

A Comparison of Various Approaches to the Back-Reaction Problem

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We compare three possible prescriptions for the back-reaction of a quantum mechanical source on the classical gravitational field to which it is coupled. These prescriptions are (i) the transition element $\langle \text{out} | T_{ik} | \text{in} \rangle$ of the energy-momentum operator between asymptotic vacuum states, (ii) the expectation value $\langle \psi | T_{ik} | \psi \rangle$, and (iii) the Born–Oppenheimer type approximation for back-reaction as used in molecular physics. It is shown that the three approaches match only when the gravitational field is varying adiabatically, and of the three, the use of expectation value provides the most accurate description of the validity of semi-classical Einstein equations. The analysis is carried out by studying the model of a quantized time-dependent oscillator coupled to a classical particle. © 1993 Academic Press, Inc.

1. INTRODUCTION

The following question arises naturally in studies of quantum field theory on a classical curved space-time. If particles are produced by a time-dependent classical gravitational field, how does one take into account the back-reaction of the produced particles on the metric? A related question, which is asked less often, is: when does this prescription for back-reaction break down? In other words, how much should the back-reaction be, before the background gravitational field ceases to be classical and a quantum gravitational description becomes unavoidable?

As an answer to the first question, a prescription for back-reaction which was suggested in earlier literature was to use the transition element $\langle \text{out} | T_{ik} | \text{in} \rangle$, T_{ik} being the energy-momentum tensor operator for the matter field (see, e.g., DeWitt [5, pp. 346–347]). Here $|\text{in}\rangle$ and $|\text{out}\rangle$ are the asymptotic vacuum states at very early and very late times (provided these states can be defined). This prescription follows naturally from the variation of an effective action. The shortcomings of this approach have been pointed out by various workers (see, e.g., [14, 4]). In general, the effective action is complex, and the back-reaction may lead to a complex metric. This is difficult to interpret, unless the imaginary part of the back-reaction happens to be negligible in a physical situation, or is dropped in an ad hoc manner.

A more plausible prescription is that the expectation value of the operator T_{ik} in a quantum state $|\psi\rangle$ should be used on the right-hand side of the semiclassical Einstein equations (the first such suggestion was probably made by Moller [17]).

This state is to be obtained by solving self-consistently the functional Schrödinger equation for the quantum field in a curved background. A third prescription suggested in some works is to adapt the Born–Oppenheimer approximation from nuclear physics to the context of quantum gravity (e.g., [16, 11, 15]).

The second question, as to what is the domain of validity of these semiclassical prescriptions, has also been addressed by some authors (some relevant papers are [6, 7, 10, 19, 20]). It is necessary, but not sufficient, to restrict oneself to the consideration of length scales longer than Planck length. There is, in addition, a constraint on the class of allowed quantum states for the matter field. It is expected that the quantum state should be such that a suitably defined dispersion of the operator T_{ik} should be small compared to the average value of T_{ik} .

In this paper, we study a simple quantum mechanical model to compare the above-mentioned prescriptions for back-reaction and their domain of validity. The model consists of a time-dependent quantized oscillator q coupled to another particle C which can be macroscopic and hence can behave classically. The oscillator can get excited from the initial ground state to a higher state, and this is the “mini-superspace” analog of particle creation. It is clear that (q, C) form a minisuperspace model of the full quantum theory of gravity, with q and C respectively representing the matter and gravitational field. Thus, our conclusions about the back-reaction problem derived from this model also hold for semiclassical general relativity.

2. BACK-REACTION OF THE TIME-DEPENDENT OSCILLATOR

We consider the back-reaction of the time-dependent oscillator q coupled to the particle C via the Lagrangian

$$L(C, q) = \frac{1}{2} M \dot{C}^2 - V(C) + \frac{1}{2} \dot{q}^2 - \frac{1}{2} \omega^2(C) q^2, \quad (1)$$

where the frequency $\omega(C)$ is a known function of C . When the classical equation for C ,

$$M \ddot{C} + V'(C) = -\omega \omega' q^2 \quad (2)$$

is solved, along with the equation $\ddot{q} = -\omega^2(C)q$, the explicit time-dependence $\omega(C(t))$ of ω becomes known, and it is in this sense that we mean q to be a time-dependent oscillator. We assume that $\omega(C)$ is a constant ω_- during $-\infty < t < T_-$ and a constant ω_+ during $T_+ < t < \infty$, so that the time-dependence of ω through C arises only during the interval $T_- < t < T_+$. We now compare three possible descriptions of the situation when the quantum-mechanical oscillator q acts as a source for a nearly classical C .

