Nonlinear density evolution from an improved spherical collapse model

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ABSTRACT

We investigate the evolution of nonlinear density perturbations by taking into account the effects of shear and angular momentum of the system. Starting from the standard spherical top hat model in which these terms are ignored, we introduce a physically motivated closure condition which specifies the dependence of these terms on δ. The modified equation can be used to model the behavior of an overdense region over a sufficiently large range of δ. The key new idea is a Taylor series expansion in $(1/\delta)$ to model the nonlinear epoch. We show that the modified equations quite generically lead to the formation of stable structures in which the gravitational collapse is halted at around the virial radius. The analysis also allows us to connect up the behavior of individual overdense regions with the nonlinear scaling relations satisfied by the two point correlation function.

Subject headings: Cosmology : theory – dark matter, large scale structure of the Universe

1. Introduction

Analytic modeling of the nonlinear phase of the gravitational clustering has been a challenging but interesting problem upon which considerable amount of attention has been bestowed in recent years. The simplest, yet remarkably successful model for nonlinear evolution is the Spherical Collapse Model (SCM, hereafter) which has been applied in the study of various empirical results in the gravitational instability paradigm. Unfortunately, this approach has serious flaws — both mathematically and conceptually. Mathematically, the SCM has a singular behavior at finite time and predicts infinite density contrasts for all collapsed objects. Conceptually, it is not advisable to model the real universe as a sphere, in spite of the standard temptations theoreticians often succumb to. These two issues are, of course, quite related. In any realistic situation, it is the deviations from spherical symmetry which lead to virialised stable structures getting formed. In conventional approaches, this is achieved by an *ad hoc* method which involves halting the collapse at the virial radius by hand and mapping the resulting nonlinear and linear overdensities to each other. This leads to the well known rule–of–thumb that when linear overdensity is about 1.68, bound structures with nonlinear overdensity of about 178 would have formed. The singular behavior, however, makes the actual trajectory of spherical system quite useless after the turn around phase — a price we pay for the arbitrary procedure used in stabilizing the system. But the truly surprising feature is that, in spite of this arbitrary nature, the SCM seems to give useful insights into the real life systems when properly interpreted. The Press–Schechter formalism (Press & Schechter 1974) for abundance of bound structures uses SCM implicitly; more recently, it was shown that the basic physics behind the nonlinear scaling relations (NSR) obeyed by the correlation function can be obtained from a judicious application of SCM. (Padmanabhan 1996a). These successes, as well as the inherent simplicity of concepts, make SCM an attractive paradigm for studying the nonlinear evolution and motivates one to ask: Can we improve the basic model in some manner so that the behavior of the system after the turn around is 'more reasonable' ?

It is clear from very general considerations that such an approach has to address fairly nontrivial technical issues. To begin with, *exact* modeling of of deviations from spherical symmetry is quite impossible since it essentially requires solving the full BBGKY hierarchy. Secondly, the concept of a radius $R(t)$ for a shell, evolving only due to the gravitational force of the matter inside, becomes ill defined when deviations from spherical symmetry are introduced. Finally, our real interest is in modeling the statistical features of the density growth; whatever modifications we make to SCM should eventually tie up with known results for the evolution of for instance, the two point correlation function. That is, we have to face the question of how best to obtain the *statistical* properties of the density field from the behavior of a *single* system.

In this paper, we try to address these problems in a limited but focussed manner. We tackle the deviations from the spherical symmetry by the retention of a term (which is usually neglected) in the equation describing the growth of the density contrast. Working in the fluid limit, we show that this term is physically motivated and present some arguments to derive an acceptable form for the same. The key new idea is to introduce a Taylor series expansion in $(1/\delta)$ (where δ is the density contrast) to model the nonlinear evolution. We circumvent the question of defining the 'radius' of the non spherical regions by working directly with density contrasts. Finally, we attempt to make the connection with statistical descriptors of nonlinear growth by using the nonlinear scaling relations known from previous work. More precisely we show that the modified equations predict a behavior for the relative pair velocity (when interpreted statistically) which agrees with the results of N-body simulations.

The paper is divided into the following sections. In Section 2, we derive the relevant equations describing the SCM, starting from the fluid equations, in order to set the stage; the physical and *ad hoc* aspects of the SCM are also summarized here. In the next section, we recast the equations in a different form and introduce two functions (i) a "rotation term" and (ii) a function $h_{SC}(\delta)$, whose asymptotic forms are easy to determine. The behavior of $h_{SC}(\delta)$ in the presence and absence of the rotation term is also detailed here. In Section 4, we present the arguments that give the functional forms for the rotation term over a large range of δ; we then go on to present the results in terms of a single collapsing body and show how this term stabilizes a collapse which would have otherwise ended up in a singularity in terms of the growth of density contrast δ with time. When this term is carried through into the equation for $R(t)$ for a single system, it can be seen the radius reaches a maximum and gracefully decreases to a constant, remaining so thereafter. In the standard spherical collapse model the radius decreases from the maximum all the way down to zero thus causing density to diverge. Section 5 summarises the results and discusses its implications.

2. The Spherical Collapse Model.

The general metric for a homogeneous, isotropic universe in 3+1 dimensions is the Friedmann-Robertson-Walker metric, given by

$$ds^2 = dt^2 - a^2(t)\left[\frac{dx^2}{1-kx^2} + x^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right] \tag{1}$$

where $a(t)$ is the scale factor and $k = 0, 1$ or -1 depending on the curvature of the Universe. This unperturbed metric can be converted into the form (see e.g. Padmanabhan 1996b)

$$ds^2 \approx \left(1 - \frac{\ddot{a}}{a}r^2\right) dT^2 - d\mathbf{r}^2 \tag{2}$$

using the transformation

$$r = a(t)x \ , \ T = t - t_o + \frac{1}{2}a\dot{a}x^2 + \mathcal{O}(x^4) \tag{3}$$

in which only terms upto quadratic order in x have been retained. Near $r = 0$, the space is flat, with deviations from flatness of order $(\ddot{a}r^2/a)$. Comparison of equation (2) with the form of the metric in the weak gravity limit, $g_{00} \approx (1 + 2\phi_N)$ (where ϕ_N is the effective Newtonian potential) shows that the unperturbed Friedmann universe has an effective gravitational potential

$$\phi_{FRW}(\mathbf{r}, t) = -\frac{1}{2}\frac{\ddot{a}}{a}r^2 \tag{4}$$

in the weak gravity limit, at scales $r^2 \ll (\ddot{a}/a)^{-1}$. In the perturbed case, the gravitational potential at any point is due to two sources : (i) The potential ϕ_{FRW} of the background Friedmann Universe and (ii) the gravitational potential $\phi(\mathbf{x}, t)$ of perturbed matter. In the Newtonian limit (*i.e.* weak gravity, $\phi \ll 1$), the superposition of these potentials is allowed and the total gravitational potential can be taken to be the sum $(\phi_{FRW} + \phi)$. Since the scales of interest in the present work are much smaller than the Hubble length and the velocities non-relativistic, the above effective Newtonian potential can be used in the following analysis. We will treat the system in the fluid limit as being made up of pressureless dust of dark matter, using a smoothed density, $\rho(t, \mathbf{x})$ and a mean velocity $\mathbf{v}(t, \mathbf{x})$. (This approximation ignores shell crossing and multi stream effects and we shall comment on them in the end.)

