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7. RELATIVISTIC COSMOLOGY

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The subject of cosmology is an admixture of imaginative ideas, intuitive predictions and hard scientific facts. This has made cosmology a case of three distinct cultures; that of astronomers, who look at the universe and find hard data, relativists – who build models of universe that range from simple to esoteric ones, and the particle physicists who can test their theories of very high energy physics only in the cosmic laboratory that was there at the very onset of the universe. This culture can well be compared with the British culture, where one has three classes: the working class, the professionals and the aristocrats.

This course will try to give certain glimpses of cosmology in these three fields.

1. Large Scale Structure of the Universe

1.1. DISTANCE SCALE

We progressively increase our distance scale as we look into distant objects and there are several orders of magnitude involved in this increment. A few typical steps on the cosmic distance ladder are:

- a) Sun's radius : 7×10^{10} cms.
- b) Interstellar distance : 3×10^{18} cms.
- c) Diameter of the Galaxy (Milky way) : 10^{23} cms.
- d) Intergalactic distance : 10^{24} cms.
- e) Size of a cluster : 10^{25} cms.
- f) Size of a supercluster : 10^{26} cms.
- g) The Hubble Radius : 10^{28} cms.

The physics may vary from one distance scale to another.

- (i) *Magnitude of H_0* : These measurements turned out to be overestimates. Now, many later researchers have tried to calculate the Hubble constant, H_0 , and have come up with different answers, due to the uncertainty in determining the actual distance of the galaxies. Tammann and Sandage have taken a sample of relatively nearby galaxies and given the value of H_0 as $50 \text{ kms}^{-1} \text{ Mpc}^{-1}$ while Van den Bergh and de Vaucoueurs have calculated the value as $100 \text{ kms}^{-1} \text{ Mpc}^{-1}$ for a more distant sample. This divergence reflects the continuing uncertainty of estimates of extragalactic distances. So, at present the value of the Hubble constant is denoted by $H_0 = 100 \text{ kms}^{-1} \text{ Mpc}^{-1} \times h_0$, where h_0 is a scale factor, $\frac{1}{2} \leq h_0 \leq 1$.
- (ii) *The Rubin-Ford Effect* : In 1975, Rubin and Ford found that there is a dipole anisotropy in the above Hubble velocity flow. They showed that instead of observing the galaxies from a cosmological rest frame we are moving with a velocity of around 500 kms^{-1} in a specified direction relative to that frame. However, there is a discrepancy still present; the anisotropy of the Hubble flow does not seem to match the anisotropy in the cosmic microwave background either in magnitude or in direction.
- (iii) *The inhomogeneity of H_0* : It was observed that H_0 (h_0) tends to increase with local distance. Calculated with respect to the local group of galaxies it gives the value of $h_0 \simeq \frac{1}{2}$. But, when we look beyond, the value of h_0 tends to unity. This can be explained by a peculiar motion of our local group of galaxies towards the centre of the Virgo Supercluster.
- (iv) *The density parameter of the universe*
In a gravitational theory the time scale is related to density ρ ,

$$H_0 \propto \frac{1}{t} \propto \sqrt{G\rho}$$

$$\text{i.e., } \frac{H_0^2}{G} \propto \rho \quad (2)$$

From calculations arising out of Einstein's General Theory of Relativity it can be shown that there is a critical density (sometimes ρ_c is called ρ_{closure})

$$\rho_c = \frac{3H_0^2}{8\pi G} \quad (3)$$

The actual density may be written as

$$\rho = \Omega_0 \rho_c, \quad (4)$$

where Ω_0 , the density parameter is a dimensionless quantity, whose value depends, theoretically upon the model we choose for the universe.

Observationally we may try to compare the actual density with the theoretically predicted value. Estimates of luminous matter density in the universe tell us that $\Omega_0 \leq 0.2$. Further, also taking into consideration the motion of matter relative to the cosmological rest frame we get $\Omega_0 \leq 0.12$.

- (v) *Dark matter* : It seems that ρ_{actual} may not be equal to $\rho_{\text{luminous matter}}$. Dynamical considerations of 'bound' objects indicate the presence of gravitating, nonluminous matter the so called dark matter. The presence of dark matter is indicated by the flat rotation curves obtained for galaxies. A rotation curve is the plot of the radial velocity and the radial distance of a test particle in the gravitational potential of a galaxy. Considering our galaxy, which is thought to be spiral shaped, it can be shown by simple Keplerian dynamics that

$$v = \sqrt{\frac{GM}{R}} \propto \frac{1}{\sqrt{R}} \quad (5)$$

But considering the motion of the hydrogen clouds which are the 'test particles' over here, we find that instead of the velocity dropping after reaching a maximum, it attains a constant value as we go further away from the galaxy (See Fig 2). This flat rotation curve extends far beyond the luminous extent of the galaxy.

The question of dark matter also arises when we consider the galaxy cluster velocities. If we consider the thermodynamics of the cluster and take its kinetic energy as $T = \sum \frac{1}{2} m_i v_i^2$, and potential energy as; $\Phi = - \sum_{i,j} \frac{m_i m_j}{r_{ij}}$, then by the Virial Theorem we expect to find

$$2T + \Phi = 0. \quad (6)$$

Actually, the velocities come out to be higher than allowed by (6). This suggests that the potential energy must be higher than that computed for luminous systems thereby indicating presence of unaccounted mass. Theoreticians have said that the value of Ω_0 can go as high as unity, but they could not succeed in explaining the result by this effect alone.

2. Models of the Universe

2.1. WEYL'S POSTULATE

Considering the non-static models of the universe, we can come up with two situations, first where the world lines of galaxies criss-cross each other in a haphazard manner (See Fig. 3a) wherein there is no order and where two world lines intersect we have colliding galaxies and large random motions.

which says that, at any given cosmological time, the universe is homogeneous and isotropic. Homogeneity implies that the geometrical property of the local neighbourhood of any point in spacetime will look similar whereas, isotropy means the universe will look similar in all directions at any point. Translating this into mathematical form (done by the method of Killing Vectors) leads to the conclusion that the space $t = \text{constant}$ has constant curvature. Let us consider the case of constant negative curvature as an example. In terms of Cartesian co-ordinates, a 3 surface of constant negative curvature is given by the equation.

$$x_1^2 + x_2^2 + x_3^2 - x_4^2 = S^2$$

where S is a constant with respect to x_1, x_2, x_3, x_4 but may depend on t .

