience. Indeed, this simplifies the consideration because one has explicit expressions for an asymptotic behavior. (Recall that the FF of GR is coordinate independent (see Sec. II).) For simplicity (see [13]) we assume that a manifold which supports the physical metric in the GF will also supports the globally auxiliary flat metric. We wish also to consider the dynamic fields as ordinary classical functions of the class C^∞ only.

Then, using the defition (3.1) and decomposition (2.10) one obtains the falloff in the gravitational potentials in the form:

$$h^{\mu\nu} = O(r^{-1}), \qquad h^{\mu\nu}{}_{,\alpha}(r^{-2}). \tag{3.3}$$

Next, no a gauge fixing exists where the $h^{\mu\nu}$ falloff would be faster than in (3.3). At last, the condition (3.2) transfers to

$$t^m_{\mu\nu} = O(r^{-3-\alpha}), \quad \alpha > 0. \tag{3.4}$$

In order to define integrals of motion (IMs) we use the Killing vectors of Minkowski space $\lambda^\mu_{(K)}$ ($K := \alpha,\ [\pi\rho]$ where the antisymmetrization in π and ρ is used) and the existence of both SET (2.7) and the differential conservation law (2.9). Then, IMs corresponding to $\lambda^\mu_{(K)}$, like in an usual field theory, are

$$P^{(K)} = \frac{1}{c} \lim_{r\to\infty} \int_{S_t} dS_0\ t^{0\mu}_{tot}\ \lambda^{(K)}_\mu \tag{3.5}$$

where the spacelike sections S_t in Minkowski space defined by $t = const$ are extrinsically and intrinsically flat; dS_0 is the element of a coordinate volume on S_t. As usual, IMs (3.5) are conserved in time if the fields have a corresponding asymptotics, i.e.,

$$\lim_{r\to\infty} \oint dS_k\ t^{k\mu}_{tot}\ \lambda^{(K)}_\mu = 0 \tag{3.6}$$

where dS_k is the element of coordinate volume on the timelike "walls" of a cylinder surrounding an isolated system in the backgropund spacetime.

By the fulfilment of Eq. (2.6), the total SET can be substituted for the left hand side of Eq. (2.6) and rewritten as

$$t^{\mu\nu}_{tot} = \frac{1}{\kappa}\ G^{\mu\nu}_L(h) = \frac{1}{\kappa}\ U^{\mu\nu\beta}{}_{,\beta} \tag{3.7}$$

where the superpotential has the form:

$$U^{\mu\nu\beta} \equiv \frac{1}{2}(h^{\mu\nu,\beta} + \eta^{\mu\nu} h^{\beta\alpha}{}_{,\alpha} - h^{\mu\beta,\nu} - \eta^{\mu\beta} h^{\nu\alpha}{}_{,\alpha}). \tag{3.8}$$

The substitution of (3.7) and (3.8) into (3.5) and using the antisymmetry $U^{\mu\nu\beta} = -U^{\mu\beta\nu}$ permit to lead the IMs to the surface integrals:

$$
\begin{aligned}
P^{(\alpha)} =& \frac{1}{c\kappa} \lim_{r\to\infty} \int_{S_t} dS_0 \left[U^{\alpha 0i}{}_{,i}\right] \\
=& \frac{1}{2c\kappa} \lim_{r\to\infty} \oint_{\partial S_t} d\sigma_i \left(h^{\alpha 0,i} + \eta^{\alpha 0} h^{i\beta}{}_{,\beta} - h^{\alpha i,0} - \eta^{\alpha i} h^{0\beta}{}_{,\beta}\right),
\end{aligned}
\tag{3.9a}
$$

$$
\begin{aligned}
P^{([mn])} =& \frac{1}{2c\kappa} \lim_{r\to\infty} \int_{S_t} dS_0 \left[(U^{n0i}x^m - U^{m0i}x^n)_{,i} + U^{m0n} - U^{n0m}\right] \\
=& \frac{1}{4c\kappa} \lim_{r\to\infty} \oint_{\partial S_t} d\sigma_i \left((h^{n0,i} - h^{ni,0} - \delta^{ni} h^{0\alpha}{}_{,\alpha})x^m + \delta^{ni} h^{m0}\right. \\
& \left. -(h^{m0,i} - h^{mi,0} - \delta^{mi} h^{0\alpha}{}_{,\alpha})x^n - \delta^{mi} h^{n0}\right),
\end{aligned}
\tag{3.9b}
$$