Given these assumptions, we can start with the fluid equations

$$\frac{\partial \rho_{\rm m}}{\partial t} + \nabla_{\rm r} \cdot (\rho_{\rm m} U) = 0 \tag{5}$$

$$\frac{\partial U}{\partial t} + (U \cdot \nabla)\, U = -\nabla\phi_{\rm FRW} - \nabla\phi \tag{6}$$

$$\nabla^2_{\rm r}\phi = 4\pi G \rho_{\rm b} \delta \tag{7}$$

which are the usual continuity equation, Euler equation and Poisson Equation, respectively; $\phi_{\rm FRW} = -(\ddot{a}/2a)r^2$ is the potential due to the background expansion, ϕ is the potential due to the perturbations and $\rho_{\rm b}$ denotes the smooth background density of matter. The spatial derivatives in the above equations are with respect to proper coordinates, $\mathbf{r} = a\mathbf{x}$. Making a coordinate transformation to co–moving coordinates given by $\mathbf{x} = \mathbf{r}/a$ and defining the peculiar velocity $\mathbf{v}$ using $U = H\mathbf{r} + \mathbf{v}$, the above equations become

$$\left(\frac{\partial \rho_{\rm m}}{\partial t}\right) + 3H\rho_{\rm m} + \frac{1}{a}\frac{\partial}{\partial x^i}\left(\rho_{\rm m} v^i\right) = 0 \quad (8)$$

$$\left(\frac{\partial v^i}{\partial t}\right) + Hv^i + \frac{1}{a}v^i\frac{\partial v^i}{\partial x^i} + \frac{1}{a}\frac{\partial \phi}{\partial x^i} = 0 \quad (9)$$

$$\nabla_{\rm x}^2\phi = 4\pi G\rho_{\rm b}a^2\delta \quad (10)$$

We next introduce a new time coordinate, $b \equiv b(t)$, which is chosen to be the growing solution of the equation

$$\ddot{b} + 2\frac{\dot{a}}{a}\dot{b} = 4\pi G\rho_b(t)b \quad (11)$$

and set $\rho_{\rm m} = \rho_{\rm b}(1+\delta)$; transforming from t to $b(t)$ and defining a velocity field $u^i = v^i/(a\dot{b})$ we obtain

$$\frac{\partial \delta}{\partial b} + \partial_i\left[u^i(1+\delta)\right] = 0 \quad (12)$$

$$\frac{\partial u^i}{\partial b} + u^k\partial_k u^i = -\frac{3\mathcal{A}}{2b}\left[\partial^i\psi + u^i\right] \quad (13)$$

$$\mathcal{A} = \left(\frac{\rho_{\rm b}}{\rho_{\rm crit}}\right)\left(\frac{\dot{a}b}{\dot{b}a}\right) \quad (14)$$

$$\nabla^2\psi = \left(\frac{\delta}{b}\right) \quad (15)$$

where the critical density, $\rho_{\rm crit}(t) = 3H^2(t)/8\pi G$, $H = \dot{a}/a$ being the Hubble parameter. Taking the divergence of the velocity field u^i and writing it as

$$\partial_i u_j = \sigma_{ij} + \epsilon_{ijk}\Omega^k + \frac{1}{3}\delta_{ij}\theta \quad (16)$$

where σ_{ij} is the shear tensor, Ω^k is the rotation vector and θ is the expansion, we can manipulate the above system of equations (see e.g. Padmanabhan 1996b) to obtain the following equation for δ

$$\frac{d^2\delta}{db^2} + \frac{3\mathcal{A}}{2b}\frac{d\delta}{db} - \frac{3\mathcal{A}}{2b^2}\delta(1+\delta) = \frac{4}{3}\frac{1}{(1+\delta)}\left(\frac{d\delta}{db}\right)^2 + (1+\delta)(\sigma^2 - 2\Omega^2) \quad (17)$$

The same equation can be written in terms of time t as

$$\ddot{\delta} - \frac{4}{3}\frac{\dot{\delta}^2}{(1+\delta)} + \frac{2\dot{a}}{a}\dot{\delta} = 4\pi G\rho_b\delta(1+\delta) + \dot{a}^2(1+\delta)(\sigma^2 - 2\Omega^2) \quad (18)$$

This equation turns out to be the same as the one for density contrast in the SCM except for the additional term $(1+\delta)(\sigma^2 - 2\Omega^2)$ due to rotation and shear. To see this explicitly, we introduce a function $R(t)$ by the definition

$$1 + \delta = \frac{9GMt^2}{2R^3} \equiv \lambda \frac{a^3}{R^3} \tag{19}$$

where M and λ are constants. Using this relation between δ and $R(t)$ and assuming a flat universe ($\mathcal{A} = 1$), equation (18) can be converted into the following equation for $R(t)$

$$\ddot{R} = -\frac{GM}{R^2} - \frac{1}{3}\dot{a}^2\left(\sigma^2 - 2\Omega^2\right)R - \frac{4\pi G}{3}(\rho + 3p)_{rest}R \tag{20}$$

where the first term represents the gravitational attraction due to the mass inside a sphere of radius R and the second gives the effect of the shear and angular momentum. The $(\rho + 3p)_{rest}$ denotes the contribution to the energy density and pressure from forms of energy *other than dust*, i.e. with equations of state different from $p = 0$. We will consider the case of a dust-dominated Universe and hence set this term to zero to obtain

$$\ddot{R} = -\frac{GM}{R^2} - \frac{1}{3}\dot{a}^2(\sigma^2 - 2\Omega^2)R \tag{21}$$

In the case of spherically symmetric evolution, for which we can set the shear and rotation terms to zero, this gives

$$\frac{d^2R}{dt^2} = -\frac{GM}{R^2} \tag{22}$$

which governs the evolution of a spherical shell of radius R, collapsing under its own gravity; M can now be identified with the mass contained in the shell; this is standard SCM.

At this point it is important to note a somewhat subtle aspect of these equations. The initial system of equations ((5), (6) and (7)) we have been working with are clearly Eulerian in nature: *i.e* the time derivatives give the temporal variation of the quantities at a fixed point in space. However, the time derivatives in equation (18), for the density contrast δ are of a different kind. Here, the observer is moving with the fluid element and hence, in this, Lagrangian case, the variation in density contrast seen by the observer has, along with the intrinsic time variation, a component (the $\mathbf{v} \cdot \nabla$ term) which arises as a consequence of his being at different locations in space at different instants of time. When the δ equation is converted into an equation for the function $R(t)$, the Lagrangian picture is retained; in SCM we can interpret $R(t)$ as the radius of a spherical shell, co–moving with the observer. The mass M within each shell remains constant in the absence of shell crossing (which does not occur in the standard SCM for reasonable initial conditions) and the entire formalism is well defined. The physical identification of R is, however not so clear in the case where the shear and rotation terms are retained, as these terms break the spherical symmetry of the system. We will nevertheless continue to think of R as the "effective shell radius" in this situation, *defined by* equation (19) governing it's evolution. Of course, there is no such ambiguity in the *mathematical* definition of R in this formalism.

Before proceeding further, let us briefly summarize the results of standard SCM. Equation (22) can be integrated to obtain $R(t)$ in the parametric form

$$R = \frac{R_i}{2\delta_i}(1 - \cos\theta) \tag{23}$$

$$t = \frac{3t_i}{4\delta_i^{3/2}}(\theta - \sin\theta) \tag{24}$$

where R_i, δ_i and t_i are the initial radius, initial δ and initial time respectively with $R_i^3 = (9GMt_i^2/2)(1+\delta_i)^{-1} \simeq (9GMt_i^2/2)$ for $\delta_i \ll 1$. Given M, there are only two independent constants *viz* t_i and δ_i. All the physical features of the SCM can be easily derived from the above solution. Each spherical shell expands at a progressively slower rate against the self gravity of the system, reaches a maximum radius and then collapses under its own gravity, with a steadily increasing density contrast. The maximum radius, $R_M = R_i/\delta_i$, achieved by the shell occurs at a density contrast $\delta = (9\pi^2/16) - 1 \approx 4.6$, which is in the "quasi-linear" regime. In case of a perfectly spherical system, there exists no mechanism to halt the infall which proceeds inexorably towards a singularity, with all the mass of the system collapsing to a single point. Thus, the fate of the shell (as described by equations (23) and (24)) is to collapse to zero radius at $\theta = 2\pi$ with an infinite density contrast; this is, of course, physically unacceptable.

