Semiclassical cosmology with a scalar field

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A self-consistent scheme for studying the semiclassical limit of a quantized scalar field in a classical Robertson-Walker metric is developed. The scalar field is chosen to be in a Gaussian state. The classical and semiclassical geometries are contrasted; and it is shown that nontrivial solutions exist for the space-time geometry which are driven by the vacuum fluctuations of the scalar field. We also discuss the stability of de Sitter space to quantum fluctuations, if the cosmological constant is larger than a critical value.

I. INTRODUCTION

In the absence of a quantum theory of gravity, the subject of quantum cosmology[1] has shed light on several important problems in classical cosmology. In a quantum cosmological model one quantizes gravitational fields with a high degree of symmetry (e.g., Robertson-Walker metric), so that they have a few, finite number of degrees of freedom. The problem is thus reduced to one in quantum mechanics, rather than one in field theory. However, the correctness of the results in quantum cosmology can be verified only by comparison with the full theory. Given the situation, it is instructive to study semiclassical cosmological models (wherein the quantum fields propagate in a classical space-time) and to investigate their classical limit. In the present paper, we shall develop one such technique for semiclassical quantization and use it to study a simple cosmological model.

Semiclassical cosmological models, of course, have been investigated before.[2] In such models, which are a part of the broader field of semiclassical gravity, one uses the modified Einstein equations

$$G_{\mu\nu} = -8\pi G \langle \psi | T_{\mu\nu} | \psi \rangle \ , \tag{1}$$

where the expectation value $\langle \psi | T_{\mu\nu} | \psi \rangle$ is evaluated in an "appropriate" quantum state for the matter field. What state $| \psi \rangle$ can be considered appropriate depends, of course, on the kind of quantum effect that one is looking for. In the previous work on this subject it is usually assumed that the field is in the ground state. Solutions to (1.1) will then tell us about the effect of *vacuum fluctuations* on the classical geometry. *However, this is obviously not the correct state to choose, if we are interested in the transition region between classical and quantum domains.* Nonzero values for $\langle \text{vac} | T_{\mu\nu} | \text{vac} \rangle$ can arise only through quantum effects, and hence will vanish when $h=0$ limit is taken. For a general state $| \psi \rangle$ one would (naively) expect the decomposition

$$\langle \psi | T_{\mu\nu} | \psi \rangle = T_{\mu\nu}^{\text{class}} + \tau_{\mu\nu} \ , \tag{2}$$

where $\tau_{\mu\nu}$ are the quantum corrections to the classical value $T_{\mu\nu}^{\text{class}}$. In physically meaningful situations, one would like to understand the effect of quantum fluctuations $\tau_{\mu\nu}$ on the classical evolution dictated by $T_{\mu\nu}^{\text{class}}$. As we shall see below, Gaussian states are very appropriate for this purpose.

An analogy with electromagnetism may be helpful in visualizing the framework behind the work presented here. Consider the electromagnetic field produced by a nonrelativistic charged particle bound to a harmonic-oscillator potential. In a suitable gauge Maxwell's equations reduce to

$$\Box \mathbf{A} = \rho \mathbf{V} \ , \tag{3}$$

where ρ is the charge density and $\mathbf{V}$ the velocity of the oscillator. The trajectory of the oscillator is

$$\mathbf{x} = \mathbf{x}_0 \sin\omega t, \quad \mathbf{V} = \omega \mathbf{x}_0 \cos\omega t \ , \tag{4}$$

for which $\mathbf{A}$ is the well-known radiation solution corresponding to an oscillating dipole.

These are classical equations. One may next consider the semiclassical limit in which the source is quantized but the field is not. In this limit, (3) will be replaced by

$$\Box \mathbf{A} = \langle \psi | \rho \mathbf{V} | \psi \rangle \ , \tag{5}$$

where $| \psi \rangle$ is an appropriate quantum state for the harmonic oscillator. We now ask: what state $| \psi \rangle$ should be chosen for the source oscillator so that the dynamical evolution mimics the classical solution? [That is, the expectation value (5) on the right-hand side is a sinusoidal function, as in (4).]

It is apparent that we *cannot* choose for $| \psi \rangle$ the ground state or any other energy eigenstate of the harmonic oscillator, because for them we will have $\langle n | \mathbf{V} | n \rangle = \langle n | \mathbf{p} | n \rangle = 0$. ($| n \rangle$ are the energy eigenstates *of the harmonic oscillator.*) From the quantum theory of a harmonic oscillator, we know that the correct classical limit can be obtained, if we choose for the right-hand side in (5), the *transition element* $\langle n | \mathbf{V} | n-1 \rangle$, rather than the *expectation value* $\langle n | \mathbf{V} | n \rangle$. But depending on $| n \rangle$, an infinity of such transition elements may be chosen, and certainly a transition amplitude is not as natural a choice as an expectation value. As is well known for the present example, the correct classical limit is obtained, if the expectation value in (5) is evaluated in the coherent state for the harmonic oscillator.

The above analysis suggests that for the case of gravity as well, we should investigate the choice of coherent state or—when such a state does not exist—a Gaussian state, for the source of gravity. In such a formalism the semiclassical limit would be approached as follows.

Consider, for example, a classical system with two degrees of freedom, described by the Lagrangian $L(Q,q,\dot{Q},\dot{q},t)$. If now, in the semiclassical limit, we want to consider q as a quantum variable, and Q as classical, then the modified equations of motion are assumed to be

$$E_L(Q,\dot{Q},\langle q\rangle,\langle\dot{q}\rangle,t)=0\ , \tag{6}$$

$$i\frac{\partial\psi(q,t)}{\partial t}=[H(q,t)+H_I(q,Q,t)]\psi(q,t)\ . \tag{7}$$

Here, E_L stands for the Euler-Lagrange equation for Q, and in this equation of motion, all functions of q are to be replaced by the corresponding expectation values in the appropriate state. The physical states for q are to be determined from Eq. (7), which is the Schrödinger equation for the wave function of the quantum variable (H_I stands for the interaction Hamiltonian). Thus the semiclassical system is to be described by a consistent solution of the above pair of coupled equations.