$$
\begin{aligned}
P^{([m0])} =& \frac{1}{2c\kappa} \lim_{r\to\infty} \int_{S_t} dS_0 \left[(U^{00i}x^m - U^{m0i}x^0)_{,i} - U^{00m}\right] \\
=& \frac{1}{4c\kappa} \lim_{r\to\infty} \oint_{\partial S_t} d\sigma_i \left((h^{00,i} - h^{ik}{}_{,k})x^m\right. \\
& \left. -(h^{m0,i} - h^{mi,0} - \delta^{mi} h^{0\alpha}{}_{,\alpha})x^0 - \delta^{mi} h^{00} + h^{mi}\right)
\end{aligned}
\tag{3.9c}
$$

where $d\sigma_i$ is the surface element on a two-sphere surounding an isolated system.

Now, let us substitute the AFST definition (3.3) into the expressions (3.9). Then, IMs (3.9a) will be finite, whereas IMs (3.9b,c) will be infinite in general. In order to avoid these undesirable results one has to use more precise definition

$$h^{\mu\nu} = O^+(r^{-1}) + O^-(r^{-\beta}), \quad \beta \geq 2, \tag{3.10a}$$

$$h^{\mu\nu}{}_{,\pi} = O^-(r^{-2}) + O^+(r^{-1-\beta}) \tag{3.10b}$$

where $(+)$ and $(-)$ mean even and odd functions with respect to changing sign of the three-vector $\vec{n} = \{x^k/r\}$. For the first time the asymptotics with taking into account an angular dependence was given in [10].

An isolated system is a real physical system. Systems of this type are known as asymptotically Minkowskian ones [20]. Their description in the FF of GR has to be invariant under Poincare transformations. Therefore, we require that the system be Poincare invariant. Then (for the detail see [13]),

$$
\begin{aligned}
h^{\mu\nu}{}_{,\pi\rho} =& O^+(r^{-3}) + O^-(r^{-2-\beta}), \\
h^{\mu\nu}{}_{,\pi\rho\sigma} =& O^-(r^{-4}) + O^+(r^{-3-\beta})
\end{aligned}
\tag{3.11}
$$

and so on. I.e., each derivative falls off faster (by on power of r) than the previous stage. Poincare invariance of the asymptotic behavior (3.10), (3.11) is in accordance with the theorems in [6].

Owing to the falloff conditions (3.10), (3.11) the SET $t^g_{\mu\nu}$ in (2.7a) falss off as r^{-4}. Then, after taking into account (3.4) one concludes that the falloff in the integrand on the left-hand side of (3.6) (see the right-hand side of Eq. (2.6)) ensures the fulfilment of Eq. (3.6). Thus, all the ten IMs (3.5) (or the same (3.9)) are conserved in time.

Note some results. We have obtained IMs (3.9) with the aid of the ordinary methods of field theories in Minkowski space. Thus, we have the well defined IMs. The falloff conditions (3.10), (3.11) ensure that the IMs are finite and that they are conserved in time. It is not difficult to see that the IMs $P^{(\alpha)}$ and $P^{([\alpha\beta])}$ are transformed as tensors under Poincare transformations in Minkowski space. Finally, the falloff conditions (3.10), (3.11) are invariant under the Poincare transformations.

Further, in order to find the weakest falloff conditions in the gravitational potentials for defining AFSTs we use the gauge invariance properties of the FF of GR.

As it is seen from (2.12), the total SET is invariant under the GTs up to a covariance divergence only. Therefore, it is not difficult to see that IMs (3.9) are invariant under the GTs up to surface terms. Thus, it is interesting to find the weakest falloff conditions in the gauge potentials ξ^α which will ensure that the values of the IMs be left unchanged under the GTs.