The substitution,

$$\begin{aligned} x^1 &= S \sinh \chi \cos \theta \\ x^2 &= S \sinh \chi \sin \theta \cos \phi \\ x^3 &= S \sinh \chi \sin \theta \sin \phi \\ x^4 &= S \cosh \chi \end{aligned}$$

gives the line element of the 3-surface as

$$d\sigma^2 = S^2 \left[\frac{dr^2}{1+r^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right]. \quad (10)$$

The cases of zero or constant positive curvature can be introduced in this equation by rewriting as:

$$d\sigma^2 = S^2 \left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right]$$

where $k = 0$ implies zero curvature,

$k = 1$ implies (+ve) curvature,

$k = -1$ implies (-ve) curvature.

Thus the line element for spacetime becomes

$$ds^2 = c^2 dt^2 - S^2 \left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right]. \quad (11)$$

$S(t)$ gives us curvature as a function of time. This line element is known as the Robertson - Walker line element as it was discovered independently by H. P. Robertson and A. G. Walker.

If the homogeneity exists both in space and time; the Cosmological Principle becomes 'perfect' and time-like Killing motions are also included and we then deal with a more restricted type of model, which uses the de Sitter metric.

The steady state cosmology came from the Perfect Cosmological Principle (PCP). Its line element is

$$ds^2 = c^2 dt^2 - e^{2H_0 t} \left[dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right], \quad (12)$$

Thus the PCP tells us the full line element purely from the symmetry considerations.

2.3. THE ENERGY TENSOR

The Weyl postulate and the cosmological principle do not tell us about the rate of expansion of the universe given by the quantity $S(t)$; at the same time they also fail to answer our question about whether the 3-space is closed or open. To deal with this problem we need a dynamical theory, such as given by Einstein's General relativity.

Considering Einstein's questions:

$$R_k^i - \frac{1}{2} g_k^i R = -\kappa T_k^i \quad \text{where } \kappa = \frac{8\pi G}{c^4} \quad (13)$$

and the Robertson - Walker line element:

$$ds^2 = c^2 dt^2 - S^2(t) \left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right]$$

we can get two sets of equations:

$$\frac{2\ddot{S}}{S} + \frac{\dot{S}^2 + kc^2}{S^2} = -\frac{8\pi G}{c^4} T_1^1 \quad (14)$$

$$\text{and, } \frac{\dot{S}^2 + kc^2}{S^2} = \frac{8\pi G}{3c^4} T_0^0 \quad (15)$$

The same equation (14) holds if T_1^1 is replaced by T_2^2 or T_3^3 . Thus we write

$$\begin{aligned} T_1^1 = T_2^2 = T_3^3 &= -\bar{p} \text{ (pressure),} \\ \text{and } T_0^0 &= u \text{ (energy density);} \end{aligned} \quad (16)$$

and ultimately we can arrive at the equation, $T_{k;i}^i = 0$, i.e.,

$$\frac{d}{dS}(uS^3) + 3pS^2 = 0. \quad (17)$$

This is the conservation equation for matter/energy. We next consider two cases of (16).

and,

$$\frac{\dot{S}^2}{S^2} - \frac{8\pi G\rho_0 S_0^3}{3S^3} = 0 \tag{27}$$

It is convenient to introduce a new parameter $q(t)$ through the relation:

$$\frac{\ddot{S}}{S} = -q(t) [H(t)]^2, \quad H(t) = \frac{\dot{S}}{S} \tag{28}$$

This parameter $q(t)$, which is dimensionless is called the "deceleration parameter".

Now, substituting the values of \ddot{S}/S we get,

$$\frac{1}{S^2} = (2q - 1)H^2 \tag{29}$$

and,

$$\rho = \left(\frac{3H^2}{4\pi G}\right) q \tag{30}$$

Now, for the present epoch,

$$\rho_0 = \left(\frac{3H_0^2}{4\pi G}\right) q_0 \equiv \rho_c \Omega_0 \tag{31}$$

and ultimately we get,

$$\Omega_0 = 2q_0 \tag{32}$$

Further, since the left hand side of equation (29) is always positive, we get,

$$q > \frac{1}{2} \text{ and thus } \Omega_0 > 1. \tag{33}$$

Thus the closed model has a density exceeding the previously defined closure density.

(iii) *Open sections; (k = -1)* : Here the space is open and unbounded. The field equations will become,

$$\frac{1}{S^2} = (1 - 2q)H^2 \Rightarrow q < \frac{1}{2}, \quad \Omega < 1. \tag{34}$$

The density remains

$$\rho_0 = \left(\frac{3H_0^2}{4\pi G}\right) q_0 < \rho_c \tag{35}$$

So, it is observed that three cases may arise depending on the value of the deceleration parameter:

$$\begin{aligned} \text{if } k &= 1; \quad q > \frac{1}{2}; \quad \rho > \rho_c, \\ \text{if } k &= -1; \quad a < \frac{1}{2}; \quad \rho < \rho_c; \\ \text{if } k &= 0; \quad \rho = \rho_c. \end{aligned}$$

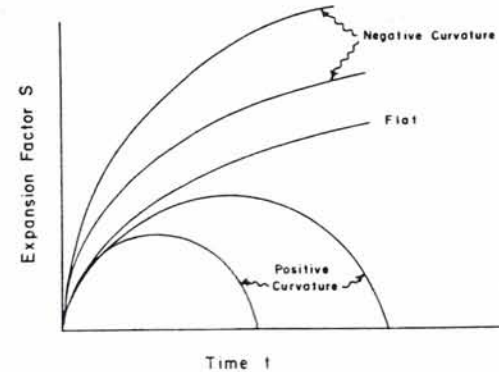


Figure 4. The three curves describe three types of Friedman models with positive, negative and zero curvature

Fig. 4 illustrates the modes of expansion of the three types of models. Notice that all the models imply that the universe had a beginning a finite time ago. Since the Friedman models are frequently used to interpret cosmological observations, we will now derive some of the observable quantities in these models.