As an example, consider the classical system described by the Lagrangian

$$L(Q,q)=\tfrac{1}{2}\dot{Q}^2+\tfrac{1}{2}\dot{q}^2-\tfrac{1}{2}gQ^2q^2\ . \tag{8}$$

The classical equations of motion for this Lagrangian

$$\ddot{Q}=-gq^2Q,\quad \ddot{q}=-gQ^2q \tag{9}$$

are to be replaced, in the semiclassical limit, by the equations

$$\ddot{Q}=-gQ\langle q^2\rangle\ , \tag{10}$$

$$i\frac{\partial\psi(q,t)}{\partial t}=(\tfrac{1}{2}\dot{q}^2+\tfrac{1}{2}gQ^2q^2)\psi(q,t)\ . \tag{11}$$

In this paper, we use the formalism described above to study the semiclassical quantization of a scalar field in a Robertson-Walker metric. We shall find that even at the level of semiclassical physics the system differs nontrivially from the classical one.

In Sec. II we develop the model and obtain a classical solution for the system. In the next section we obtain a Gaussian solution for the scalar field, in a background Robertson-Walker geometry. We use these results in Sec. IV to obtain a solution for the semiclassical geometry, and compare it with the classical solution. In particular, we find that there exist oscillatory solutions for the geometry which are driven solely by the vacuum fluctuations of the scalar field. We also show that when a constant potential is included, the equations of motion admit static solutions for the geometry. In Sec. V we study the back reaction of the scalar field on de Sitter space, and discuss its stability.

II. FRIEDMANN UNIVERSE WITH A SCALAR SOURCE: THE CLASSICAL SOLUTION

The system we shall investigate consists of the $K=+1$ Friedmann-Robertson-Walker (FRW) space-time with a massless scalar field as its source. The space-time is described by the metric

$$ds^2=\Omega^2(t)\left[dt^2-\frac{dr^2}{1-r^2}-r^2(d\theta^2+\sin^2\theta\,d\Phi^2)\right]\ . \tag{12}$$

The action for this system is given by

$$A=(16\pi G)^{-1}\int R\sqrt{-g}\,d^4x+\tfrac{1}{2}\int[\phi^i\phi_i-V(\phi)]\sqrt{-g}\,d^4x\ . \tag{13}$$

The isotropy and homogeneity of the space-time implies that the scalar field is a function of time alone, and therefore A reduces to the form

$$A=-\tfrac{1}{2}m\int dt[\dot{\Omega}^2-\Omega^2-\Omega^2\dot{\eta}^2+\Omega^4V(\eta)]\ , \tag{14}$$

where

$$m=3\pi/2G,\quad \eta=(4\pi G/3)^{1/2}\phi,\quad V(\eta)=\frac{4\pi G}{3}V(\phi)\ . \tag{15}$$

The action in (14) describes a system with two degrees of freedom, which makes it a relatively simple action to study in the semiclassical limit. We note that the conformal factor Ω has dimensions of length, and that the choice of units $\hbar=c=1$ makes the scalar field η a dimensionless variable.

Upon the varying of Ω and η the action in (14) leads to the classical equations of motion:

$$\ddot{\Omega}=-(\dot{\eta}^2+1)\Omega+2\Omega^3V(\eta)\ , \tag{16}$$

$$\frac{d}{dt}\left[\Omega^2\frac{d\eta}{dt}\right]=-\tfrac{1}{2}\Omega^4\frac{dV(\eta)}{d\eta}\ . \tag{17}$$

If instead, we were to write down the Einstein equations corresponding to the action in (13), and then specialize to the metric in (12) we would have obtained Eq. (16) *and the constraint equation:*

$$h=\dot{\Omega}^2+\Omega^2-\Omega^2\dot{\eta}^2-\Omega^4V(\eta)=0\ . \tag{18}$$

Equations (16) and (18) together imply (17), as may be verified by differentiating (18) and using (16). However, (16) and (17) do not lead to (18), but instead give

$$\dot{\Omega}^2+\Omega^2-\Omega^2\dot{\eta}^2-\Omega^4V(\eta)=\text{const}\ . \tag{19}$$

In this crucial aspect, the system described by the action (14) is different from the full gravity-scalar system of Eq. (13). One can obtain (18) by including one more degree of freedom in the metric. Consider, for example, the line element

$$ds^2=\Omega^2(t)\left[N^2(t)dt^2-\left[\frac{dr^2}{1-r^2}+r^2(d\theta^2+\sin^2\theta\,d\Phi^2)\right]\right], \tag{20}$$

for which A becomes

$$A=-\tfrac{1}{2}m\int dt[N\dot{\Omega}^2-N^{-1}\Omega^2-N\Omega^2\dot{\eta}^2+N^{-1}\Omega^4V(\eta)]\ . \tag{21}$$

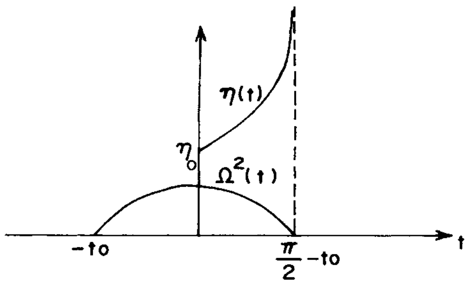

FIG. 1. The classical solution for the conformal factor $\Omega^2(t)$, and the scalar field $\eta(t)$. The scalar field blows up at the "big crunch" at $t=\pi/2-t_0$.