An important point is that the expression $G^L_{\mu\nu}(h)$ on the left-hand side of Eq. (2.6) describes the linear field of the spin 2. It is well known that $G^L_{\mu\nu}(h)$ is invariant under the linear in ξ^α and $h^{\mu\nu}$ part of GTs (2.11a)

$$h'^{\mu\nu} = h^{\mu\nu} + L_\xi(\sqrt{-\gamma}\gamma^{\mu\nu}). \tag{3.12}$$

Now, recall that the IMs in (3.9) have been obtained with the use of the operator $G^L_{\mu\nu}$ and that, at the same time, the second term on the right hand side of Eq. (2.12) is the operator $G^L_{\mu\nu}$ acting on $\Delta_\xi h^{\mu\nu}$. Then, it is clear that the term $L_\xi(\sqrt{-\gamma}\gamma^{\mu\nu})$ in (3.12) can be excluded from $\Delta_\xi h^{\mu\nu}$ in (2.11a) under the consideration.

Now, turn to search for the restrictions on ξ^α. The following evedent requirements are used:

(i) All the dynamic variables $h^{\mu\nu}$ and ϕ^A satisfy the Einstein equations (2.6), (2.8) and $h^{\mu\nu}$ have the falloff (3.10), (3.11). Otherwise, they are arbitrary.

(ii) Initial components of the gauge field ξ^α, $\xi^\alpha{}_{,\beta}$, $\xi^\alpha{}_{,\beta\gamma}, \ldots$ in (2.11) are arbitrary independent quantities at each of points in Minkowski space. Only it is required that they be symmetric with respect to the lower indeces.

(iii) The condition (ii) is complemented by the requirement that the components ξ^α, $\xi^\alpha{}_{,\beta}$, $\xi^\alpha{}_{,\beta\gamma}, \ldots$ change as tensor components under Poincare transformations.

(iv) The functions ξ^α in (2.11a) are ordinary classical ones of the class C^∞ and each derivative of ξ^α falls off faster (by one power of r) than previous stage. Thus, the gauge transformated components $h'^{\mu\nu}$ will have also the same important properties as $h^{\mu\nu}$ have.

Let the falloff conditions in ξ^α have the form:

$$\xi^\alpha = O^-(r^{1-\varepsilon}) + O^+(r^{1-\delta}). \tag{3.13}$$

In order to the four-momentum $P^{(\alpha)}$ be invariant under the GTs one has to assume (see (3.9a)) that the odd part of $(\Delta_\xi h^{\mu\nu})_{,\alpha}$ falls off faster than the odd part of the r^{-2} falloff, i.e.,

$$O^-((\Delta_\xi h^{\mu\nu})_{,\alpha}) < O^-(r^{-2}). \tag{3.14}$$

Keeping in mind p. (ii) consider all the terms of the type $\xi\xi_{,\alpha\beta\gamma}$ (we do not write out upper indeces) in $(\Delta_\xi h^{\mu\nu})_{,\alpha}$ as independent ones. Then, with the use of p. (iv) from (3.14) one obtains the following restrictions on the ξ^α falloff in (3.13):

$$\varepsilon > \frac{1}{2}, \quad \delta > \frac{1}{2}. \tag{3.15}$$

Owing to these conditions and to the requirements (i) - (iv) all the rest terms in $(\Delta_\xi h^{\mu\nu})_{,\alpha}$ as well as the terms of the type $\xi\xi_{,\alpha\beta\gamma}$ do not give a contribution into $P^{(\alpha)}$. Analogiously, the requirement of conserving $P^{([\alpha\beta])}$ under the GTs gives the restrictions

$$\varepsilon + \delta > 2, \quad \delta > 1, \quad \varepsilon \geq 0 \;\; if \;\; \beta > 2, \quad \varepsilon > 0 \;\; if \;\; \beta = 2 \tag{3.16}$$

on the falloff conditions (3.13). Combining (3.15) and (3.16) one gets the restrictions

$$\varepsilon + \delta > 2, \quad 1 \geq \varepsilon > \frac{1}{2}, \quad \delta > 1 \tag{3.17}$$

which ensure the invariance both $P^{(\alpha)}$ and $P^{([\alpha\beta])}$ under the GTs. The condition $\varepsilon \leq 1$ expresses the fact that for a real isolated system the falloff in the gravitational potentials can not be faster than in (3.3), or more exactly in (3.10), (3.11).