2.5. THE RED SHIFT, LUMINOSITY DISTANCE AND HUBBLE'S LAW

(i) *The red shift* : Consider the propagation of null rays in the Robertson Walker line elements given by.

$$ds^2 = c^2 dt^2 - S^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right].$$

Suppose, our observer is at $r = 0$ and receives the signal at t_0 from a source galaxy G_1 at co-ordinates (r_1, θ_1, ϕ_1) and the time of emission of

- (i) When we are talking about flux, the concept of unit time appears. In an expanding universe time gets elongated as per formula (38). So, to include this effect we must write

$$l = \frac{L}{4\pi r_1^2 S^2(t_0) \cdot (1+z)}$$

where $1+z$ is the correction factor.

- (ii) The frequency, of each photon received by us gets reduced by a factor of $1+z$:

$$\nu_{\text{received}} = \frac{\nu_{\text{emitted}}}{1+z}$$

Hence, the corresponding energy is reduced by the factor $(1+z)$ so that

$$l = \frac{L}{4\pi r_1^2 S^2(t_0) (1+z)^2} \equiv \frac{L}{4\pi D^2} \text{ (say)}. \quad (41)$$

Comparing the above formula with the Euclidean case, the 'luminosity distance' is defined by

$$D = r_1 S(t_0) (1+z). \quad (42)$$

r_1 is found, after some calculations to be

$$r_1 = \frac{\sqrt{2q_0 - 1}}{q_0^2(1+z)} \left[q_0 z + (1 - q_0) (1 - \sqrt{1 + 2zq_0}) \right] \quad (43)$$

Therefore,

$$D = \frac{c}{H_0 q_0^2} \left[q_0 z + (q_0 - 1) (\sqrt{1 + 2zq_0} - 1) \right] \quad (44)$$

If we take, $q_0 = 1$, then

$$D = \frac{c}{H_0} z. \quad (45)$$

which in our familiar Hubble's formula. Therefore, for this particular model ($q_0 = 1$) we get, $D \propto z$. In general, however, the $D - z$ relation is not linear as shown in the Fig. 5.

2.6. HORIZONS

There are two kinds of horizons in cosmology. The particle horizon limits our observations of the past while the event horizon limits our ability to communicate in the future.

THE SYNTHESIS OF HELIUM AND OTHER NUCLEI

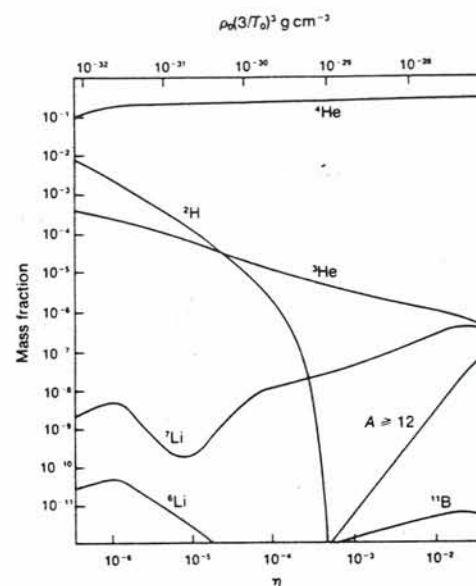


Figure 5. A plot giving nuclear abundances of light elements as functions of baryon density in the universe

- (i) *The particle horizon*: This typically arises in cosmological models like the Friedman models which imply that the universe had a beginning a finite time ago.

Consider an observer at $r = 0$, $t = t_0$ receiving signals from the distant galaxies. From (36) we see that the maximum value of R.H.S. is attained when $r = r_L$ where,

$$\int_0^{r_L} \frac{dr}{\sqrt{1 - kr^2}} = \int_0^{t_0} \frac{dt}{S(t)}. \quad (46)$$

This means that for $r > r_L$ the signal cannot reach the observer, provided the integral on the right hand side converges, as happens for almost all Friedman models. The surface $r = r_L$ defines the 'particle horizon'. Anything inside the particle horizon is connected with the observer causally. Note that r_L is the coordinate distance. The metric

relativistic particles, other than radiation and include them in the energy-momentum tensor term in the Einstein equations.

From our knowledge of the strong and weak interactions, it can be said that, the particles that could have been present were,

$$\gamma, p, e, \bar{p}, \bar{e}, n, \bar{n}, \nu_e, \nu_\mu, \bar{\nu}_e, \bar{\nu}_\mu, \dots, \pi^+, \pi^-, \pi^0, \text{ etc.}$$

We have to consider the universe as a mixture of these particles, which are moving very fast and randomly thereby colliding with each other. This scenario is far from Weyl's postulate! There is a large scale random motion indicating that the pressure term which we have neglected previously (dust model) is important. At the temperature mentioned some of the above particles will be relativistic. Let us now consider the thermodynamic properties of these particles.

3.1. THE DISTRIBUTION FUNCTIONS

Assuming ideal gas approximation and thermodynamic equilibrium it is possible to write down the distribution function for any given species of particles. Let us use the symbol A to denote the typical species ($A = 1, 2, \dots$) $n_A(P)dP$ denotes the number density of species A in the momentum range $(P, P + dP)$, where,

$$n_A(P)dP = \frac{g_A}{2\pi^2\hbar^3} P^2 \left[\exp\left(\frac{E_A(P) - \mu_A}{kT} \pm 1\right) \right]^{-1}. \quad (55)$$

where, T = temperature of the distribution; g_A = number of spin states of the species A ; k = Boltzmann constant; μ_A = chemical potential and

$$E_A^2 = c^2 p^2 + m_A^2 c^4. \quad (56)$$

As we are concentrating on a relatively small region of space, we can consider the space to be locally flat and we can use special theory of relativity instead of the general theory. Now, the total number of particles is given by

$$N_A = \int_0^\infty n_A dP, \quad (57)$$

Energy density,

$$\rho_A = \int E_A n_A dP. \quad (58)$$

pressure,

$$p_A = \frac{1}{3} \int \frac{p^2}{E_A} n_A dP. \quad (59)$$

and entropy:

$$S_A = \frac{p_A + \rho_A}{T}. \quad (60)$$

In equation (55) the (+)ve sign is for fermions, e.g., protons and electrons and the (-)ve sign is for bosons e.g., γ, π^+, π^- . As we are working under the ideal gas approximation we assume that between two collisions a particle is acted on by no force, i.e., the particle executes free motion between collisions. Initially, let us put $\mu_\gamma = 0$, since,