Varying Ω, η, and N now gives Eqs. (16)–(18), respectively. The constraint equation is thus a statement of the reparametrization invariance of the theory under the coordinate change

$$t \to t' = \int N(t)dt \ . \tag{22}$$

Thus the classical physics of the gravity-scalar field system is correctly described only by the set of equations (16), (17), and the constraint equation (18). For zero potential, the solution to Eqs. (16) and (17), which satisfies the constraint equation (18) is given by

$$\Omega^2 = \Omega_0{}^2 \sin[2(t+t_0)] \ , \tag{23}$$

$$\eta = \tfrac{1}{2}\ln\left(\left|\frac{\tan(t+t_0)}{\tan t_0}\right|\right) + \eta_0, \quad t>0 \ , \tag{24}$$

where $\Omega_0{}^2$, t_0, and η_0 are constants. The behavior of Ω^2 and η in the period ($t=0$, $t+t_0=\pi/2$) is shown in Fig. 1. Since Ω^2 is a positive quantity, the argument of the sine function in (23) lies in the range $(0,\pi)$, i.e., from one singularity to the next one. We shall return to the case of a constant potential in Sec. IV.

III. SEMICLASSICAL ANALYSIS: QUANTIZATION OF THE SCALAR FIELD

Having described the classical system, we are now ready to investigate the action (14) in the semiclassical limit. By this we mean (a) quantization of the scalar field in an arbitrary classical FRW geometry and (b) a study of how this quantization modifies the space-time geometry in the semiclassical limit. Since the scalar field is a function of time alone, we are essentially dealing with a problem in quantum mechanics, rather than that in field theory.

Our semiclassical system is described by a wave function $\psi(\eta,t)$ for the scalar field and a classical function $\Omega(t)$. In accordance with the general scheme outlined in the introduction, we assume that the semiclassical limit is described by the equations

$$\ddot{\Omega} = -(\langle \dot{\eta}^2\rangle + 1)\Omega + 2\Omega^3\langle V(\eta)\rangle \ , \tag{25}$$

$$i\hbar\frac{\partial\psi}{\partial t} = \hat{H}(\eta)\psi(\eta,t) \ . \tag{26}$$

Here, (25) is obtained from (16) by replacing all quantities depending on η by the corresponding expectation values in the quantum state ψ. The state $\psi(\eta,t)$, on the other hand, has to be found by solving Eq. (26). Thus Eqs. (25) and (26) constitute a set of coupled equations for the functions $\Omega(t)$ and $\psi(\eta,t)$; $\hat{H}(\eta)$ stands for the Hamiltonian of the scalar field, obtained from the action (14).

In the present section, we solve Eq. (26) for the wave function $\psi(\eta,t)$ in an arbitrary (classical) $\Omega(t)$, and then compute the relevant expectation values. These expectation values are functions of Ω. In the next section we use these expressions in Eq. (25) to solve for Ω as a function of time. We will then compare this semiclassical solution for $\Omega(t)$ with the classical solution obtained in Sec. II.

A. Gaussian solution for the scalar field

Since we are interested in a solution $\psi(\eta,t)$ which mimics the classical evolution, we shall limit ourselves to a Gaussian form for the wave function. That is, we look for solutions to (26) of the form

$$\psi(\eta,t) = A(t)\exp\{-B(t)[\eta-f(t)]^2\} \ . \tag{27}$$

From Eq. (14) we have that

$$\hat{H}(\eta) = \tfrac{1}{2}m[\Omega^2\dot{\eta}^2 + \Omega^4 V(\eta)] \ , \tag{28}$$

and, for the momentum conjugate to η,

$$\pi_\eta = m\Omega^2\dot{\eta} \ . \tag{29}$$

The commutation rule $[\eta,\pi_\eta]=i\hbar$ can be realized in the η representation by taking

$$\pi_\eta \to -i\hbar\partial/\partial_\eta \ . \tag{30}$$

The Schrödinger equation (26) becomes

$$i\hbar\frac{\partial}{\partial t}\psi(\eta,t) = \left[-\frac{\hbar^2}{2m\Omega^2}\frac{\partial^2}{\partial\eta^2} + \tfrac{1}{2}m\Omega^4 V(\eta)\right]\psi(\eta,t) \ . \tag{31}$$

We also assume that the potential has a quadratic form given by

$$V(\eta) = \alpha\eta^2 + \beta\eta + V_0 \ . \tag{32}$$

Using the expressions from (27) and (32) in (31), and comparing coefficients of like powers of η leads to the following equations:

$$i\hbar\left[\frac{\dot{A}}{A} - \dot{B}f^2 - 2Bf\dot{f}\right] = -\frac{2\hbar^2}{m\Omega^2}B^2f^2 + \frac{\hbar^2 B}{m\Omega^2} + \tfrac{1}{2}m\Omega^4 V_0 \ , \tag{33}$$

$$i\hbar(2f\dot{B} + 2B\dot{f}) = \frac{4\hbar^2}{m\Omega^2}B^2 f + \tfrac{1}{2}m\Omega^4\beta \ , \tag{34}$$

$$-i\hbar\dot{B} = \frac{-2\hbar^2}{m\Omega^2}B^2 + \tfrac{1}{2}m\Omega^4\alpha \ . \tag{35}$$

Defining a new variable $\chi(t)$ by

$$B \equiv -\frac{im\Omega^2}{2\hbar}(\dot{\chi}/\chi) \tag{36}$$

simplifies Eq. (34) to

$$\frac{d}{dt}\left[\Omega^2\frac{d\chi}{dt}\right]+\Omega^4\alpha\chi=0\ . \tag{37}$$

We shall return to the solution of these equations in Sec. III B. Here we just list some expressions which we shall require later.