a. The In–Out Effective Action

In the path-integral description for the quantum mechanics of the system (1), the kernel is

$$K = \int \mathcal{D}C \exp[iS(C)] I(C), \quad (3)$$

where

$$S(C) = \int_{t_1}^{t_2} dt [\frac{1}{2} M \dot{C}^2 - V(C)],$$

$$I(C, q_1, q_2; t_1, t_2) = \int_{q_1, t_1}^{q_2, t_2} \mathcal{D}q \exp i \int dt [\frac{1}{2} \dot{q}^2 - \frac{1}{2} \omega(C) q^2].$$
(4)

If we want to rid $I(C)$ of its dependence on the end-points q_1, q_2 , the integration in (4) has to be performed from $t_1 \rightarrow -\infty$ to $t_2 \rightarrow \infty$. Then the in-out effective action $W(C)$ is defined as

$$e^{iW(C)} \equiv \lim_{t_1 \rightarrow -\infty} \lim_{t_2 \rightarrow \infty} \int \mathcal{D}q \exp[iI(C)].$$
(5)

As is well known (e.g., [21]), the path-integral on the right-hand side of (5) is, apart from a normalization constant, equal to the vacuum persistence amplitude $\langle \text{out} | \text{in} \rangle$, which gives the amplitude for the system to be in the $|\text{out}\rangle$ vacuum, if it has started in the $|\text{in}\rangle$ vacuum.

In the limit where C behaves classically, we can do a saddle-point approximation in the kernel

$$K = \int \mathcal{D}C \exp[iS(C) + iW(C)],$$
(6)

so that the effective equation of motion for C is given by

$$\frac{\delta}{\delta C} [S(C) + W(C)] = 0.$$
(7)

Thus the in-out effective action for the model of the oscillator is

$$W(C) = -i \ln \langle \text{out} | \text{in} \rangle.$$
(8)

Here, $\langle \text{in} |, |\text{out} \rangle$ are the ground states of the oscillator defined in the time-domains $t < T_-$ and $t > T_+$, respectively:

$$\langle q | \text{in} \rangle \equiv \psi_-(q, t) = \left(\frac{\omega}{\pi}\right)^{1/4} \exp - \left[\frac{i}{2} \omega_- t + \frac{q^2}{2\omega} \right],$$
(9)

and

$$\langle q | \text{out} \rangle \equiv \psi_+(q, t) = \left(\frac{\omega_+}{\pi}\right)^{1/4} \exp - \left[\frac{i}{2} \omega_+ t + \frac{q^2}{2\omega_+} \right].$$
(10)

The transition element is easily calculated using the Schrödinger equation for the oscillator,

$$i \frac{\partial \psi}{\partial t} = -\frac{1}{2} \frac{\partial^2 \psi}{\partial q^2} + \frac{1}{2} \omega^2(C) q^2 \psi.$$
(11)

It can be shown (see, e.g., [18]) that in the quadratic potential, the initial state $\psi(q, t)$ given in (9) evolves into a gaussian state,

$$\psi(q, t) = N(t) \exp \left[-\frac{a(t)}{2} (q - \bar{q}(t))^2 \right]. \quad (12)$$

Substituting (12) in (11) and defining ξ through the relation $a(t) \equiv -i\dot{\xi}/\xi$ shows that $\bar{q}(t) = 0$ for all times, that

$$N(t) = (\omega / \pi)^{1/4} \xi^{-1/2}(t) \quad (13)$$

and that $\xi(t)$ satisfies the same equation as a classical time-dependent oscillator does,

$$\ddot{\xi} + \omega^2 [C(t)] \xi = 0, \quad \xi(t \rightarrow -\infty) = e^{i\omega t}. \quad (14)$$

The boundary condition as $\xi \rightarrow -\infty$ determines both ξ and $\dot{\xi}$ in the asymptotic past and hence fixes the solution uniquely. Note that in this equation, $C(t)$ is not necessarily a solution of the classical equation (2); rather, it is an arbitrary trajectory. $\xi[C(t)]$ and hence $a[C(t)]$ are functionals of $C(t)$, and we will often indicate this dependence simply as $\xi(t)$, $a(t)$.