1. Photons can be emitted or, absorbed in any number, so their number is a free quantity, reconciled by the fact: $\mu_\gamma = 0$.
2. Normally the total number density of particles is \ll the photon number density, so we can ignore the chemical potential of the particles.

$$\begin{aligned} g_A \text{ the number of spin states} &= 2 \text{ for electron} \\ &= 1 \text{ for neutrino} \\ &= 2 \text{ for photon} \end{aligned}$$

For photons, the thermodynamic quantities are:

$$\begin{aligned} N_\gamma &= \frac{2.404}{\pi^2} \left(\frac{kT}{c\hbar} \right)^3, \\ \rho_\gamma &= \frac{\pi^2 (kT)^4}{15 (c\hbar)^3} = 3p_\gamma, \\ S_\gamma &= \frac{4\pi^2 k}{45} \left(\frac{kT}{c\hbar} \right)^3. \end{aligned} \quad (61)$$

We can arrive at a simplified picture for particles in the High Temperature Approximation (HTA) (or, *The relativistic Approximation*)

- (a) *High temperature approximation*: Here the kinetic temperature satisfies the condition

$$T \gg T_A \equiv m_a c^2 / k. \quad (62)$$

We make two rules regarding the determination of the thermodynamic property of the particles:

- i. For bosons, multiply the photon values by $\frac{g_A}{2}$.
- ii. For fermions, multiply the photon values by $\frac{3g_A}{8}$ for N_A and, multiply the photon values by $\frac{7g_A}{16}$ for ρ_A, S_A . Thus the effective spin state number g for computing total energy of a family of relativistic particles is obtained by adding to the total spin state number g_b of bosons $7/8$ times the total spin state number g_f of fermions.

$$G = \text{weak interaction constant} = 1.4 \times 10^{-49} \text{erg.cm}^{-3} \quad (70)$$

Now, the number density of the participating $e^\pm \sim \left(\frac{kT}{ch}\right)^3$ (since, these particles are relativistic at this temperature). The number density of muons μ^\pm on the other hand is given by $\sim T^{3/2} \exp(-\frac{T_\mu}{T})$ where, the exponential term arises due to low temperature approximation. Notice that the muons are non-relativistic because of their comparatively larger masses. Thus the typical neutrino reaction rate becomes

$$\eta = G^2 \hbar^{-7} c^{-6} kT^5 \exp(-\frac{T_\mu}{T}). \quad (71)$$

We must now consider the rate at which a typical volume expands. According to the field equations

$$H^2 = \frac{\dot{S}^2}{S} = \frac{8\pi G}{3c^2} u \approx \frac{16\pi^3 G}{90\hbar^3 c^5} (kT)^4. \quad (72)$$

Thus, the ratio of the reaction rate to expansion rate given by,

$$\begin{aligned} \frac{\eta}{H} &\sim G^{-1/2} \hbar^{-11/2} G^2 c^{-7/2} (kT)^4 \exp(-\frac{T_\mu}{T}) \\ &\sim \left(\frac{T}{10^{10} K}\right)^3 \exp(-\frac{10^{12} K}{T}) \\ &\sim T_{10}^3 \exp(-\frac{1}{T_{12}}), \end{aligned} \quad (73)$$

where T_n is the temperature expressed in units of 10^n K.

As T decreases, the exponential term becomes dominant, and a stage comes when $\eta/H \ll 1$, and at this temperature the neutrinos decouple from leptons.

Earlier, when the calculation was done using the simple weak interaction theory and the Feynman - Gellmann cross-section formula it was found that the decoupling temperature was $\sim 10^{11} K$. Later, when the Electro-weak theory came about with the introduction of neutral current, the temperature was reduced to $\sim 10^{10} K$. This is because by opening additional channels, the Electro-weak theory keeps the neutrinos interacting for a longer period.

Now, the neutrinos are not in thermal equilibrium, so, when they are decoupled their distribution function is determined by looking at each neutrino. The momentum and energy of each neutrino scale down with expansion as $1/S$. Therefore, if we take each neutrino in the original distribution and scale it down, then we will see that the temperature also falls off as $1/S$. So, the neutrino temperature falls off as for rest of the matter.

But if the neutrinos were massive $p \propto \frac{1}{S}$ but $E \propto \frac{1}{S}$ since, $E^2 = c^2 p^2 + m^2 c^4$. Thus the temperature no longer scales as $1/S$. Massive neutrinos eventually become non-relativistic. There is, however, going to

be a difference later on in the temperatures of the neutrinos and the photons even in the case of neutrinos of zero rest mass. This phenomenon is explained below:

Let us consider the dynamics of the universe, from $T = 10^{12} K$ to $T = 10^9 K$. In this phase, we have photons, e^\pm pairs and the neutrinos, each with a distribution function given earlier in the high temperature approximation. Therefore,

$$\rho_{\text{matter}(\text{rel})} = \rho_{\nu_e} + \rho_{\bar{\nu}_e} + \rho_{\nu_\mu} + \rho_{\bar{\nu}_\mu} + \rho_{e^+} + \rho_{e^-} + \rho_\gamma.$$

To know the g -factor, we have,

$$g = 1(\nu_e) \ 1(\bar{\nu}_e) \ 1(\nu_\mu) \ 1(\bar{\nu}_\mu) \ 2(e^+) \ 2(e^-) \ \& \ 2(\gamma)$$

Thus, the effective ' g ' = $\frac{7}{8}g_f + g_b = 9$, and hence

$$\rho = \frac{9}{2} aT^4. \quad (74)$$

So, the temperature - time relationship will get altered from (52) to

$$T = \left(\frac{c^2}{48\pi aG}\right)^{1/4} t^{-1/2}. \quad (75)$$

$$\text{i.e., } T_{10} = 1.04 \times t_s^{-1/2}. \quad (76)$$

Until now the temperature has not come down to the rest temperature of the electron, $T_e \simeq 6 \times 10^9 K$. So, the question arises, what happens between $10^{10} K$ and $10^9 K$?