Corresponding to the wave function in Eq. (27), the probability density is given by

$$|\psi(\eta,t)|^2=N(t)\exp\left[-\frac{[\eta-\overline{\eta}(t)]^2}{2\sigma_\eta^2(t)}\right], \tag{38}$$

where

$$N(t)=|A(t)|^2\exp\left[-\left[(Bf^2+B^*f^{*2})-\frac{(Bf+B^*f^*)^2}{B+B^*}\right]\right], \tag{39}$$

$$\overline{\eta}(t)=\frac{Bf+B^*f^*}{B+B^*}\ , \tag{40}$$

and

$$\sigma_\eta^2(t)=\tfrac{1}{2}(B+B^*)^{-1}\ . \tag{41}$$

The normalization of $\psi(\eta,t)$ in (38) implies that

$$N(t)=[2\pi\sigma_\eta^2(t)]^{-1/2}\ . \tag{42}$$

Comparing this with (39) we see that it is sufficient to solve Eqs. (34) and (35) for $B(t)$ and $f(t)$, because then $A(t)$ would automatically get fixed up to a phase factor. Moreover, it can be shown that $A(t)$ obtained this way matches with that obtained by solving Eq. (33). Thus we will not need to work with Eq. (33) explicitly.

The quantities $\langle\dot\eta^2\rangle$ and $\langle\dot\eta\rangle^2$ will be required in our further computation. In the Gaussian state (27) they can be easily calculated to be

$$\langle\dot\eta^2\rangle=\frac{4\hbar^2}{m^2\Omega^4}|B|^2\left[\sigma_\eta^2+\frac{4|B|^2}{(B+B^*)^2}(\mathrm{Im}f)^2\right], \tag{43}$$

$$\langle\dot\eta\rangle^2=\frac{16\hbar^2|B|^4}{m^2\Omega^4}\frac{(\mathrm{Im}f)^2}{(B+B^*)^2}\ . \tag{44}$$

B. Gaussian solution with constant potential

For the sake of generality, we have written down Eqs. (33)–(35) for the Gaussian solution in the case of a quadratic potential. In the present paper, however, we shall work only with a constant potential, or its subcase, a zero potential.

For a constant potential, we have in (32) that $\alpha=\beta=0$. Then Eqs. (34) and (35) imply that

$$\dot f=0,\quad f=\text{const}\ . \tag{45}$$

Equation (37) gives

$$\chi(t)=\chi_1+\chi_2F(t)\ , \tag{46}$$

where

$$F(t)=\int_0^t d\tau/\Omega^2(\tau)\ , \tag{47}$$

and χ_1 and χ_2 are arbitrary constants. We note that $F(t)$ is a real function. From (36) it follows that

$$B(t)=-\frac{im}{2\hbar}\frac{1}{\chi+F(t)}\ , \tag{48}$$

where

$$\chi=\chi_1/\chi_2\ . \tag{49}$$

We impose the condition that $B(t=0)$ is real, which implies that χ is an imaginary number. Defining a quantity σ_0^2 by

$$\sigma_0^2=\frac{i\hbar}{2m}\chi \tag{50}$$

and using Eqs. (40) and (41) gives

$$\overline{\eta}=\eta_1+\eta_2F(t)\ , \tag{51}$$

$$\sigma_\eta^2(t)=\sigma_0^2+\frac{1}{\sigma_0^2}\left[\frac{\hbar}{2m}\right]^2F^2(t)\ , \tag{52}$$

where

$$\eta_1=\mathrm{Re}(f),\quad \eta_2=(\hbar/2m)\,\mathrm{Im}(f)/\sigma_0^2\ . \tag{53}$$

Using Eqs. (48) and (50), $B(t)$ may be rewritten as

$$B(t)=\frac{\sigma_0^2-(i\hbar/2m)F(t)}{\sigma_0^2+(\hbar/2m)^2F^2(t)}\ . \tag{54}$$

The expectation values which we calculated in (43) and (44) can now be written down for the case of constant potential, using σ_η^2 and $B(t)$ from above, as

$$\langle\dot\eta\rangle^2=4\hbar^2(\mathrm{Im}f)^2/m^2\sigma_0^4\Omega^4\ , \tag{55}$$

and

$$\langle\dot\eta^2\rangle=\tilde\Omega_0^4/\Omega^4\ , \tag{56}$$

where

$$\tilde\Omega_0^4=\frac{4\hbar^2}{m^2\sigma_0^2}\left[1+\frac{(\mathrm{Im}f)^2}{\sigma_0^2}\right]. \tag{57}$$

Thus Eqs. (45), (51), (52), and (54) describe the wave function for the case of constant potential, and Eqs. (55) and (56) provide the expectation values needed in the next section. We note that the constant value of the potential V_0 does not appear anywhere in the wave function. However, since it does appear in the semiclassical equation (25) for $\Omega(t)$, it has a nontrivial role to play in the full system of equations. This would necessitate that when we study the semiclassical geometry, the cases of zero potential and a constant nonzero potential are to be considered separately.

We note in passing that a redefinition of the time coordinate by

$$T=F(t) \tag{58}$$

transforms $\eta(T)$ to the coordinate of a free particle.

After having obtained the expectation values $\langle\dot\eta^2\rangle$ and $\langle\dot\eta\rangle^2$ in a Gaussian state, we are now ready to investigate the effect of quantization on the classical geometry.

IV. THE SEMICLASSICAL GEOMETRY

The evolution of the geometry in the semiclassical limit is described by Eq. (25) for the conformal factor $\Omega(t)$. In the present section we first obtain the semiclassical solution for the case of zero potential and discuss the classical limit of the gravity-scalar system. We then give a qualitative discussion of the classical and semiclassical solutions for a constant potential. Next we show that the gravity-scalar system with a constant potential admits of static solutions for the conformal factor. We then discuss the implications of the various results.

A. Semiclassical solution with zero potential

By setting the potential to zero, Eq. (25) reduces to

$$\ddot{\Omega} = -(\langle \dot{\eta}^2 \rangle + 1)\Omega \ . \tag{59}$$

The expression for $\langle \dot{\eta}^2 \rangle$, found in (56) for the case of constant potential, is independent of the value of the constant, and in particular holds for $V_0 = 0$. Using this expression in (59) gives

$$\ddot{\Omega} = -\left[\frac{\tilde{\Omega}_0{}^4}{\Omega^4} + 1\right]\Omega \ , \tag{60}$$

with $\tilde{\Omega}_0{}^4$ as in (57).