In real systems, however, the implicit assumptions that : (i) matter is distributed in spherical shells and (ii) the non-radial components of the velocities of the particles are small will break down long before the infinite densities are reached. Instead, we expect the collisionless dark matter to reach virial equilibrium. After virialization, $|U| = 2K$, where U and K are, respectively, the potential and kinetic energies; the virial radius can be easily computed to be half the maximum radius reached by the system.

The virialization argument is clearly physically well-motivated for real systems. However, as mentioned earlier, there exists no mechanism in the standard SCM to bring about this virialization; hence, one has to put in by hand the assumption that, as the shell collapses and reaches a particular radius, say $R_M/2$, the collapse is halted and the shell remains at that radius thereafter. This arbitrary introduction of virialization is clearly one of the major drawbacks of the standard SCM and takes away its predictive power in the later stages of evolution. We shall now see how the retention of the angular momentum term in equation (21) can serve to stabilize the collapse of the system, thereby allowing us to model the evolution towards

$R_{vir} = R_M/2$ smoothly.

3. The $h_{\rm SC}(\delta)$ function.

As detailed in the previous section, the primary defect of the standard SCM is the *ad hoc* nature of the stabilization of the shell against its collapse under gravity which arises on account of the assumption of perfect spherical symmetry, implicit in the neglect of the shear and rotation terms. We hence return to equation (17), retain the shear and rotation terms and recast the equation into a form more suitable for analysis. Let us consider a dust-dominated, $\Omega = 1$ Universe, for which $b(t) \equiv a(t)$. Using logarithmic variables, $D_{\rm SC} \equiv \ln(1+\delta)$ and $\alpha \equiv \ln a$, equation (17) can be written in the form (The subscript 'SC' stands for 'Spherical Collapse'.)

$$\frac{d^2 D_{\rm SC}}{d\alpha^2} - \frac{1}{3}\left(\frac{dD_{\rm SC}}{d\alpha}\right)^2 + \frac{1}{2}\frac{dD_{\rm SC}}{d\alpha} - \frac{3}{2}\left(\exp(D_{\rm SC}) - 1\right) - a^2(\sigma^2 - 2\Omega^2) = 0 \tag{25}$$

It is convenient to introduce the quantity, S, defined by

$$S \equiv a^2(\sigma^2 - 2\Omega^2) \tag{26}$$

which we shall hereafter call the "rotation term". The consequences of the retention of the rotation term are easy to describe qualitatively. We expect the evolution of an initially spherical shell to proceed along the lines of the standard SCM in the initial stages, when any deviations from spherical symmetry, present in the initial conditions, are small. However, once the maximum radius is reached and the shell recollapses, these small deviations are amplified by a positive feedback mechanism. To understand this, we note that all particles in a given spherical shell are equivalent due to the spherical symmetry of the system. This implies that the motion of any particle, in a specific shell, can be considered representative of the motion of the shell as a whole. Hence, the behavior of the shell radius can be understood by an analysis of the motion of a single particle. The equation of motion of a particle in an expanding universe can be written as

$$\ddot{\mathbf{X}}_i + 2\frac{\dot{a}}{a}\dot{\mathbf{X}}_i = -\frac{\nabla\phi}{a^2} \tag{27}$$

where $a(t)$ is the expansion factor of the locally overdense "universe". The $\dot{\mathbf{X}}_i$ term acts as a damping force when it is positive; *i.e* while the background is expanding. But when the overdense region reaches the point of maximum expansion and turns around, this term becomes negative, acting like a *negative* damping term, thereby amplifying any deviations from spherical symmetry which might have been initially present.

Non-radial components of velocities build up, leading to a randomization of velocities which finally results in a virialised structure, with the mean relative velocity between any two particles balanced by the Hubble flow. It must be kept in mind, however, that the introduction of the rotation term changes the behavior of the solution in a global sense and it is not strictly correct to say that this term starts to play a role *only after* recollapse, with the evolution proceeding along the lines of the standard SCM until then. It is, nevertheless, reasonable to expect that in early times when the term is small the system will evolve as standard SCM, to reach a maximum radius but will fall back smoothly to a constant size, unlike standard SCM.

The rotation term S is, in general, a function of a and $\mathbf{x}$, especially since the derivatives in equation (18) are total time derivatives, which, for an expanding Universe, contain partial derivatives with respect to both $\mathbf{x}$ and t separately. Handling this equation exactly will take us back to the full nonlinear equations for the fluid and — of course — no progress can be made. Instead, we will make the *ansatz* that the rotation term depends on t and $\mathbf{x}$ only through $\delta(t, \mathbf{x})$.

$$S(a, \mathbf{x}) \equiv S(\delta(a, \mathbf{x})) \equiv S(D_{\rm SC}) \tag{28}$$

In other words, S is a function of the density contrast alone. This ansatz seems well motivated because the density contrast δ can be used to characterize the SCM at any point in its evolution and one might expect the angular momentum term to be a function only of the system's state, at least to the lowest order. Further, the results obtained with this assumption appear to be sensible and may be treated as a test of the *ansatz* in its own framework.

To proceed further systematically, we *define* a function $h_{\rm SC}$ by the relation

$$\frac{dD_{\rm SC}}{d\alpha} = 3h_{\rm SC} \tag{29}$$

For consistency, we shall assume the ansatz $h_{\rm SC}(a, \mathbf{x}) \equiv h_{\rm SC}(\delta(a, \mathbf{x}))$. The definition of $h_{\rm SC}$ allows us to write equation (25) as

$$\frac{dh_{\rm SC}}{d\alpha} = h_{\rm SC}^2 - \frac{h_{\rm SC}}{2} + \frac{1}{2}\left(\exp(D_{\rm SC}) - 1\right) + \frac{S(D_{\rm SC})}{3} \tag{30}$$

Dividing (30) by (29), we obtain the following equation for the function $h_{\rm SC}(D_{\rm SC})$

$$\frac{dh_{\rm SC}}{dD_{\rm SC}} = \frac{h_{\rm SC}}{3} - \frac{1}{6} + \frac{1}{6h_{\rm SC}}\left(\exp(D_{\rm SC}) - 1\right) + \frac{S(D_{\rm SC})}{9h_{\rm SC}} \tag{31}$$

If we know the form either of $h_{SC}(D_{SC})$ or $S(D_{SC})$, this equation allows us to determine the other. Then, using (29) one can determine D_{SC}. Thus, our modification of the standard SCM essentially involves providing the form of $S_{SC}(D_{SC})$ or $h_{SC}(D_{SC})$. We shall now discuss several features of such a modeling in order to arrive at a suitable form.