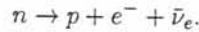
Earlier we had $e^+ + e^- \rightleftharpoons 2\gamma$ but now only annihilations take place $e^+ + e^- \rightarrow 2\gamma$. So the net effect is that the energies residing in e^\pm are given to the photons but not to the neutrinos as they have already decoupled. Thus the photon temperature shoots up above the neutrino temperature. One particular reaction may be possible: $e^+ + e^- \rightarrow \nu + \bar{\nu}$, but its probability is very small. At $10^{10} K$ we had a situation when $T_\gamma = T_\nu \propto \frac{1}{S}$, but as more energy is pumped into the photons we expect $T_\gamma > T_\nu$. How do we estimate the effect quantitatively?

We have seen previously that $\sigma = S^3(p + \rho)/T$ is conserved. So, if we can calculate σ for $10^{10} K$ and $10^9 K$ then we can get an idea by what fraction the ratio T_γ/T_ν has risen above unity. In the initial relativistic phase,

$$\begin{aligned} \sigma_i &= \frac{4S_i^3}{3T_i} (\rho_{\bar{e}} + \rho_e + \rho_\gamma) \\ &= \frac{11}{3} \frac{S^3}{T_i} aT_i^4 = \frac{11}{3} (T_i S_i)^3. \end{aligned}$$

(Remember that $1 - X_n$ denotes the proton fraction)

As the universe expands and cools down to lower temperatures the pairs disappear, but it is still possible for the neutron to decay via the reaction,



As the temperature goes lower the neutrons and proton do not remain free but combine to form deuterium atoms which can capture neutrons and protons to form higher nuclei.

So, ultimately it becomes a many-body problem which we consider next.

3.5. THE PRIMORDIAL NUCLEOSYNTHESIS

A typical nucleus Q is described by two quantities: A = atomic mass and Z = atomic number; and is written as



This nucleus has Z protons and $A-Z$ neutrons. If m_Q is the mass of the nucleus, its binding energy is given by

$$B_Q = [Zm_p + (A - Z)m_n - m_Q]c^2. \quad (81)$$

We use the approximation $m_p \approx m_n = m$, and write

$$X_p = \frac{N_n}{N_N}, X_n = \frac{N_n}{N_N}, X_Q = A \frac{N_Q}{N_N}, \quad (82)$$

where N_N = number density of all nucleons of which N_p is in the form of free protons, N_n as free neutrons and N_Q as bound nuclei Q . The X 's are called mass fractions of their respective species.

As we are now concerned with relative number densities, we can no longer ignore the chemical potentials. So,

$$N_Q = g_Q \left(\frac{m_Q K T}{2\pi\hbar^2} \right)^{3/2} \exp\left(\frac{\mu_Q - m_Q c^2}{K T}\right), \text{ etc.} \quad (83)$$

We know that the chemical potential satisfy the relation

$$\mu_Q = (A - Z)\mu_n + Z\mu_p. \quad (84)$$

Now, using (82) for n, p and Q and (80), (83) to eliminate the values μ_Q, μ_n, μ_p we get,

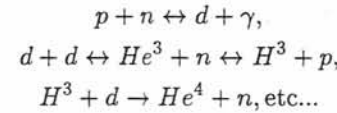
$$X_Q = \frac{1}{2} g_Q A^{5/2} X_p^Z X_n^{A-Z} \xi^{(A-1)} \exp\left(\frac{B_Q}{K T}\right), \quad (85)$$

$$\text{where, } \xi = \frac{1}{2} N_N \left(\frac{m K T}{2\pi\hbar^2} \right)^{-3/2}. \quad (86)$$

T must drop to a low enough value to make $\exp(B/KT)$ large enough to compensate for the smallness of $\xi^{(A-1)}$. This occurs for nucleus Q when T has dropped down to

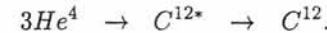
$$T_Q \sim \frac{B_Q}{k(A-1)|\ln\xi|}. \quad (87)$$

We apply the above formula to He^4 , its binding energy being $4.3 \times 10^{-5} \text{ ergs} = 28.29 \text{ Mev}$, to find that $T_Q \approx 3 \times 10^9 \text{ K}$. At this low temperature the number densities of participating nucleons is so low, however, that the probability of four body collisions leading to the formation of helium is very small. So, we try using two body collisions to describe the buildup of heavier nuclei, and He^4 is formed as a result of the chain reactions :



Although at this temperature nucleosynthesis does proceed rapidly enough, it cannot go beyond He^4 since, there are no stable nuclei with $A = 5$ or, 8.

The primordial process therefore cannot give us heavier nuclei. So, the observed abundances of the heavier nuclei are due to the evolution of stars, where further nucleosynthesis can occur beginning with the reaction



Although it is a three - body reaction its rarity is compensated for by its being a resonant reaction. This resonant reaction was theoretically predicted by Hoyle in the mid-1950s.

(i) *Abundances of light nuclei* : Returning to the primordial scenario, the bulk of nucleosynthesis is over at $\sim 8 \times 10^8 \text{ K}$; and at this temperature all the neutrons have been gobbled up by helium nuclei. Denoting the mass fraction of the primordial helium by Y_P , we get $Y_P = 2X_n$. To this must be added the helium generated in stars, say Y_S before we can compare with the presently observed mass fraction of cosmic helium. We can write,

$$Y_{\text{obs}} = Y_P + Y_S. \quad (88)$$

Observations put an upper limit on Y_P in the range 0.23 - 0.24. The theoretical value should depend on the total number of nucleons in the universe. We define a parameter η related to the nucleon density :

$$\eta = \left(\frac{\rho_N}{2.7 \times 10^{-26} \text{ gm.cm}^{-3}} \right) \left(\frac{3}{T} \right)^3. \quad (89)$$

densities of free electrons (N_e), protons ($N_p = N_e$) and H atoms (N_H) at a given temperature we take the help of Saha's ionization equation,

$$\frac{N_E^2}{N_H} = \left(\frac{m_e K T}{2\pi\hbar^2} \right)^{3/2} \exp(-B/KT) \quad (95)$$

where, m_e = electron mass ; B = Binding energy of hydrogen nuclei ($= 13.59\text{eV}$). Let, $x = \frac{N_e}{N_B}$ (N_B = total baryonic density). Then because $N_H = N_B - N_e$, we have from (94)

$$\frac{x^2}{1-x} = \frac{1}{N_B} \left(\frac{m_e K T}{2\pi\hbar^2} \right)^{3/2} \exp(-B/KT). \quad (96)$$