Thus we obtain

$$\begin{aligned} \langle \text{out} | \text{in} \rangle &= \lim_{t \rightarrow \infty} \int_{-\infty}^{\infty} dq \psi_+^*(q, t) \psi(q, t) \\ &= \lim_{t \rightarrow \infty} (4\omega_+ \omega_-)^{1/4} \xi^{-1/2}(t) (\omega_+ + a(t))^{-1/2}, \end{aligned} \quad (15)$$

and up to a constant,

$$W(C) = \lim_{t \rightarrow \infty} \left\{ \frac{i}{2} \ln \xi[C(t)] + \frac{i}{2} \ln(\omega_+ + a[C(t)]) \right\}. \quad (16)$$

In the adiabatic limit, $\omega(t)$ changes slowly with time (i.e., $(\dot{\omega}/\omega) \ll 1$), we have from (14) that $\xi(t) \propto \exp i \int_{-\infty}^t \omega(t) dt$, and we obtain $W(C) = -\frac{1}{2} \int_{-\infty}^{\infty} \omega(C) dt$. Thus $(-\omega(C)/2)$ may be identified as the effective Lagrangian, which gives the adiabatic back-reaction as $\ddot{C} + V'(C) = -\omega'(C)/2$. In this limit, the oscillator stays in the instantaneous ground state for all times, and the net change in its energy is $(\omega_+ - \omega_-)/2$. There are no transitions to excited states and $W(C)$ is real.

When the evolution of $C(t)$ is non-adiabatic, $\xi(t)$ will evolve into a superposition of positive and negative frequency solutions, and the general solution in the domain $t > T_+$ can be written as

$$\xi(t, t > T_+) = C_1 e^{i\omega_+ t} + C_2 e^{-i\omega_+ t}. \quad (17)$$

The effective action will now be complex, and its imaginary part measures the probability of transition to the excited states. In general, it will not be possible to

identify an effective Lagrangian from (16). To obtain the back-reaction equation, we return to the original definition of W and note from (5) that

$$\frac{\delta W}{\delta C} = \omega\omega' \frac{\langle \text{out} | q^2 | \text{in} \rangle}{\langle \text{out} | \text{in} \rangle}, \tag{18}$$

so that the back-reaction equation is

$$\ddot{C} + V'(C) = -\omega\omega' \frac{\langle \text{out}, t | q^2 | \text{in}, t \rangle}{\langle \text{out}, t | \text{in}, t \rangle}. \tag{19}$$

Here, by $|\text{in}, t\rangle$ we mean that the state $|\text{in}\rangle$ has been evolved forward to time t , and by $\langle \text{out}, t|$ we mean that state at time t which becomes $\langle \text{out}|$ as $t \rightarrow \infty$. Thus, from (12),

$$|\text{in}, t\rangle = (\omega_-/\pi)^{1/4} \xi^{-1/2}(t) \exp\left[-\frac{a(t)}{2} q^2\right]. \tag{20}$$

Similarly,

$$\langle \text{out}, t| = (\omega_+/\pi)^{1/4} \xi^{-1/2}(t) \exp\left[-\frac{b(t)}{2} q^2\right], \tag{21}$$

where $b(t) = -i\dot{\eta}/\eta$, η satisfies the same equation as ξ , and the boundary condition $\eta(t \rightarrow \infty) = \exp(i\omega_+ t)$. Then we obtain

$$M\ddot{C} + V'(C) = -\frac{\omega\omega'}{a(t) + b^*(t)} \equiv -B_R - iB_I. \tag{22}$$

In the adiabatic limit, $a(t) = b(t) = \omega(t)$, and we obtain the result we saw above. In the first-order deviation from adiabaticity, we have $a = b = (\omega + i\dot{\omega}/2\omega)$, and curiously enough the back-reaction is still real. However, note that deviation from adiabaticity is being expressed as a power series expansion of the trajectory in $\dot{\omega}/\omega$, and not all trajectories may be obtainable as such a power-series expansion about the adiabatic trajectory. Thus, in general, the back-reaction in (22) will be complex. (An instructive example of a complex effective action and its non-analytic behaviour in \hbar has been discussed by Brown [3]).

The procedure of using the in-out effective action for defining back-reaction should be considered valid so long as the imaginary part of the back-reaction in (22) is small compared to $V'(C) + B_R$. We also notice that, in general, there will be a back-reaction through the term B_R even when B_I is negligible. B_R accounts for back-reaction from level shifts, while B_I accounts for back-reaction from transitions. In the field theory language, the analog of level shift is the reactive part of the effective action, while transitions are analogous to particle creation. Thus we may say that as long as the particle-creation does not affect the trajectory $C(t)$ significantly, the use of this effective action procedure is valid.