The behavior of $h_{SC}(D_{SC})$ can be qualitatively understood from our knowledge of the behavior of δ with time. In the linear regime ($\delta \ll 1$), we know that δ grows linearly with a; hence h_{SC} increases with D_{SC}. At the extreme non-linear end ($\delta \gg 1$), , the system "virializes", i.e. the proper radius and the density of the system become constant. On the other hand, the density ρ_b, of the background falls like t^{-2} (or a^{-3}) in a flat, dust-dominated Universe. The density contrast δ is defined by $\delta = (\rho/\rho_b - 1) \sim \rho/\rho_b$ (for $\delta \gg 1$) and hence

$$\delta \propto t^2 \propto a^3 \tag{32}$$

in the non-linear limit. Equation(29) then implies that $h_{SC}(\delta)$ tends to unity for $\delta \gg 1$. Thus, we expect that $h_{SC}(D_{SC})$ will start with a value far less than unity, grow, reach a maximum a little greater than one and then smoothly fall back to unity. [A more general situation discussed in the literature corresponds to $h \to$ constant as $\delta \to \infty$, though the asymptotic value of h is not necessarily unity. Our discussion can be generalised to this case and we plan to explore this in a future work.]

This behavior of the h_{SC} function can be given another useful interpretation whenever the density contrast, δ has a monotonically decreasing relationship with the scale, x, with small x implying large δ and vice-versa. Then, if we use a local power law approximation $\delta \propto x^{-n}$ for $\delta \gg 1$ with some $n > 0$ then $D_{SC} \propto \ln(x^{-1})$ and

$$h_{SC} \propto \frac{dD_{SC}}{d\alpha} \propto -\frac{d\ln(\frac{1}{x})}{d\ln a} \propto \frac{\dot{x}a}{\dot{a}x} \propto -\frac{v}{\dot{a}x}$$

where $v \equiv a\dot{x}$ denotes the mean relative velocity. Thus h_{SC} is proportional to the ratio of the peculiar velocity to the Hubble velocity. We know that this ratio is small in the linear regime (where the Hubble flow is dominant) and later increases, reaches a maximum and finally falls back to unity with the formation of a stable structure; this is another argument leading to the same qualitative behavior of the h_{SC} function.

Note that, in the standard SCM (for which $S = 0$), equation (31) reduces to

$$3h_{SC}\frac{dh_{SC}}{dD_{SC}} = h_{SC}^2 - \frac{h_{SC}}{2} + \frac{\delta}{2} \tag{33}$$

The presence of the linear term in δ on the RHS of the above equation causes $h_{\rm SC}$ to increase with δ, with $h_{\rm SC} \propto \delta^{1/2}$ for $\delta \gg 1$. If virialization is imposed as an *ad hoc* condition, then $h_{\rm SC}$ should fall back to unity discontinuously — which is clearly unphysical; the form of $S(\delta)$ must hence be chosen so as to ensure a smooth transition in $h_{\rm SC}(\delta)$ from one regime to another.

4. The rotation term

We will now derive an approximate functional form for the rotation function from physically well motivated arguments. If the rotation term is retained in equation (21), we have

$$\frac{d^2R}{dt^2} = -\frac{GM}{R^2} - \frac{H^2R}{3}S \tag{34}$$

where $H = \dot{a}/a$. Let us first consider the late time behavior of the system. When the virialization occurs, it seems reasonable to assume that $R \to$ constant and $\dot{R} \to 0$. This implies that, for large density contrasts,

$$S \approx -\frac{3GM}{R^3H^2} \qquad (\delta \gg 1) \tag{35}$$

Using $H = \dot{a}/a = (2/3t)$, and equation (19)

$$S \approx -\frac{27GMt^2}{4R^3} = -\frac{3}{2}(1+\delta) \approx -\frac{3}{2}\delta \qquad (\delta \gg 1) \tag{36}$$

Thus the rotation term tends to a value of $(-3\delta/2)$ in the non-linear regime, when stable structures have formed. This asymptotic form for the rotation term, however, is insufficient to model the behavior of $S(\delta)$ over the larger range of density contrast (especially the quasi linear regime) which is of interest to us. Since the rotation term tends to the above asymptotic form at late times, the residual part, *i.e* the part that remains after the asymptotic value has been subtracted away, can be expanded in a Taylor series in $(1/\delta)$ without any loss of generality. Retaining the first two terms of expansion, we write the complete rotation term as

$$S(\delta) = -\frac{3}{2}(1+\delta) - \frac{A}{\delta} + \frac{B}{\delta^2} + \mathcal{O}(\delta^{-3}) \tag{37}$$

[It can be easily demonstrated that the first order term in the Taylor series alone is insufficient to model the turnaround behavior of the h function. Hence we shall include the next higher order term and use the form in equation (37) for the rotation term. The signs are chosen for future convenience since it will turn out that $A > 0$ and $B > 0$.] In fact, for sufficiently large δ, the evolution depends only on the combination $q \equiv (B/A^2)$. This is most easily seen by writing the equation (17) [specialized to a $\Omega = 1$, matter dominated universe] replacing the rotation term $(\sigma^2 - 2\Omega^2)$ with the above form. When we take the limit of large δ *i.e* $\delta \gg 1$ and rescale δ to δ/A we get

$$\frac{d^2\delta}{db^2} + \frac{3}{2b}\frac{d\delta}{db} - \frac{4}{3}\frac{1}{\delta}\left(\frac{d\delta}{db}\right)^2 = -\frac{1}{a^2} + \frac{B}{A^2}\frac{1}{a^2\,\delta} = -\frac{1}{a^2} + \frac{q}{a^2\delta} \tag{38}$$

From the form of the equation it is clear that the constants A and B occur in the combination $q = B/A^2$ and hence the nonlinear regime is modeled by a one parameter family for the rotation term.

Equation (34) can be written as

$$\ddot{R} = -\frac{GM}{R^2} - \frac{1}{3}\frac{4R}{9t^2}\left[-\frac{27GMt^2}{4R^3} - \frac{A}{\delta} + \frac{B}{\delta^2}\right] \tag{39}$$

Using $\delta = 9GMt^2/2R^3$ and $B = qA^2$ we may express equation (39) completely in terms of R and t. We now rescale R and t in the form $R = r_{vir}y(x)$ and $t = \beta x$ where r_{vir} is the final virialised radius (that is $R \to r_{vir}$ for $t \to \infty$ and $\beta^2 = (8/3^5)(A/GM)r_{vir}^3$ so as to obtain an equation for $y(x)$ to be

$$y'' = \frac{y^4}{x^4} - \frac{27}{4}q\frac{y^7}{x^6} \tag{40}$$

We can integrate this equation to find a form for $y_q(x)$ parametrised by q using the physically motivated boundary conditions $y = 1$ and $y' = 0$ as $x \to \infty$, which is simply an expression of the fact that the system reaches the virial radius r_{vir} and remains there thereafter.

The results of numerical integration of this equation for a range of q values is shown in figure (1). As expected on physical grounds, the function has a maximum and gracefully decreases to unity for large values of x [The behavior of $y(x)$ near $x = 0$ is irrelevant since the original equation is valid only for $\delta \geq 1$ at least]. For a given value of q it is possible to find the value x_c at which the function reaches it's maximum as well as the ratio $y_{max} = R_{max}/r_{vir}$. The time t_{max} at which the system will reach the maximum radius is related to x_c by the relation $t_{max} = \beta x_c = t_0(1 + z_{max})^{-3/2}$ where $t_0 = 2/(3H_0)$ is the present age of the universe and z_{max} is the redshift at which the system turns around. Figure 3 shows the variation of x_c and $y_{max} \equiv (r_{vir}/R_{max})^{-1}$ for different values of q. The entire evolution of the system in modified spherical collapse model (MSCM) can be expressed in terms of

$$R(t) = r_{vir}\, y_q(t/\beta) \tag{41}$$

where $\beta = (t_0/x_c)(1 + z_{max})^{-3/2}$.

