

## FRIEDMANN UNIVERSE IN A QUANTUM GRAVITY MODEL

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It is shown that, in a model theory of gravity, which quantises only the conformal part, the Robertson–Walker universe has a nonsingular evolution. The method also shows that there arises a lower bound to the physical length scale in any static metric with positive curvature.

*1. Introduction.* Gravitational field, as is well known [1] has only two degrees of freedom. An approximate (model) theory of quantum gravity can be constructed by treating one of them, viz. the conformal part of the metric, as a quantum variable. (For an excellent discussion on the degrees of freedom of gravity see [2].) This method has led to many interesting results in the past (see refs. [3–7]). Recently, this method was used [8] to discuss the scale factor of Friedmann universe. It was shown that there exists a lower bound to the length scale. However, this analysis suffered from the following short comings: (i) the effect of matter was not included (ii) one could not explicitly solve the “Schrödinger” equation, hence one could not discuss the evolution of the system.

We show, in this note, that a much more complete discussion is possible if we use the overall conformal part (rather than the scale factor) of the metric. Since the RW metrics have only one degree of freedom (as is clear from a superspace analysis, see ref. [9]) either of these conformal parts can be taken to be the independent quantum variable. This, in some sense, represents a choice of the gauge. We present this analysis in the next section and generalize this result to an arbitrary static metric with positive curvature in section 3.

*2. Formalism.* Consider a set of metrics represented in the form,

$$ds^2 = [1 + \phi(t)]^2 [c^2 dt^2 - (1 - kr^2)^{-1} dr^2 - r^2(d\theta^2 + \sin^2 \theta d\psi^2)] . \quad (1)$$

We shall assume the source to be pressure free dust. Classically Einstein’s equation leads to  $\phi(t) = -\cos \omega t$  with  $\omega^2 = kc^2$ . Similar results exist for  $k < 0$  in terms of hyperbolic functions; we shall, however, discuss  $k > 0$  and quote the final results for  $k < 0$ . The curvature factor  $k$  is related to the energy density by,

$$k = 8\pi G\rho/3c^2 \sim \text{cm}^{-2} . \quad (2)$$

We can obtain the quantized version of the theory by treating  $\phi(t)$  as a quantum variable. The physics is contained in the amplitude, given by the path integral (for a detailed discussion of this method see refs. [3–8], especially ref. [6]).

$$G[\phi_2 t_2; \phi_1 t_1] = \int D\phi(t) \exp \frac{i}{\hbar} J[\phi(t)] , \quad (3)$$

where

$$J[\phi(t)] = (c^4/16\pi G) \int R\sqrt{-g} d^4x + \int L_m\sqrt{-g} d^4x + \text{surface terms} . \quad (4)$$

We shall choose the matter to be pressure free dust and add suitable surface terms to remove second derivatives. This procedure leads to the result (see refs. [3,6]; this can also be obtained by transforming eq. (6) of ref. [8]) that (bar denotes quantities pertaining to the metric with  $\phi = 0$ )

$$J[\phi(t)] = (c^4/16\pi G) \int_v (\bar{R}\phi^2 - 6\dot{\phi}^2)\sqrt{-g} dt d^3\bar{x} + \text{terms independent of } \phi(t) \tag{5}$$

$$= -f \int_{t_1}^{t_2} \frac{1}{2} M(\dot{q}^2 - \omega^2 q^2) dt, \tag{6}$$

where the following substitutions have been made

$$a^2 \equiv 4/|k| = (\text{'radius' of the universe})^2, \tag{7}$$

$$M \equiv ac^2/2G \sim (\text{'mass' of the universe}), \tag{8}$$

$$\omega \equiv 2c/a \sim (\text{sec})^{-1} \quad (\text{for } k > 0), \tag{9}$$

$$q = \frac{1}{2} a\phi \sim (\text{cm}). \tag{10}$$

The quantity  $f$  is related to the fraction of the total volume of the universe (in the closed model). If one restricts the domain of observation within  $0 < r < L$ , it is given by the following expression. (The choice of  $L$  is arbitrary in models without a particle horizon. However in standard RW models, it seems physically meaningful to restrict the region of observation within the particle horizon. This point requires further investigation. However, our main results are independent of  $f$ .)

$$f = \frac{3}{4} \int_0^{L/a} (1 \mp x^2)^{-1/2} x^2 dx \tag{11}$$

(- for  $k > 0$ , + for  $k < 0$ ).

The transition from (5) to (6) follows by substituting  $\bar{R} = 6k$  and  $\sqrt{-g} = (\sin \theta) r^2 (1 - kr^2)^{-1/2}$  and rearranging the terms. The propagator in (3) can now be trivially calculated, since the action in eq. (6) corresponds to that of a harmonic oscillator with mass  $M$  and frequency  $\omega$  [for negative  $k$  (open model), this must be taken to be imaginary;  $\omega = 2ic/a$ , but this does not lead to any problem in evaluating the kernel].