The substitution of (3.13) and (3.17) into (2.11a) and taking into account (3.10), (3.11) give

$$h'^{\mu\nu} = O^+(r^{-\varepsilon}) + O^-(r^{-\delta}). \tag{3.18}$$

As is seen, the asymptotics (3.10) is destroied. However, values of the IMs constructed with using (3.18), as it was shown, will be rest previous. Moreover, if the second step of the GTs with (3.13), (3.17) is made,

$$h''^{\mu\nu} = h'^{\mu\nu} + \Delta_\xi h'^{\mu\nu},$$

then the asymptotics (3.18) and values of $P^{(K)}$ do not change again, and so on.

Thus, for a real isolated system with the falloff conditions (3.17), (3.18) it is easy to conclude the following. The IMs are well defined by the expressions (3.9) and they have finite values which are conserved in time. The system is described in the manner which is covariant under Poincare transformations in Minkowski space.

For an isolated system the leading term in the expressions of the gravitational field at spi must be the Newtoinian potential. Then, on the one hand, in the framework of the GF of GR every solution to the Einstein equations can be led with the help of a coordinate transformation to a form with the leading Schwarzshild metric at spi [10]. On the other hand, there is not a coordinate system for which the metric falls off faster than the Schwarzshild metric. In the framework of the FF of GR the above means that any initial asymptotics (3.17), (3.18) corresponding to an isolated system can be led to the falloff conditions (3.10), (3.11) and not faster them with the help of the GTs.

IV. DISCUSSION

In this section we compare the obtained results with some well known ones. Many of the characteristic properties of the field approach to an investigation of AFSTs were considered and discussed in [13]. Here, only we discuss the weakest falloff conditions (3.18) under the restrictions (3.17). This conditions can be transformed into those in the dynamic metric $g^{\mu\nu}$ by the simple way with the use of (2.10) and, then, can be compared with the others.

For the most part, we compare our results with those in the remarkable work [6] where studing the question about the weakest falloff conditions was carried out very carefully with the use of an elegant mathematical technique. In this regard, we use a more simple technique and restrict our consideration to classical functions of the class C^∞.

The Einstein equations in the form (2.6) play the central role in [6]. We use also this form. However, we treat the field $h^{\mu\nu}$ as a field of the spin 2 with self-interaction in the background Minkowski space more definitely. We treat $t^{tot}_{\mu\nu}$ as the total metric SET of all the dynamic fields in Minkowski space. Thus, in spite of the fact that Minkowski space has really an auxiliary character, we give it more fundamental sense than in [6]. Recall that in [6] Eq. (2.6) is treated as a simple decomposition of the Einstein equations in GF after decomposing the dynamic metric $g^{\mu\nu}$ into $h^{\mu\nu}$ and $\eta_{\mu\nu}$.

In [6], in order to find the weakest falloff conditions it was required a finiteness of the four-momentum that was defined with the use of the right-hand side of Eq. (2.6) and of the asymptotic symmetries. These conditions have the form of the $r^{-1/2}$ falloff and have been obtained without using the ordinary r^{-1} falloff as an initial one. The conservation laws in time have been also obtained. After using the left-hand side of Eq. (2.6) and the constraints it was shown that the ADM expression for the four-momentum is also finite in the case of the $r^{-1/2}$ falloff.

On the other hand, in [6] a special attention was paid to the transformation which is obtained in the following way. Two asymptotically flat metrics connected by a coordinate transformation are compared. In the case of each metric the Minkowski background metric is selected. Then, the transformation from the dynamic part $h^{\mu\nu}$ in the one case to the dynamic part $h'^{\mu\nu}$ in the other case is considered. (In our approach, the same transformation is defined in the explicit form of the closed GTs (2.11). It will be easy to see if in (2.13) and (2.15) the metrices $(\sqrt{-\gamma}\gamma^{\mu\nu})(x)$ and $(\sqrt{-\gamma}\gamma^{\mu\nu})(x')$ are substituted by the Minkowski metric $\eta^{\mu\nu}$. As is seen, we have more concrete expressions which are more convenient for explicit calculations.) The requirement to the four-momentum be invariant under this transformation gives the same restriction by the $r^{-1/2}$ falloff.