In SCM, the conventional value used for (r_{vir}/R_{max}) is (1/2) which is obtained by enforcing the virial condition that $|U| = 2K$, where U is the gravitational potential energy and K is the kinetic energy. It must be kept in mind, however, that the ratio (r_{vir}/R_{max}) is not really constrained to be *precisely* (1/2) since the actual value will depend on the final density profile and the precise definitions used for these radii. While we expect it to be around 0.5, some amount of variation, say between 0.25 and 0.75, cannot be ruled out theoretically.

Figure (2) shows the parameter (r_{vir}/R_{max}) (dashed line) plotted as a function of $q = B/A^2$ obtained by numerical integration of the equation (34) with the ansatz (37). It also shows the dependence of x_c (solid line) or equivalently t_{max} on the value of q. It can be seen that by choosing a suitable value for q one can obtain a suitable value for the (r_{vir}/R_{max}) ratio and vice versa. Using the equation (29) and definition of $\delta \propto t^2/R^3$ we can obtain

$$h_{\rm SC}(x) = 1 - 3\frac{x}{y}\frac{dy}{dx} \qquad (42)$$

which gives the form of $h_{\rm SC}(x)$ for a given value of q which in turn determines the function $y_q(x)$. Since δ can be expressed in terms of x, y and x_c as $\delta = (9\pi^2/2x_c^2)x^2/y^3$ this allows us to implicity obtain a form for $h_{SC}(\delta)$ determined only by the value of q. Figure 3 shows the behavior of h_{SC} functions obtained by integrating equation (31) backwards, assuming that $h_{SC} \to 1$ for δ large. The labels for the curves are the values for the ratio $q = B/A^2$ and (r_{vir}/R_{max}). It is seen that all the curves have the same turnaround behavior expected on the basis of the physical arguments presented in the earlier section.

If the functional form for the h_{SC} — determined from N-body simulations, say — is used as a further constraint, we should be able to obtain the values of q. The major hurdle in attempting to do this is the fact that the available simulation results are given in terms of the averaged two point correlation function $\bar{\xi}$ and averaged pair velocity $h(a, x)$ defined by

$$\bar{\xi} = \frac{3}{r^3}\int_0^r \xi(x,a)x^2 dx \; ; \qquad h(a,x) = -\frac{\langle v(a,x)\rangle}{\dot{a}x} \qquad (43)$$

where the two-point correlation function ξ is defined as the Fourier transform of the power spectrum $P(k)$ of the distribution. The results published in the literature assumes that $h(a, x)$ depends on a and x only through $\bar{\xi}(a, x)$, that is, $h(a, x) \equiv h[\bar{\xi}(a, x)]$. This assumption has been invoked in several papers in the past (See e.g. Hamilton et al. 1991, Nityananda & Padmanabhan 1994, Mo et al. 1995, Padmanabhan 1996a, Padmanabhan & Engineer 1998) and seems to be validated by numerical simulations. The fitting formula for $h(\bar{\xi})$. can be obtained from related fitting formulas available in the literature(e.g. Hamilton et al,1991). These are, however, statistical quantities and are not well defined for an isolated overdense region. Hence we have to first make the correspondence between $h_{SC}(\delta)$ and $h(\bar{\xi})$ which we do as follows.

It is possible to show by standard arguments that (Nityananda & Padmanabhan 1994)

$$\frac{d\bar{\xi}}{d\alpha} = 3h(1+\bar{\xi}) \tag{44}$$

that is,

$$\frac{dD}{d\alpha} = 3h \tag{45}$$

where $D = \ln(1+\bar{\xi})$ and $\alpha = \ln a$. Equation (45) is very similar to equation (29), which defines the function $h_{SC}(\delta)$ except for the different definitions of D and D_{SC} in terms of $\bar{\xi}$ and δ respectively. This suggests that one can obtain a relation between $h_{SC}(\delta)$ and $h(\bar{\xi})$ by relating the density contrast δ of an isolated spherical region to the two-point correlation function $\bar{\xi}$ averaged over the distribution at the same scale. We need to find a mapping between $\bar{\xi}$ and δ which is valid in a statistical sense.

Gravitational clustering is known to have three regimes in its growing phase usually called "linear", "quasi linear" and "non–linear" respectively. The three regimes may be characterized by values of density contrast as $\delta \ll 1$ in linear regime, $1 < \delta < 100$ in the quasi linear regime and $100 < \delta$ in nonlinear regime, say. The three regimes have different rates of growth for various quantities of interest such as δ, $\bar{\xi}$ and so on. In the linear regime, it is well known that the density contrast δ grows proportional to the scale factor, a. This implies that the power spectrum, $P(k) \equiv |\delta_k|^2$ (where δ_k is the Fourier mode corresponding to $\delta(x)$) grows as a^2. Consequently, $\bar{\xi}$, which is related to $P(k)$ via a Fourier transform, also grows as a^2, *i.e.* as the square of the density contrast. In the quasi- linear and non-linear regimes, the density contrast does not grow linearly with the scale factor and the relation between δ and $\bar{\xi}$ is not so clearly defined. The quasi-linear regime may be loosely construed as the interval of time during which the high peaks of the initial Gaussian random field have collapsed although mergers of structures have not yet begun to play an important role. (This idea was used in Padmanabhan 1996a to model the nonlinear scaling relations successfully). If we consider a length scale smaller than the size of the collapsed objects, the dominant contribution to $\bar{\xi}$ (at this scale) arises from the density profiles centered on the collapsed peaks. Using the relation

$$\rho \simeq \rho_b\,(1+\bar{\xi}) \tag{46}$$

for density profiles around high peaks one can see that $\bar{\xi} \propto \delta$ in this regime. In the nonlinear regime, both δ and $\bar{\xi}$ have the forms $\delta(a,x) = a^3 F(ax)$, $\bar{\xi} = a^3 G(ax)$ where x is comoving and $r = ax$ is proper coordinate. When the system is described by Lagrangian coordinates (which correspond to proper coordinate $r = ax$, i.e at constant r) $\bar{\xi}$ is proportional to δ. Thus, the relation $\bar{\xi} \propto \delta$ appears to be satisfied in all regimes, except at the very linear end. Since we are only interested in the $\delta > 1$ range, we use $\bar{\xi} \approx \delta$ and compare equations (44) and (29) to identify

$$h_{SC}(\delta) \approx h(\bar{\xi}) \tag{47}$$

It is now straightforward to choose the value of q such that the known fitting function for h function is reproduced as closely as possible. We use the original function given by (Hamilton et al,1991) to obtain the following expression, for $h(\bar{\xi})$:

$$h(\bar{\xi}) = \frac{2}{3}\left(\frac{d\ln \mathcal{V}(\bar{\xi})}{d\ln(1+\bar{\xi})}\right)^{-1} \tag{48}$$

where $\mathcal{V}(\bar{\xi})$ is given by the following fitting function

$$\mathcal{V}(\bar{\xi}) = \bar{\xi}\left(\frac{1 + 0.0158\,\bar{\xi}^2 + 0.000115\,\bar{\xi}^3}{1 + 0.926\,\bar{\xi}^2 - 0.0743\,\bar{\xi}^3 + 0.0156\,\bar{\xi}^4}\right)^{1/3} \tag{49}$$

Figure 4 shows the simulation data represented by the fit (solid line) (Hamilton et al,1991) and the best fit obtained in our model (dashed line) obtained for $q \simeq 0.02$. Given the value of q, the equation for y can be integrated. Figure 5 shows the plot of scaled radius $y_q(x)$ vs x with $q = 0.02$. The figure also shows an accurate fit for this solution of the form

$$y(x) = \frac{x + ax^3 + bx^5}{1 + cx^3 + bx^5} \tag{50}$$

where a, b, c take the values $a = -3.6$,$b = 53$ and $c = -12$, respectively. The fit is shown as a dashed line. This fit along with values for r_{vir} and z_{max} completely specifies our model through equation 41. It can be observed that (r_{vir}/R_{max}) is approximately 0.65. It is interesting to note that the value that is obtained for the (r_{vir}/R_{max}) ratio is not very widely off the usual value of 0.5 used in standard spherical collapse model,*in spite of the fact that no constraint was imposed on this value ab initio in arriving at this result.*

Finally figure 6, compares the nonlinear density contrast δ in modified SCM (dashed line)with that in standard SCM (solid line) by plotting both against the linearly extrapolated density contrast δ_L. It can be (for a given system with the same z_{max} and r_{vir}) seen that at the epoch where the standard SCM model has a singular behavior our model has a smooth behavior with $\delta \approx 110$. This is not widely off from the value usually obtained from the *ad hoc* procedure applied in the standard spherical collapse model. In a way, this explains the unreasonable effectiveness of standard SCM in the study of nonlinear clustering.