A similar conclusion about the relation between the real and imaginary parts of W follows from demanding energy conservation. From (22) it follows that

$$\left\{ \frac{1}{2} \dot{C}^2 + V(C) \right\} \Big|_{-\infty}^{+\infty} = - \int_{-\infty}^{+\infty} dt \dot{C} \frac{\omega \omega'}{a + b^*} = -L_{\text{eff}}(+)+L_{\text{eff}}(-), \quad (23)$$

where $L_{\text{eff}}(-) = -\omega_-/2$, and $L_{\text{eff}}(+)$ is the effective Lagrangian in the limit $t \rightarrow \infty$. In general, $L_{\text{eff}}(+)$ will be complex and it is meaningless to talk of a conserved energy. In the adiabatic limit, $L_{\text{eff}}(+)= -\omega_+/2$, and the change $(\omega_+ - \omega_-)$ in the oscillator's energy balances the change in the energy of mode C . More generally, we can talk of a conserved energy as long as $\text{Im}(L_{\text{eff}}(+))$ is small compared to the other terms in (23).

b. Determination of Back-Reaction from the Expectation Value of the Hamiltonian

Instead of the back-reaction equation (22), consider the alternate proposal

$$\begin{aligned} M\ddot{C} + V'(C) &= \frac{\partial}{\partial C} \langle \text{in} | h(q) | \text{in} \rangle \equiv \frac{\partial}{\partial C} \langle \psi(t) | h(q) | \psi(t) \rangle \\ &= \frac{\partial}{\partial C} \left\{ \frac{|a|^2 - \omega^2}{4a_R} \right\}. \end{aligned} \quad (24)$$

The initial state $|\text{in}\rangle$ defined at $t < T_-$ evolves into the state $|\psi(t)\rangle$, and this back-reaction is real. It has certain other features which make $\langle \text{in} | h | \text{in} \rangle$ a natural candidate for back-reaction, unlike $\langle \text{in} | h | \text{out} \rangle$. The first feature is energy conservation. Integration of (24) gives

$$\left\{ \frac{1}{2} M\dot{C}^2 + V(C) \right\} \Big|_{-\infty}^{+\infty} = \langle \psi(t) | h(q) | \psi(t) \rangle \Big|_{-\infty}^{+\infty}. \quad (25)$$

Evidently, the change in the energy of mode C is balanced by the change in the average energy of q , as the state evolves from T_- to T_+ . In the adiabatic limit,

$$\{ \langle \psi | h | \psi \rangle \} \Big|_{-\infty}^{+\infty} = \frac{1}{2}(\omega_+ - \omega_-), \quad (26)$$

which is the energy change due to level shift. If the evolution of C is not adiabatic, there will be transitions to higher levels (the equivalent of particle production). The back-reaction in (24) thus takes into account both level-shift and particle production. While the real and imaginary parts of the effective action back-reaction (22) are respectively a measure of level-shift and probability of particle production, together they do not give the back-reaction due to the net change in the average energy of q . On the other hand, the average energy back-reaction (24) cannot distinguish between level-shifts and particle production, for which purpose we resort to the effective action. A direct comparison of the two forms of back-reaction (22) and (24) shows that they match only in the adiabatic limit; outside this limit they do not match, in general.

Independent support for (24) comes from the Schrödinger equation corre-

sponding to the Lagrangian (1). Recall that the effective action back-reaction (22) has been derived from the path-integral corresponding to (1) in the saddle-point approximation. A similar derivation for the $\langle \text{in} | h | \text{in} \rangle$ back-reaction of (24) has been given by Jordan [14] and Calzetta and Hu [4]. Moreover, it can be shown that a semiclassical theory with this form of back-reaction can be derived as an approximation, starting from the full Schrödinger equation for C and q ,

$$i\dot{f} = Ef = \left\{ -\frac{1}{2M} \frac{\partial^2}{\partial C^2} + V(C) - \frac{1}{2} \frac{\partial^2}{\partial q^2} + \frac{1}{2} \omega^2(C) q^2 \right\} f. \quad (27)$$

This derivation of the back-reaction crucially requires $V(C)$ to be proportional to M , a requirement fulfilled in the gravitational case and not necessary in the effective action approach. For details of the derivation, we refer the reader to Padmanabhan and Singh [19]. Here we only outline the results and discuss some more recent related developments.