We can solve for x as a function of T . The results show that x drops sharply from 1 to ~ 0 around the temperature 10^3K . For example, if we consider $\Omega_0 h_0^2 = 0.1$, then $x = 0.003$ at temperature $T = 3000\text{K}$. The consequence of this lowering of temperature will be that the electrons will be bound and will not be available for scattering the photons. The universe will then become transparent. The photon background will be decoupled from matter and cooled with a Planckian temperature $T \propto S^{-1}$. The cosmic microwave background has a planckian spectrum with the temperature $\sim 2.7\text{K}$. If this is the relic of the hot early epochs then the above 'recombination epoch' turns out to have a redshift $\sim 10^3$.

We know the universe changed over from being radiation dominated to matter - dominated around the same epoch. This instance of both these phenomena occurring at around the same time appears a coincidence that cannot be explained in a natural way.

Another puzzling result is the observed ratio of photons to baryons

$$\frac{N_\gamma}{N_B} = 4.57 \times (\Omega_0 h_0^2)^{-1} (T_0/3)^2. \quad (97)$$

This ratio has been conserved since the time the universe became essentially transparent. Why this ratio has this value and not any other is still a mystery. Why are there so many photons for every baryon in the universe? The clue to this issue may lie in even earlier epochs, (*i.e.*, prior to the nucleosynthesis epoch) which we will consider next.

4. The very early Universe

The history of the universe based on the current ideas is diagrammatically represented in Fig.6. To study the origin of the baryons and to get insight into the ratio (97) we have to resort to ideas in particle physics and the analysis of different interactions : (i) Q.E.D. (ii) Weak interactions and (iii) Strong interactions at very high energies.

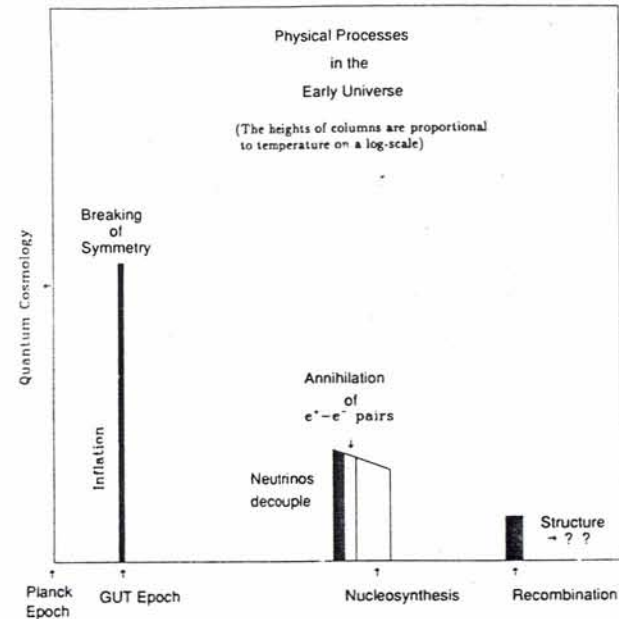


Figure 6. The time evolution of the standard hot big bang model. The time axis is on a logarithmic scale. The crucial epochs are the Planck epoch, the GUTs epoch, the neutrino decoupling epoch, the e^\pm annihilation epoch, the nucleosynthesis epoch, the recombination epoch and the epoch when discrete structures begin to appear. This last stage is still poorly understood

In 1968 the electromagnetic and weak interactions were unified through the symmetry group $SU(2)$ and we got the electro - weak interaction. It was found that the energy at which this unification takes place is around 100GeV , which has been tested through man - made accelerators, by the detection of massive bosons W^\pm and Z . Extrapolating theoretically it has been predicted that, if we increase our energy to $\sim 10^{15}\text{GeV}$, then all the 3 basic interactions may be unified. This theory, often called the 'Grand Unified Theory' is not yet known but is expected to be based on the gauge group symmetries, the simplest one being given by the group $SU(5)$.

The temperature corresponding to the GUT era is $\sim 10^{27} - 10^{26}\text{K}$. So, from the above diagram, we can say that nothing of notable importance happened between 10^{26}K and 10^{16}K . Another energy range which is important is that in the quantum gravity epoch. Then we can construct a

Putting in numbers one can show that

$$(\Omega - 1) \simeq 4 \times 10^{-15} T_{\text{MeV}}^{-2} (\Omega_0 - 1). \quad (102)$$

$$T \sim 10^{15} \text{GeV} = 10^{18} \text{MeV}$$

so that,

$$\Omega - 1 \cong 4 \times 10^{-15-36} (\Omega_0 - 1) < 10^{-49} \quad [\Omega_0 \sim 10].$$

This means, at the GUT epoch the universe was extremely fine tuned near $\Omega = 1$.

If we go to the Planck epoch then $\Omega - 1 < 10^{-57}$. On the other hand, if we say that there was no such fine tuning, then why do we find the universe so close to the flat model ($\Omega_0 = 1$) now?

(b) *The horizon problem* : Let us consider the particle horizons, at small t . Regions which were in physical communication would have established homogeneity. This scale is set by the horizon radius $\approx ct$, where t = age of the universe. At the GUTs epoch, when $T = 10^{28} \text{K}$, $t = 10^{-37} \text{sec}$ then $r \simeq 10^{-27} \text{cm}$. So, if we consider a region of this size then there will be homogeneity. The question arises that as the universe expands what will be the extent of homogeneity now? As the temperature has come down from 10^{28}K to 3K , the scale factor has increased by 3×10^{27} . So the size of the homogenous region at present turns out to be $\sim 3 \times 10^{-27} \times 10^{27} = 3 \text{cms}$. But from the observations of CMBR we are finding homogeneity in a region whose radius is of the order of 10^{28}cms . Working backwards, we see that though particle horizon was small, homogeneity was all pervading. So how did two disconnected regions 'come to know' of each other's condition? This is known as the horizon problem.