5. Results and Conclusions

In this paper, we have shown how the Taylor expansion of a term in the equation for the evolution of the density contrast δ in inverse powers of δ allows us to have a more realistic picture of spherical collapse, which is free from arbitrary abrupt "virialization" arguments. Beginning from a well motivated ansatz for the dependence of the "rotation" term on the density contrast δ we have shown that a spherical collapse model will gracefully turn around and collapse to a constant radius with a density contrast of the order 100 at the same epoch when a standard model reaches a singularity. Figure 6 shows clearly that the singularity, which is avoided — in our model — due to enhancement of deviations from spherical symmetry and consequent generation of strong non radial motions. We derive an approximate functional form for the rotation term starting from a physically reasonable assumption that the system reaches a constant radius. This assumption allows us to derive an asymptotic form for the rotation term with the residual part adequately expressed by keeping only the first and second order terms in a Taylor series in $(1/\delta)$. It is shown that there is a scaling relation between the coefficients of the first and second order terms, essentially reducing it to a one parameter family of models.

The form of the h function published in the literature, along with a tentative mapping from δ to $\bar{\xi}$ in the non–linear and quasi–linear regimes allow us to further constrain our model bringing it in concordance with the available simulation results. Further it is shown that that this form for the rotation term is sufficient to model the turnaround behavior of the spherical shell and leads to a reasonable numerical value for density contrast at collapse.

There are several new avenues suggested by this work which we plan to pursue in the future.

(i) The assumption $h \to 1$, $R \to r_{vir}$ is equivalent to "stable clustering" in terms of the statistical behaviour. Since this result remains unproven conclusively in simulations and is often questioned, it would be interesting to see the effect of changing this constraint to $h \to$ constant for $t \to \infty$.

(ii) The technique of Taylor series expansion in $(1/\delta)$ seems to hold promise. It would be interesting to try such an attempt with the original fluid equations and (possibly) with more general descriptions.

(iii) It must be stressed that we used the $\delta - \bar{\xi}$ mapping — possibly the weakest part of our analysis, conceptually — only to fix a value of q. We could have used some high resolution simulations to actually study the evolution of a realistic overdense region. We conjecture that such an analysis will give results in conformity with those obtained here. We plan to investigate this — thus eliminating our reliance $\delta - \bar{\xi}$

mapping — in a future work.

(iv) Since we used the fluid picture from the initial stages it is clear that we have ignored the effects of multi–stream flow in the system which can act as a supporting term against gravitational collapse, just like rotation. We have also not modeled the detailed behavior of the rotation term at the very linear end as well as examined the question in the context of other cosmologies. These issues are also being studied.

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6. Figures

Fig. 1.— The figure shows the plots of the function $y_q(x)$ for some values of q. The x axis has scaled time, x and y axis is the scaled radius y. (See text for discussion)

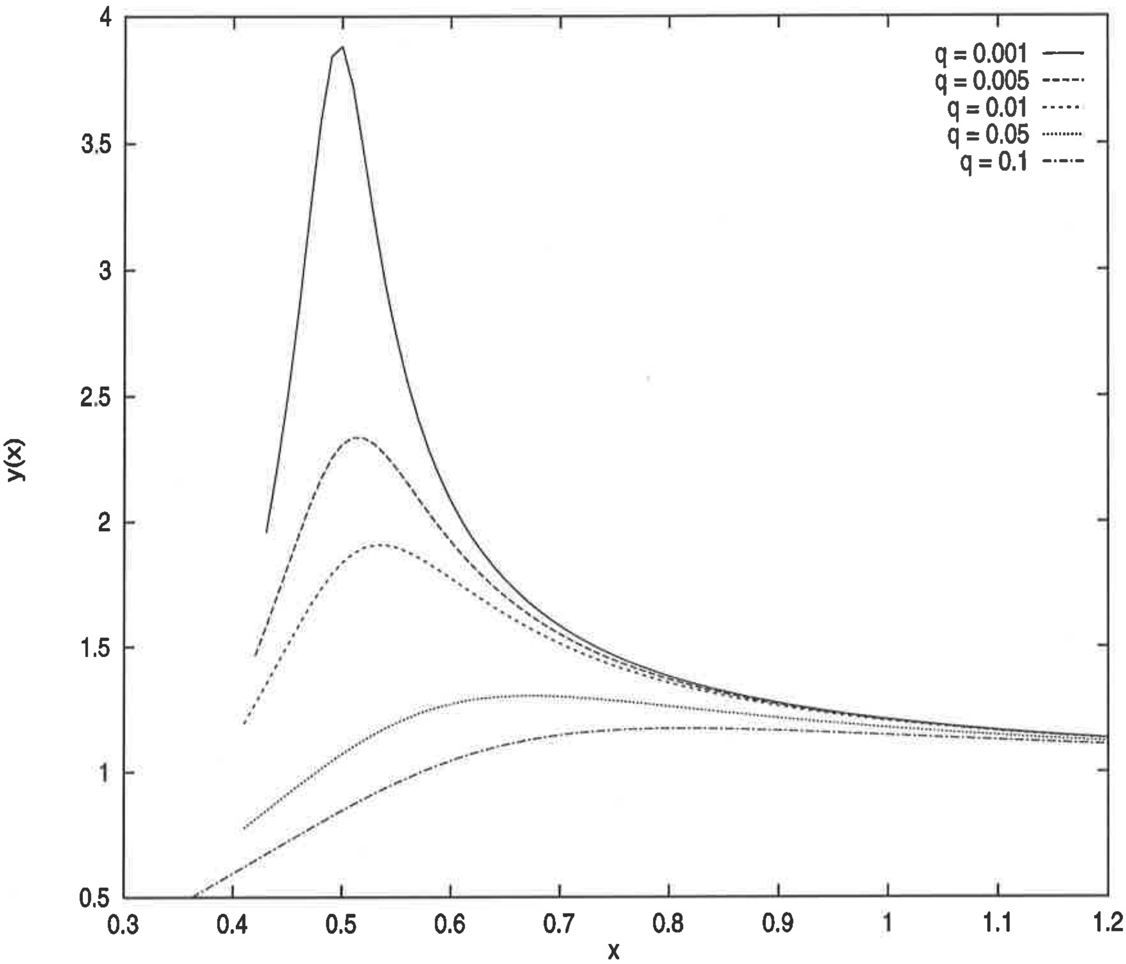
Fig. 2.— The figure shows the parameters (r_{vir}/R_{max}) (dashed line) and x_c (solid line) as a function of $q = B/A^2$. The figure clearly demonstrates that the single parameter description of the rotation function is constrained by the value that is chosen for the ratio r_{vir}/R_{max}.