We next recall that the classical system obeys the constraint equation (18). At the semiclassical level, the constraint equation is modified to

$$\dot{\Omega}^2 + \Omega^2 - \Omega^2 \langle \dot{\eta}^2 \rangle - \Omega^4 \langle V(\eta) \rangle = 0 \ . \tag{61}$$

(Such a modification is in accordance with the general scheme we have adopted for writing the semiclassical equations of motion.) For the case of zero potential, the constraint equation can be written, using (56), as

$$\dot{\Omega}^2 + \Omega^2 - \frac{\tilde{\Omega}_0{}^4}{\Omega^2} = 0 \ . \tag{62}$$

Equations (60) and (62) together give the solution for $\Omega(t)$ as

$$\Omega^2 = \tilde{\Omega}_0{}^2 \sin[2(t+t_0)] \ . \tag{63}$$

The form of the solution is similar to that of the classical solution (23), but as we shall now show, the amplitude gets modified due to quantum corrections. The constant $\tilde{\Omega}_0{}^4$ in (57) can be rewritten, using first (55), and then (29), as

$$\tilde{\Omega}_0{}^4 = (\langle \pi_\eta \rangle^2 + 4\hbar^2/\sigma_0{}^2)/m^2 \ , \tag{64}$$

or in a more suggestive form as

$$\tilde{\Omega}_0{}^4 = \Omega_0{}^4 + \sigma_\pi{}^2/m^2 \ , \tag{65}$$

where

$$\Omega_0{}^4 = \langle \pi_\eta{}^2 \rangle / m^2, \quad \sigma_\pi{}^2 = 4\hbar^2/\sigma_0{}^2 \ . \tag{66}$$

Since $\langle \pi_\eta \rangle$ can be identified with the classical value of π_η, we can interpret $\sigma_\pi{}^2/m^2$ as the semiclassical correction. We see that the quantum correction modifies the amplitude of the classical oscillatory solution, but leaves its evolution unchanged. Also, putting $\hbar = 0$ reproduces the classical solution, as was to be expected. The solution for $\Omega(t)$ can now be written explicitly as

$$\Omega^2(t) = (\Omega_0{}^4 + \sigma_\pi{}^2/m^2)^{1/2} \sin[2(t+t_0)] \ , \tag{67}$$

where the constant $\Omega_0{}^4$ should be interpreted to be the same as in the classical solution (23).

Of particular interest is the case for which

$$\langle \pi_\eta \rangle = 0 \ . \tag{68}$$

This is seen to follow from (55) by taking $(\mathrm{Im} f) = 0$, which because of (51) and (53) implies that

$$\bar{\eta} = \text{const} = \eta_1 \ . \tag{69}$$

More simply, if the momentum has a zero expectation value, the expectation value $\bar{\eta}$ is constant. Classically, $\pi_\eta = 0$ would imply the absence of any source to drive the expansion: $\Omega_0{}^2 = 0$ and Ω^2 stays at zero forever. However, from (67) we find that quantum fluctuations provide an oscillatory solution, with the amplitude given by

$$\tilde{\Omega}_0{}^2 = \sigma_\pi / m = 2\hbar / \sigma_0 m \ , \tag{70}$$

even when the classical contribution is absent. Using m from (15), this may be rewritten as

$$\tilde{\Omega}_0{}^2 = (4/3\pi) L_p{}^2 / \sigma_0 \ . \tag{71}$$

We recall from (52) that σ_0 is a dimensionless number which is a measure of the initial spread of the Gaussian. It would be natural to take Planck time as the "initial value of time," so that a semiclassical theory may be valid. Thus σ_0 can be expected to be of the order of unity in Planck units, and we find that quantum fluctuations provide a natural oscillation for the geometry, with an amplitude of the order of Planck length.

After having found the semiclassical solution for $\Omega^2(t)$, as in Eq. (63), we can use this result to compute $F(t)$, which was defined in Eq. (47). This, when used in (51) and (52), gives

$$\bar{\eta} = \eta_1 + (\eta_2 / 2\tilde{\Omega}_0{}^2) \ln \left[\left| \frac{\tan(t+t_0)}{\tan t_0} \right| \right], \quad t \geq 0 \tag{72}$$

and

$$\sigma_\eta{}^2 = \sigma_0{}^2 + \frac{1}{4\sigma_0{}^2} (\hbar/2m)^2 \left[\ln \left| \frac{\tan(t+t_0)}{\tan t_0} \right| \right]^2 , \quad t \geq 0 \ . \tag{73}$$

As expected, $\bar{\eta}$ has the same form as the classical solution in (24). The behavior of $\sigma_\eta{}^2$ in the period $(t=0, t=\pi/2 - t_0)$ is shown in Fig. 2.

From Eqs. (63), (72), and (73) we note that at $t = \pi/2 - t_0$, the conformal factor goes to zero ("big crunch"), and both $\bar{\eta}$ and $\sigma_\eta{}^2$ become infinite. The blowing up of the spread $\sigma_\eta{}^2$ shows that the semiclassical theory ceases to be valid near the singularity.

This completes the discussion of the semiclassical equations for the case of zero potential. We shall next consider the case of a constant potential.

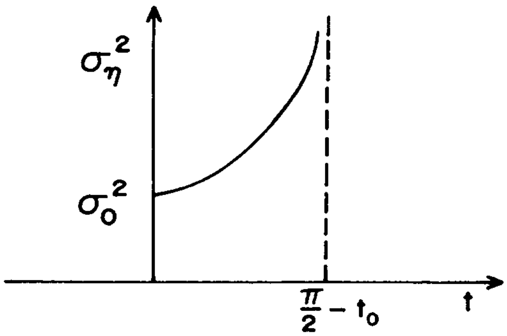

FIG. 2. The time evolution of the Gaussian spread for the scalar field in the semiclassical limit. The spread becomes singular at the "big crunch."