(c) *The monopole problem* : This problem arose from a property of group theory. When a large local group such as $SU(5)$, (under the breakdown of the symmetry) splits into smaller groups, including the $U(1)$ group, e.g.,

$$SU(3) \rightarrow \text{strong interaction}$$

$$SU(2)_L \rightarrow \text{weak interaction}$$

$$U(1) \rightarrow \text{electromagnetic interaction}$$

$$SU(5) = SU(3) \times SU(2)_L \times U(1)$$

then some solutions are inevitably generated that will be like the 'magnetic monopole solution'. Dirac had studied the modification of

Maxwell's equations for monopole charges and derived quantization conditions on the monopole charge. If it is formed at the GUT epoch then one would expect at least one monopole of mass \mathcal{M} in a volume $\frac{4\pi}{3} r_H^3$ at the GUTs epoch, r_H being the horizon radius. As the universe expands, what would be the density of monopole matter now? At the GUTs epoch it was

$$\rho_{\mathcal{M}} = \frac{3\mathcal{M}}{4\pi r_H^3} \sim 1.5 \times 10^{65} \text{gm.cm}^{-3}.$$

This is scaled down to

$$(\rho_{\mathcal{M}})_0 = 5 \times 10^{-18} \text{g.cm}^{-3}. \quad (103)$$

which though small, is much greater than ρ_c (closure density) = $2 \times 10^{-29} \text{g.cm}^{-3}$. The monopoles are indestructible and can easily destroy the magnetic field in the inter-galactic region (10^{-6}Gauss). They are clearly absent, contrary to the above prediction.

(d) *The entropy problem* : This is again related to the photon to baryon ratio. The entropy density s in the early universe is given by

$$s = \frac{2\pi^2}{45} T^3 g(T). \quad (104)$$

Also, during expansion $S^3 s = \text{constant} = \mathcal{F}$. Thus for the present epoch,

$$\mathcal{F}_0 = S_0^3 s_0 \approx 10^{87} \gg 1. \quad (105)$$

Why should such a large number be present at the early epoch, when photons and other particles were on equal footing?

(e) *The domain wall problem* : There are certain particles, which play mediating role in the symmetry breaking process of the GUT - type interaction; they are known as Higgs bosons (ϕ). When the symmetry breakdown occurs there can be two values of $\phi \rightarrow \phi_1$ and ϕ_2 , say, and the very nature of phase transitions is such that we have continuous regions of $\phi = \phi_1$ and $\phi = \phi_2$ with the boundary wall as a discontinuity. This is the domain wall and if it intersects some part of our observing universe we will see some large scale astronomical effect, which is not observed. In other words, there is no observational evidence for any topological discontinuity of space.

In 1981, Guth proposed a modification of the standard big bang scenario that claimed to cure it of these defects. We will consider this interesting idea next.

Here ρ_{other} is the classical radiation term of the SBB. It drops as $1/S^4$. Let

$$\rho(\phi = 0) = \epsilon_0 = \text{constant.} \quad (113)$$

So, the dominating term is $\rho(\phi)$ and, we will have

$$\frac{\dot{S}^2}{S^2} \sim \frac{8\pi G\epsilon_0}{3}$$

$$\Rightarrow S \propto e^{H_0 t} \quad \text{where } H_0^2 = \frac{8\pi G\epsilon_0}{3}. \quad (114)$$

This solution has been in cosmology for a very long time, under various contexts.

In 1917, de-Sitter considered the Λ term in the Einstein equations, *i.e.*,

$$R_{ik} - \frac{1}{2}g_{ik}R + \Lambda g_{ik} = -\frac{8\pi G}{c^4}T_{ik}. \quad (115)$$

where Λ = cosmological term, denotes some sort of repulsive force.

In 1948 while formulating the Steady State Theory Hoyle introduced certain term on the right hand side of (115) instead of the Λ term to arrive at the de-Sitter model.

The Λ term was also anticipated by W. H. McCrea in 1950 from processes in the vacuum, although particle theorists were not prepared for it! The difference now is in the order of magnitude, the present Λ is 10^{108} times greater than the previous ones thought out by cosmologists!

Thus as T drops below T_c , for a while the universe is in the $\phi = 0$ state which now has the status of a false vacuum. While in this state the universe will temporarily expand exponentially (thus the word 'inflation', *i.e.*, something faster than 'expansion'). The inflation is over and the universe reverts to SBB after the phase transition is completed.

Due to this inflation a bubble is formed at $\phi = \phi_0$ and in this region phase transition goes on to a lower state while outside $\phi = 0$. So, a bubble was obtained by inflating a small region, and it may not occur simultaneously everywhere. All the bubbles thus formed are expected to collide and coalesce after which we will get a universe which will expand as per the Friedmann prescription.

It is worth pointing out that in 1966, while considering the concept of baryon creation in the C-field cosmology, Hoyle and Narlikar had arrived at the notion of 'Friedmann Bubbles' in the ambient de-Sitter (steady state) universe in much the same way as in the present inflationary scenario.

We now consider some details of Guth's model.

- (a) *The entropy problem* : The inflation can account for the large entropy. Suppose the expansion by inflation occurs over a time τ . This leads to an increase of S by $Z = \exp H_0\tau$. To make it as high as $S^3 s = 10^{87}$, we need $Z^3 = 10^{87}$, *i.e.*, $Z = 10^{29}$. This implies $H_0\tau = 29 \ln 10 \cong 67$. Whatever H_0 we may have, after 67 Hubble epochs S^3 will be very high and the energy dumping process will provide the extra entropy.
- (b) *The flatness problem* : In the Friedmann model k/S^2 term ($k = \pm 1$) could have been important. However, with the sudden inflation the scale factor S has increased by $\sim 10^{29}$ and the value of k/S^2 has been reduced by 10^{58} . So, even if we started from a large curvature term there will be no fine tuning necessary to arrive at the 'nearly flat' universe.
- (c) *The horizon problem* : Due to inflation the small size particle horizon is increased by a factor of 10^{29} , which solves the problem. So, we need sufficient time to inflate but we also need the bubbles to coalesce. In the Guth model there were problems on the second count. So, a newer version was proposed.