Thus, in [6] the statement that the metric falls off faster than $r^{-1/2}$ is used twice. First, it is used to show that the four-momentum be finite in a given Minkowski space. Second, the four-momentum is unchanged when the set off allowed backgrounds is restricted also by the $r^{-1/2}$ falloff. As it is appeared, in [6] the weakest falloff conditions were obtained by the two ways which are thought as independent ones.

Let us outline our way briefly. We construct the IMs with the use of the total SET $t^{tot}_{\mu\nu}$ and the Killing vectors of Minkowski space. The ordinary r^{-1} falloff is used as initial one. Further, we convince ourselves that for the r^{-1} falloff the IMs are finite and are coserved in time. Under this consideration we use the rules of an ordinary field theory in Minkowski space only. Then, we find the weakest falloff conditions in the gauge potentials such that the IMs are unchanged under the GTs. After that we use this restrictions on the GTs to find the weakest falloff conditions in the gravitational potentials. In the result, the definition of IMs (3.9), the finiteness of the IMs and their conservation in time are also the case for the $r^{-1/2}$ falloff.

Thus, in comparison with the standard geometrical approach (in particular, with the approach in the work [6]) the properties of our study are the following:

1. The IMs and the conservation laws in time are defined as well as in any field theory in Minkowski space. Thus, our approach is more simple and more illustrative than an ordinary geometrical approach.

2. We obtained the weakest falloff conditions in the gravitational potentials by a unified manner. These conditions are connected with GTs only and are defined by the latters only. The finiteness of the IMs and their conservation in time are consequences of invariance of the IMs under the GTs with the restrictions found on the gauge potentials.

3. In the work [6], with using the harmonic coordinates (and also the TT-coordinates) it was shown the following. For the system with the weakest falloff conditions a so-called

"wave" part can be selected. It does not give a contribution into a value of the four-momentum. A value of the four-momentum is defined so-called by the "Newtonian potential" part which falls off like r^{-1}. As it is appeared, the conclusions about the "wave" and the "Newtonian potential" parts were obtained in the particular coordinate systems. In comparison, we shown for an isolated system in the general case that any falloff which is faster than $r^{-1/2}$ and slower than r^{-1} has a purely gauge part which does not give a contribution into values of the IMs. The values of the IMs are defined by the r^{-1} part only. Indeed, the values of the IMs are defined by the expression $G^L_{\mu\nu}(h)$ which has the behavior $O^+(r^{-3}) + O^-(r^{-2-\beta})$ at spi for the field configuration (3.18). This behavior is just defined by the asymptotics (3.10a) and follows from our method, where the r^{-1} falloff is an initial one.

4. The Hamiltonian description and $3+1$ decomposition are very popular to study AFSTs. It is used either elements of this formalism (as example, in [6]) or this formalism in full measure (as example, in [5]). In contrast, our approach is Lagrangian and four-dimentional in full measure. Only we use the $3+1$ decomposition in order to define the IMs on the spacelike hypersurfaces S_t and to obtain their conservation laws in time, i.e., under the transition from the one S_t to another.

5. In [6], the four-momentum was studied only. We study all the ten IMs. In the result, our approach takes into account an angular dependence in asymptotics of the gravitational potentials and gives more detailed results. In [5] the Hamiltonian approach was applied to AFSTs and the Regge-Teitelboim concept [10] was developed. The closure condition for the Poisson brackets of the generators of the ten IMs was used as a key assumption therein. The results of [5] would coincide with ours (3.18) under the conditions (3.17) in full if the erroneous (see [13]) restriction $|\varepsilon - \delta| \leq 1$ was excluded.

Especially we stress the fact that we consider isolated systems in GR as well as in the other works which we have noted in Introduction. Thus, our conclusions do not have to differ (and do not differ) from the conclusions in these works in principle. However, our results were obtained in the essentially different way from others. This permits to consider the problem from the other viewpoint and in the other details.

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