B. Semiclassical solution with constant potential

The case of a constant potential is of interest mainly because the constant V_0 can be looked upon as the cosmological constant term in Einstein's equations. A particular case is that of a constant potential, with η set as zero de Sitter space, which we shall discuss in Sec. V. Here we shall summarize some general features of the classical and semiclassical geometries with a constant potential.

When a constant potential is included, the semiclassical equation (25) becomes

$$\ddot{\Omega} = -(\langle \dot{\eta}^2 \rangle + 1)\Omega + 2\Omega^3 V_0 \quad (74)$$

and the constraint equation is modified to

$$\dot{\Omega}^2 + \Omega^2 - \Omega^2 \langle \dot{\eta}^2 \rangle - \Omega^4 V_0 = 0 . \quad (75)$$

The expectation value $\langle \dot{\eta}^2 \rangle$ for a constant potential is as in (56), using which the above equations for $\Omega(t)$ can be integrated to give

$$2(t+t_0) = \int_0^{\Omega^2} \frac{d(\Omega^2)}{(\tilde{\Omega}_0^4 - \Omega^4 + \Omega^6 V_0)^{1/2}} . \quad (76)$$

The classical solution for $\Omega(t)$ in the presence of a potential is also of the form (76), except that the term $\tilde{\Omega}_0^4$ has a quantum correction, as is evident from its form in (57). The general solution to (76) is complicated, and uninteresting; we shall content ourselves with a study of the approximate behavior. This can be obtained from Eq. (75), which may be written as

$$\tfrac{1}{2}\dot{\Omega}^2 + \tilde{V}(\Omega) = 0 , \quad (77)$$

where

$$\tilde{V}(\Omega) = \frac{1}{2}\left[\Omega^2 - V_0\Omega^4 - \frac{\tilde{\Omega}_0^4}{\Omega^2}\right] . \quad (78)$$

We recall from Eq. (65) that $\tilde{\Omega}_0^4$ (which is completely independent of V_0), can be rewritten as

$$\tilde{\Omega}_0^4 = (\langle \pi_\eta \rangle^2 + \sigma_\pi^2)/m^2 , \quad (79)$$

where the contribution σ_π^2 is due to quantum corrections. Using this form of $\tilde{\Omega}_0^4$ in (78) we see that as long as $\langle \pi_\eta \rangle \neq 0$ the potential $\tilde{V}(\Omega)$ will be of the same form in the classical and semiclassical case. Thus we conclude that as long as the field has a nonzero momentum, the semiclassical evolution of $\Omega(t)$ will be of the same form as in the classical case, and will differ only in the value of $\tilde{\Omega}_0^4$. This is analogous to the case of zero potential.

The interesting physics arises, again as in the case of zero potential, when $\langle \pi_\eta \rangle = 0$. Then $\tilde{V}(\Omega)$ can be written as

$$\tilde{V}(\Omega) = \tfrac{1}{2}[\Omega^2 - V_0\Omega^4 - (\sigma_\pi^2/m^2)/\Omega^2] . \quad (80)$$

Since σ_π^2 vanishes in the classical limit, the third term is of quantum origin. It is now obvious that the potential $\tilde{V}(\Omega)$ is very different in the classical and semiclassical cases. The form of the potential in these two cases has been plotted in Fig. 3. From the shape of the potential we find that in the classical case ($\sigma_\pi^2 = 0$) the point $\Omega^2 = 0$ is a stable minimum, and the behavior is oscillatory about $\Omega^2 = 0$. However, after quantum correction, $\Omega^2 = 0$ is not an equilibrium point.

The point $\Omega^2 = 0$ is thus seen to be unstable to quantum

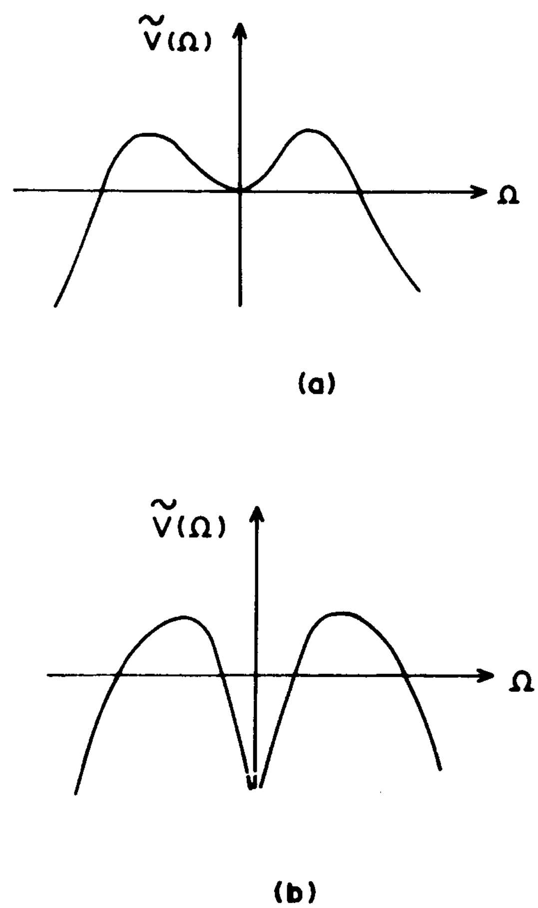

FIG. 3. (a) The effective classical potential $\tilde{V}(\Omega)$ in which $\Omega(t)$ evolves. (b) How this potential is modified by the inclusion of quantum fluctuations of the scalar field.

fluctuations. This result is of course true for the case of zero potential as well, as may be seen from Eq. (80). One may say that quantum fluctuations "disallow" the static, singular solution $\Omega=0$, $\dot{\Omega}=\ddot{\Omega}=0$. This may be reasoned as follows. When the source for $\Omega(t)$, namely, the scalar field, is quantized, the uncertainty principle forbids simultaneous measurement of η and the conjugate momentum π_η. This then gives rise to an uncertainty relation for $\Omega(t)$ and the conjugate momentum π_Ω (which is equal to $-m\dot{\Omega})^3$, so that it is not possible to talk of a state with $\Omega=0$, $\dot{\Omega}=0$. Thus one may understand the instability of the singular solution in terms of the uncertainty principle.