This kernel can be used to study the quantum evolution of the system provided the wave function  $\psi(q)$  is known at a time  $t_1$ , so that the integral

$$\psi(qt_2) = \int G(qt_2; q't_1) \psi(q') dq' \tag{12}$$

can be performed. One cannot specify  $\psi(q')$  to be a completely localized delta function, since this will make the future evolution totally uncertain. The best

choice is a gaussian wave packet with the minimum spread allowed by the uncertainty principle. Also we expect the average value for  $\phi(t)$  to follow the classical trajectory  $(-\cos \omega t)$ . As is well known there exists such a nonspreading wave packet solution to the harmonic oscillator given by [10] (up to a phase)

$$\psi(q, t) = (M\omega/\pi\hbar)^{1/4} \times \exp[-(M\omega/2\hbar)(q + q_0 \cos \omega t)^2], \tag{13}$$

so that

$$|\psi(q, t)|^2 = (M\omega/\pi\hbar)^{1/2} \times \exp[-(M\omega/\hbar)(q + q_0 \cos \omega t)^2] \quad (q_0 = \sqrt{f\bar{a}}). \tag{14}$$

[This wavefunction is propagated without change of shape by the kernel via eq. (12)]. The expectation value for the conformal factor is given by, (in this state)

$$\langle [1 + \phi(t)]^2 \rangle = (1 - \cos \omega t)^2 + f^{-1} L_p^2/a^2,$$

where  $L_p$  is the Planck length,  $L_p = (G\hbar/c^3)^{1/2}$ . Thus the evolution of the quantum universe is described, on the average, by the metric,

$$ds^2 = [(1 - \cos \omega t)^2 + f^{-1} L_p^2/a^2] \times [c^2 dt^2 - (1 - kr^2)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)] = s^2(\eta)[d\eta^2 - d\chi^2 - \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2)],$$

where,  $\eta = 2ct/a$ ,  $\chi = \sin^{-1}(r/a)$  and the expansion factor is,

$$s^2(\eta) = \frac{1}{4} a^2 (1 - \cos \eta)^2 + f^{-1} L_p^2. \tag{16}$$

Thus the quantum fluctuations modify the classical evaluation near the singularity and lead to a nonzero value for  $s(0)$ .

$$s(0) \sim L_p \text{ near } \eta = 0. \tag{17}$$

Thus the Planck length sets a lower bound to the scale factor, as noted previously. This analysis, however goes beyond and gives an explicit expression for the scale factor evolution. This scale factor has a well defined classical limit, which gives one confidence in the whole approach. This has to be compared with previous works (refs. [3,5,7]), where a background metric *with a scale factor of classical evolution* is retained from the start. In those approaches we consider quantum fluctuations around a classical geometry while here we derive classical geometry as a limiting case of quantized Friedman universe.

The state considered in eq. (13) is not a stationary state. One can expand this state in terms of the eigenstates of harmonic oscillator to ascertain which states contribute most. It turns out (using the results of eq. (13.23) of ref. [10]) that most of the contribution arises from the state with "energy"

$$E \approx Mc^2 . \tag{18}$$

This gives a simple interpretation to the concept of "energy" in the present case. However more investigation is required in this direction.

The results are directly generalizable to the open model with  $k < 0$ . For example the expansion factor reads, (and gives the same limit as  $t \rightarrow 0$ )

$$s^2(\eta) = \frac{1}{4} a^2 (1 - \cosh \eta)^2 + f^{-1} L_P^2 . \tag{19}$$

Lastly we wish to draw attention to the dependence of the quantum correction on  $f$  (the fraction of the total volume where the quantum fluctuations are considered).

As expected, the quantum fluctuations increase when smaller and smaller regions are considered. This may have relevance to the mini exploding structures suggested by Wheeler.

**3. Generalization.** Consider any family of metrics which can be represented in the form,

$$ds^2 = \Omega^2(t) ds_{\text{background}}^2 ,$$

where,  $ds_{\text{bg}}^2$  is static and has positive curvature. Then by performing an analysis similar to the above one can show that there exist well defined stationary states for  $\Omega(t)$ . Further this scale factor has a lower bound

given by

$$\langle \Omega^2 \rangle > (4f)^{-1} (L_P/a)^2 ,$$

where  $a$  and  $f$  are defined through the following expressions:

$$B = \int_{\nu} \sqrt{-\bar{g}\bar{g}^{00}(\bar{x})} d^3\bar{x} \equiv \frac{4\pi}{3} a^3 f ,$$

$$A = \int_{\nu} \sqrt{-\bar{g}\bar{R}(\bar{x})} d^3\bar{x} \equiv 6B/a^2 .$$

The detailed implications of this result are under investigation.

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