4.4. THE NEW INFLATIONARY MODEL

In Guth's scenario the bubbles tend to increase around according to the Friedmann expansion rate ($t^{1/2}$) whereas the outside will expand as $e^{H_0 t}$. The problem therefore was that the bubble centres are moving away and they don't coalesce. We need the rate of bubble formation to be large compared to that of the expansion of the universe, but, it was found that the rate was not so fast. To ensure the desired effect, it needed again a fine tuning of parameters which we want to avoid.

So a new idea came from Linde and from Albrecht and Steinherdt, to look at a different potential, given by Coleman and Weinberg :

$$V(\phi, T) = \frac{25}{16}\alpha^2 \left[\phi^4 \ln \frac{\phi^2}{\sigma^2} + \frac{1}{2}(\sigma^4 - \phi^4) \right] + \frac{18}{\pi^2} T^4 \int_0^\infty x^2 \ln \left(1 - \exp \left[-(x^2 + \frac{5}{12}\phi^2 g^2 / T^2)^{1/2} \right] \right) dx. \quad (116)$$

At $T = 0$, $V(\phi, 0)$ has a maximum at $\sigma = 1.2 \times 10^{15} \text{ GeV}$, and,

$$\alpha^2 = \text{strong coupling constant} \\ = g^2/4 = \frac{1}{45} \text{ for } SU(5). \quad (117)$$

and we get, (see Fig.9),

$$V(0) = \frac{25}{32}\alpha^2 \sigma^4 > V(\sigma) = 0. \quad (118)$$

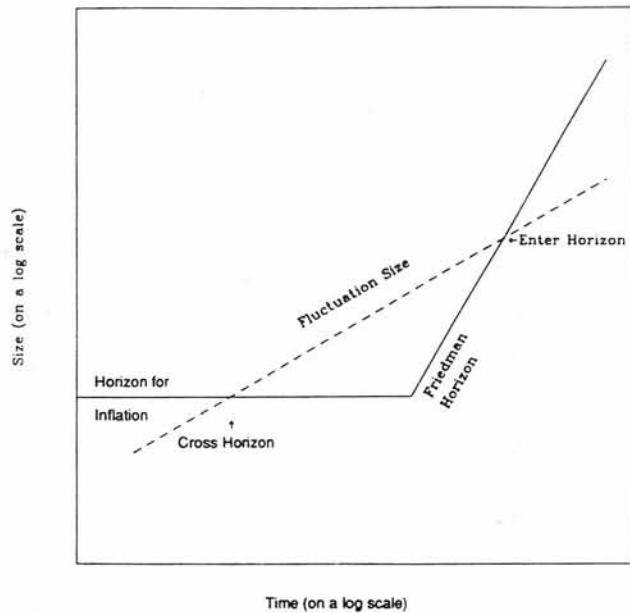


Figure 10. The fluctuation size is shown in relation to the 'horizon'. First the fluctuation size grows and 'crosses' the horizon (which is constant in size) during inflation. Subsequently the universe switches over to the Friedmann state when the horizon overtakes the fluctuation size. This is when the fluctuation 'enters' the horizon.

Fig.10 illustrates what happens to these fluctuations in relation to the actual horizon.

In metric term k corresponds to the wavelength $2\pi S(t)/k \equiv \lambda$. λ goes out of the horizon and then comes back in as the universe changes over from the de Sitter to the Friedmann model. Let $t_1 =$ time at which $\lambda H_0 = 1$. We then find that at t_1 , $\delta\phi(k) = \text{constant}$ (not dependant on k). Therefore, whatever fluctuations we start with when λ leaves the horizon it is causally disconnected (t_1) and when it reenters (t_2) they will come into the Friedmann horizon with the same value.

Thus, after t_2 , $\delta\phi$ is again almost independent of k , and one can relate $\delta\phi$ to the density contrast

$$\frac{\delta\rho}{\rho}|_{t=t_2} = \sqrt{\frac{4\gamma}{3\pi^3}} \ln\left(\frac{H_0}{k}\right)^{3/2} \quad (124)$$

where,

$$V_{\text{eff}}(\phi) = V_{\text{eff}}(0) - \frac{1}{2}\gamma\phi^4, \quad \text{at } T = 0. \quad (125)$$

This work was done by Guth and Pi and leads to the important conclusion that density fluctuations are scale independent.

From an analysis of galaxy correlation functions Zeldovich and Harrison in the 1970s had independently suggested that there should be a scale invariant spectrum of matter fluctuations in the universe.

There, is however, a problem with the above calculation. As the de Sitter model has only an event horizon and the Friedmann model has only a particle horizon, so the concept of 'causal' connection is quite vague in the above argument. Further the actual $\delta\rho/\rho$ turns out to be ~ 50 in this calculation while the observed value $< 10^{-4}$ (from limits on the fluctuations of CMBR).

Thus the saga of inflation continues with further inputs from particle physics. One such idea is of chaotic inflation in which the universe begins to inflate at the Planck epoch. Consider a field ϕ at $t_P \simeq \sqrt{\frac{G\hbar}{c^3}}$ and then find $\Delta\phi$ from quantum fluctuations. It is found to behave in a chaotic manner, since, quantum domain does not allow us to fix our initial condition precisely.

Finally, the Λ -term of the inflationary stage is given by

$$\Lambda = (10^{15}\text{GeV})^4 \text{Mp}^{-2}, \quad (126)$$

while its present observational value is $\leq (10^{-12}\text{GeV})^4 \text{Mp}^{-2}$. This means when the transition took place the Λ term dropped by a huge factor of $\sim 10^{108}$. This is another fine tuning factor !!

In short,

"Inflation has not lived up to it's original promise."