The analysis of the stability of de Sitter space also makes use of the potential in (80) and we shall return to this point in Sec. V.

C. Static solutions for the semiclassical geometry

It is well known that the classical Einstein equations with a cosmological constant term have a static solution, which has dust as its source, and which goes by the name of "Einstein's universe." Although such a solution is now only of academic interest, we would like to point out that in the semiclassical limit, such solutions exist, with the vacuum fluctuations playing the role of source.

If in the semiclassical equations with constant potential, (74) and (75), we put $\dot{\Omega}$ and $\ddot{\Omega}$ as zero, these equations reduce to [after using the expression for $\langle \dot{\eta}^2 \rangle$ from (56)]

$$-(\tilde{\Omega}_0^4+\Omega^4)+2\Omega^6 V_0=0\ , \tag{81}$$

$$\tilde{\Omega}_0^4-\Omega^4+\Omega^6 V_0=0\ . \tag{82}$$

These equations have a consistent solution if we choose

$$\tilde{\Omega}_0^4=\tfrac{4}{27}V_0^2\ , \tag{83}$$

and the corresponding constant value of the conformal factor is given by

$$\Omega_c^2=\sqrt{3}\tilde{\Omega}_0^2=\tfrac{2}{3}V_0\ . \tag{84}$$

This is the static solution with its source as a scalar field in a Gaussian state. In particular, we can choose the scalar field to be in its ground state. Using the corresponding expression for $\tilde{\Omega}_0^2$ from (71), we get that

$$\Omega_c^2=(4\sqrt{3}/3\pi)(L_p^2/\sigma_0)=\tfrac{2}{3}V_0\ . \tag{85}$$

As discussed in Sec. IV A, it is natural to assume that the dimensionless parameter σ_0 is of order unity. We then get the result that when the cosmological constant is of the order L_p^{-2}, quantum fluctuations give rise to a static solution for the geometry, with an amplitude L_p. However, one must note that since this solution was obtained by setting both $\dot{\Omega}$ and $\ddot{\Omega}$ at zero, it is (absolutely) static. Although the solution is interesting, it is not clear what it has to do with the real world.

This completes our discussion of the semiclassical geometry. We have demonstrated the solution which has the correct classical limit, and as a by-product, we have looked at the effect of vacuum fluctuations on the geometry. In all the problems considered in this section, we find that quantum fluctuations give rise to nontrivial effects which are absent in the classical limit.

V. THE DE SITTER SOLUTION AND ITS STABILITY

de Sitter space-time is the solution to Einstein equations with a cosmological constant term and without matter. The space-time is of interest because of its inherent high degree of symmetry, and also because the existence of a de Sitter phase (inflation) in the early Universe can help explain some long-standing cosmological problems. Moreover, it is also possible to look upon the cosmological constant as the net vacuum energy of the various matter fields. In that case, it is indeed a puzzle as to why the observed net vacuum energy density is smaller than 10^{-120} L_p^{-2}. The various attempts made so far to get rid of the Λ term[4,5] generally involve fine-tuning of coupling constants in the theory or have other problems of their own.

With the help of the semiclassical formalism developed in this paper, we now show that de Sitter space is unstable to quantum fluctuations[6,7] if the cosmological constant exceeds an upper bound. We are thus able to set a bound on the value of V_0, albeit a very weak one.

The classical equations for $\Omega(t)$, with a constant potential and no scalar field are

$$\ddot{\Omega}=-\Omega+2\Omega^3 V_0\ , \tag{86}$$

$$\dot{\Omega}^2+\Omega^2-\Omega^4 V_0=0\ . \tag{87}$$

These equations have a solution, namely, the de Sitter solution, which is given by

$$\Omega(t)=V_0^{-1/2}\sec t,\quad -\pi/2<t<\pi/2\ . \tag{88}$$

The metric (12) corresponding to the above $\Omega(t)$ can be brought to a more familiar form by the transformation

$$\tau=V_0^{-1/2}\,\mathrm{arcsinh}^{-1}(\tan t)\ . \tag{89}$$

In the new time coordinate the metric becomes

$$ds^2=d\tau^2-V_0^{-1}\cosh^2(V_0^{1/2}\tau)\times\left[\frac{dr^2}{1-r^2}+r^2(d\theta^2+\sin^2\theta\, d\Phi^2)\right]\ , \tag{90}$$

which is the familiar $K=+1$ coordinate system on the de Sitter manifold.

Let us now consider the effect of vacuum fluctuations of the scalar field on the de Sitter solution. To do this, we resort to the semiclassical equations (74) and (75). $\langle\dot{\eta}^2\rangle$ is as in (56), and since we are considering the scalar field to be in its ground state, the constant $\tilde{\Omega}_0^4$ is as in Eq. (71).

The stability analysis is straightforward if we write the constraint equation in the form of an energy-conservation equation, as in (77), with $\tilde{V}(\Omega)$ as in (78). We begin by noting that for the de Sitter solution (88) the range of the conformal factor is

$$V_0^{-1/2}<\Omega<\infty\ . \tag{91}$$

$\Omega(t)$ starts at infinity at $t=-\pi/2$, reaches a minimum $(=V_0^{-1/2})$ at $t=0$, then again starts increasing, reaching infinity at $t=\pi/2$.

Let us next recall the form of the potentials shown in Fig. 3, which are the shapes of $\tilde{V}(\Omega)$ of (78) for the classical and semiclassical case. The classical potential

($\tilde{\Omega}_0^2=0$), has a peak at $\Omega^2=\frac{1}{2}V_0$. On comparing with the range of $\Omega(t)$ in (91), we find that the minimum reached by $\Omega(t)$ *lies to the right of the peak of the classical potential in Fig. 3(a)*. Now we show that quantum fluctuations can push this minimum to the left of the peak.