Fig. 3.— The figure shows the plots of $h = < v > /\dot{a}x$ function obtained for various values of q. The curves in this figure have been labeled with the value of q parameter and the (r_{vir}/R_{max}) ratios. (Further discussion in text)

Fig. 4.— The figure shows the best fit curve for the h function (dashed line) to the simulation data (solid line). The simulation results are obtained from (Hamilton et al,1991) and the fit is obtained by adjusting the value of q parameter for best fit.

Fig. 5.— The figure shows a plot of the scaled radius of the shell y as a function of scaled time x (solid line) and the fitting formula $y = (x + ax^3 + bx^5)/(1 + cx^3 + bx^5)$ with $a = -3.6$, $b = 53$ and $c = -12$ (dashed line) (See text for discussion)

Fig. 6.— The figure shows a plot of the nonlinear density contrast δ of SCM (solid line) and the nonlinear density contrast in modified SCM (dashed line) plotted against the linearly extrapolated density contrast δ_L. (See text for discussion)



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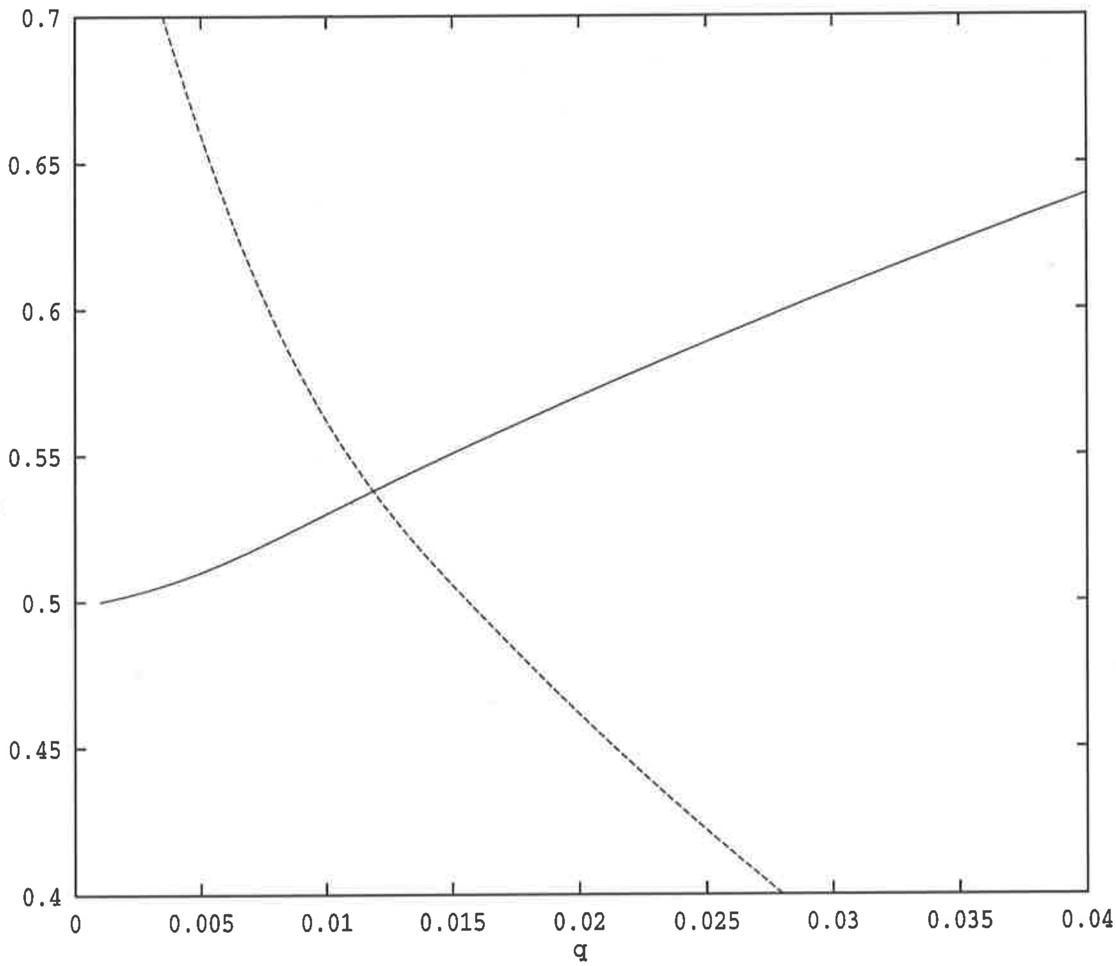

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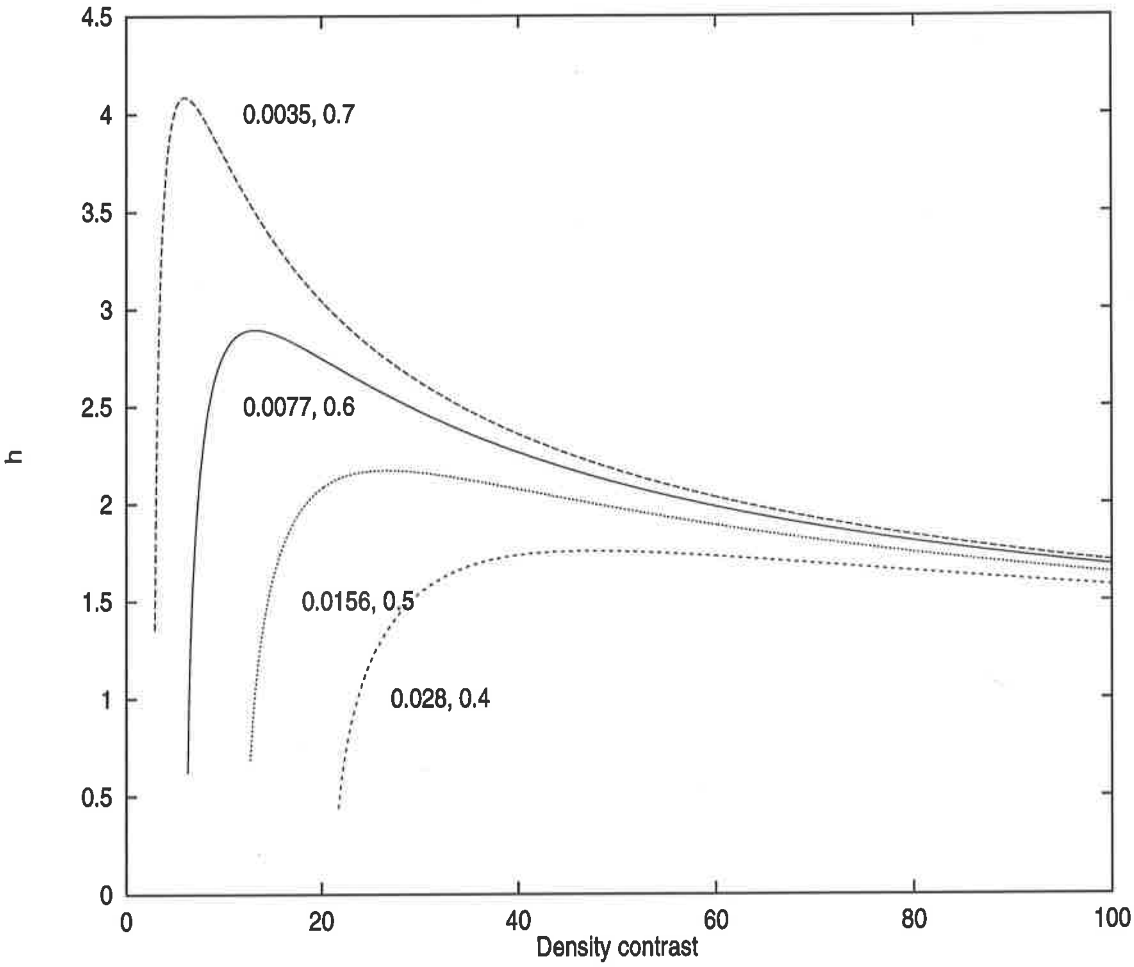