The semiclassical potential of Fig. 3(b) has maxima, as may be found from (78), at Ω_m given by

$$\Omega_m{}^4(2V_0\Omega_m{}^2-1)=\tilde{\Omega}_0{}^4 . \tag{92}$$

Making the transformation

$$\Omega_m{}^2\equiv\frac{1}{2V_0}(1+\epsilon) \tag{93}$$

changes the above equation to

$$(1+\epsilon)^2\epsilon/4=\tilde{\Omega}_0{}^4V_0{}^2 . \tag{94}$$

We note that this equation is satisfied only by positive ϵ. Along with (93), this means that the quantum correction shifts the peak to larger values of Ω (in absolute value). Now, if $\epsilon>1$, we will have from (93) that

$$\Omega_m{}^2>1/V_0 , \tag{95}$$

and after the quantum correction, the peak will lie to the right of the minimum of the classical de Sitter path ($\Omega^2=1/V_0$). From the potential in Fig. 3(b), it is then obvious that, for $\epsilon>1$, the de Sitter solution is unstable to quantum fluctuations and will roll down to $\Omega=0$. The stability of de Sitter solution requires that $\epsilon<1$, or from (94) that

$$\tilde{\Omega}_0{}^4V_0{}^2<1 . \tag{96}$$

Using $\tilde{\Omega}_0{}^4$ from (71), we may put this in the form

$$L_p{}^2V_0<\sigma_0 . \tag{97}$$

This sets an upper bound on the cosmological constant V_0, if de Sitter space is to be stable to quantum fluctuations. By the now familiar argument that $\sigma_0\sim 1$, we get that $V_0\lesssim L_p{}^{-2}$. This provides a natural value $L_p{}^{-2}$ for V_0 at Planck time. A possible mechanism for the decay of the "cosmological constant," starting from this value, could be through particle creation. We hope to investigate this possibility in future.

While this effect may be very suggestive, the reader should be strongly cautioned about two issues related to Eq. (97).

(a) The semiclassical—or for that matter, any other—approximation breaks down at $t=t_{\text{Planck}}$. One cannot take the above result too seriously until the limits of validity of our approximation are worked out rigorously. There is no assurance, at this stage, that the result will survive the rigors of a full quantum theory.

(b) Because of the above reason, the term "instability" may be misleading. If we are dealing with the epochs of $t=t_{\text{Planck}}$, then we really have no clear idea of possible physical consequences. There will be some effect, of course, but to call this an "instability" could be an oversimplification.

[The reader might feel that it is possible to assume the validity of our approximation from some time $t=t_0\gg t_{\text{Planck}}$, and take σ_0 to be the dimensionless spread of the wave function at $t=t_0$. The result (97) will now, of course, be applicable, and we may call the effect an instability. Unfortunately, our result loses most of its interest in such a situation, because any reasonable theory will lead to a σ_0 at $t=t_0$ much larger than unity (since we expect $\sigma_0\gtrsim 1$ at $t=t_{\text{Planck}}$), and completely undetermined. This renders the resulting constraint on V_0 practically useless.]

VI. CONCLUSIONS

In this paper we have developed a semiclassical formalism for a gravity-scalar system—quantization of the scalar field in a classical geometry—so that the correct classical limit is obtained. To keep the analysis simple, we considered a background space-time which is sufficiently symmetric, so that the problem becomes one in quantum mechanics. This, of course, does not take away the significance of the results.

We first obtained the classical solution to Einstein's equations for a massless scalar field in a $K=+1$ Robertson-Walker universe. It was then shown that quantization of the scalar field in a classical geometry modifies the amplitude of oscillation due to quantum corrections, but leaves the form of the evolution unchanged. We also showed that the vacuum energy by itself can provide a nontrivial oscillating solution for the geometry, with an amplitude of the order of a Planck length.

Our analysis based on the semiclassical formalism is valid only after Planck time. This is testified by the divergence of the Gaussian spread $\sigma_\eta{}^2$ [Eq. (73)] at the singularity. A correct description before Planck time, therefore, requires the full quantization of the gravity-scalar system. There has been more than one approach to this problem.[8] Now that we have a semiclassical description which has the correct classical limit, the next step would be to obtain the semiclassical system as a limit of the fully quantized one. Such an investigation is still under way.

[1]For some of the reviews in quantum cosmology, see James B. Hartle, in *High Energy Physics, 1985,* proceedings of the Theoretical Advanced Study Institute in Elementary Particle Physics, New Haven, Connecticut, 1985, edited by M. J. Bowick and F. Gursey (World Scientific, Singapore, 1985), Vol. 2; T. Padmanabhan (unpublished).

[2]For a detailed discussion, see, N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, 1982).

[3]A result which supports this reasoning may be found in T. Padmanabhan and T. R. Seshadri, Int. J. Mod. Phys. A (to be published).

[4]L. F. Abbott, Phys. Lett. **150B**, 427 (1986); A. D. Dolgov, in *The Very Early Universe*, proceedings of the Nuffield Workshop, Cambridge, 1982, edited by G. W. Gibbons and S. T. C. Siklos (Cambridge University Press, Cambridge, England, 1983).

[5]F. Wilczek, in *The Very Early Universe* (Ref. 4); T. P. Singh, T. Padmanabhan, and T. R. Seshadri (unpublished).

[6]Throughout this section, by "unstable" we mean that the conformal factor $\Omega(t)$ is driven to the singularity $\Omega=0$.

[7]The quantum instability of de Sitter space has been investigated by many authors before. See, for example, L. H. Ford, Phys. Rev. D 31, 710 (1985), and the references cited therein.

[8]The difference is mainly due to the handling of the constraint equation. For comparison, see Ref. 1.