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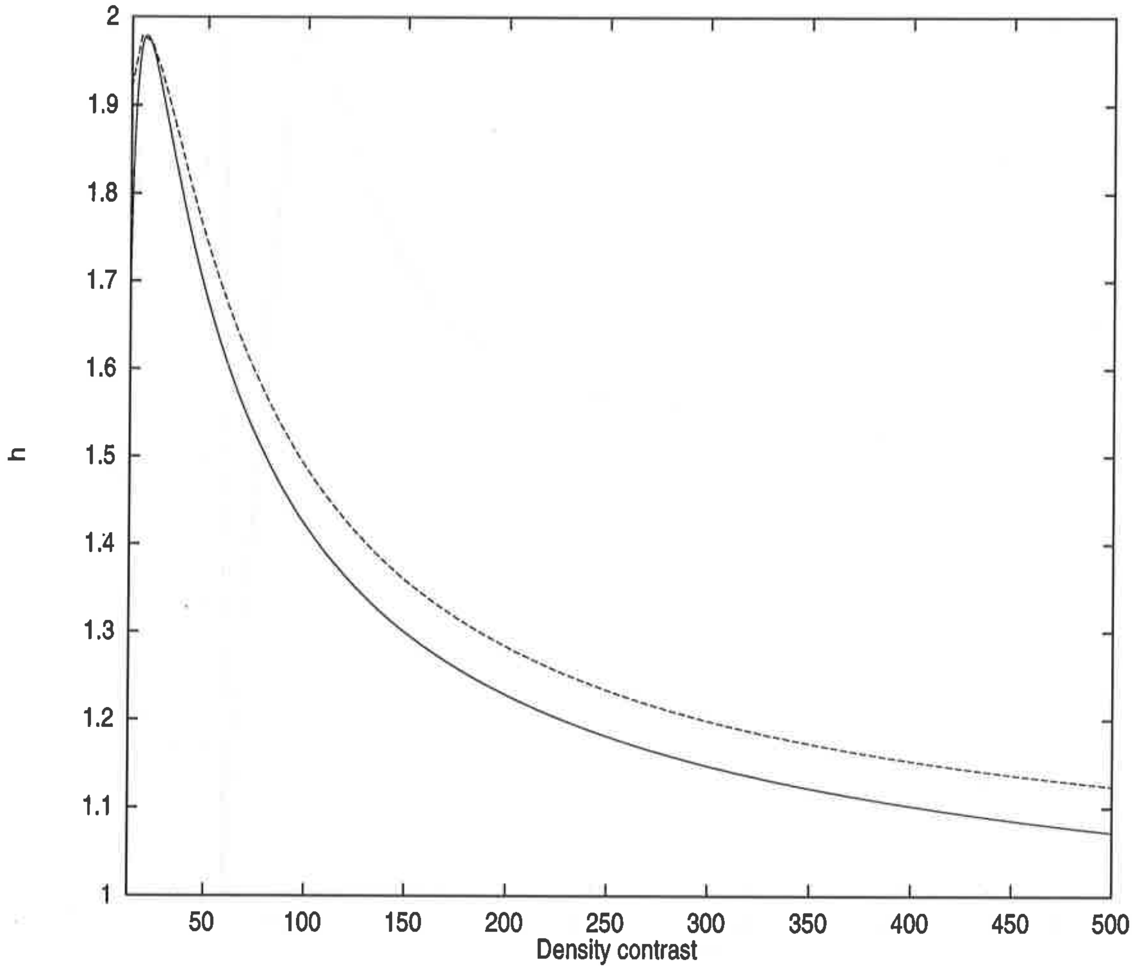

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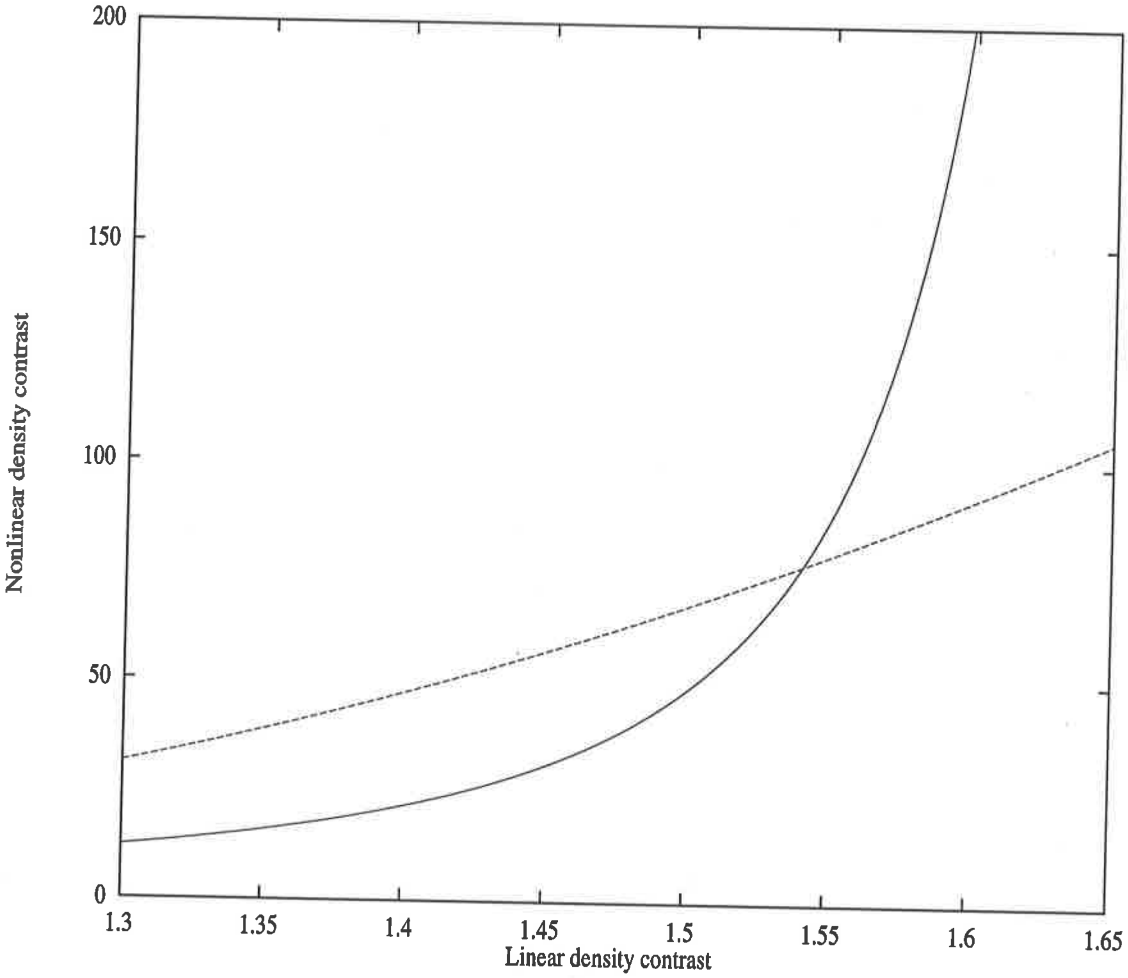

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