We performed a semiclassical expansion of the wave-function f of (27) in powers of M and constructed a Wigner distribution function for a given state. This distribution $F_M(C, P)$ gives the correlation between the position and momentum of the particle C . It has the form

$$F_M = \frac{M}{2\pi P_{\text{cl}}} \int_{-\infty}^{\infty} d\lambda \exp(-i\lambda[P - P_{\text{cl}}]) \times \int_{-\infty}^{\infty} dq R^2(q, C) \exp(i\lambda\beta'(q, C)). \quad (28)$$

Here, P_{cl} is the classical momentum for C , as determined by (2) with no source term on the right-hand side. R and β are the amplitude and phase of the wave-function ψ of Eq. (11): $\psi = R \exp(i\beta)$, and prime denotes derivative with respect to C .

This form of the Wigner function is not difficult to understand. If $F_M(C, P)$ shows a strong correlation between C and P in phase space (e.g., F_M may be close to a delta function $\delta(P - P_{\text{cl}})$, with position C in phase space arbitrary), we can say that the motion of C is along a classical trajectory. In the above integral, we note that if there was no q integration, then F_M has a peak on the classical momentum: this is precisely the Wigner function for a WKB state and shows how WKB states describe classical behaviour. The classical trajectories associated with a WKB wave-function are normal to the surface of constant WKB phase, and the momentum on a trajectory is the gradient of the phase, which from the Hamilton–Jacobi equation is equal to the classical momentum.

In the presence of a source q in (28), F_M will in general not show a classical behaviour. However, since F_M depends on the state $\psi(q, C)$ defined at any one time it is clear that any semiclassical behaviour which is recovered will involve expectation values and not transition elements. The back-reaction equation (24) will follow from (28) if and only if the q integration yields

$$\int_{-\infty}^{\infty} dq R^2(q, C) \exp(i\lambda\beta'(q, C)) = \exp[i\lambda \langle \psi | \beta' | \psi \rangle]. \quad (29)$$

That is, the expectation value of the exponential should be the exponential of the expectation value. This will then show that F is a delta function $\delta(P - P_{cl} - \beta')$. Such peaking of the momentum is equivalent to the semiclassical equation (24) [19].

It can be shown that (29) is a stringent requirement on the quantum state ψ and holds true if and only if either of the following two conditions are satisfied: (i) The probability density $R^2(q, Q)$ is a delta function $\delta(q - q_0)$, which means that q is peaked around a trajectory; this makes the state $\psi(q)$ a quasi-classical state, and thus the semiclassical equation is the same as the classical equation; the more interesting case is (ii). The background C evolves adiabatically, so that the state ψ is a stationary or a quasi-stationary state; in this case the phase β' is independent of q and can be pulled out of the integral over q ; moreover, $\langle \psi | \beta' | \psi \rangle = \beta'$ and the semiclassical equations (24) follow.

A simple proof for these statements, which was first pointed out to us by R. Nityananda and S. Sridhar, is the following: If β' is independent of q , then (29) is trivially true. Clearly, β' is independent of q for stationary states (definable for a static background) and for quasi-stationary states (definable when the background is no longer static, but changing only adiabatically). It is very likely these are the only classes of states for which β' is independent of q . In the more general case, when β' depends on q , we take the Fourier transform of (29) w.r.t. λ and find that $R^2(q)$ is proportional to a delta function $\delta(q - q_0)$.

Thus the adiabaticity constraint holds for the average energy equation as well. When the evolution of C is non-adiabatic, there will be level transitions (particle production), and the Wigner function argument will not yield a delta-function peak for F_M . A quasi-classical description will hold for C so long as the non-adiabatic contribution to $\langle \psi | \hbar | \psi \rangle$ is negligible compared to the adiabatic contribution $\omega(C)/2$.

The Wigner function as discussed here, has small distance fluctuations (on the size \hbar in phase space). These can make the Wigner function negative and thus prevent it from being a true probability distribution. In any discussion of this distribution there is a tacit assumption that a coarse-graining has been performed in phase space. For instance, it can be demonstrated (see, e.g., [22]) that only after such a coarse-graining does the Wigner function for a WKB-state show a delta-function peak on the classical trajectory. However, this averaging procedure does not affect our results on back-reaction, as the same coarse-graining can be applied to the two-particle Wigner function F_M of Eq. (28), and the conclusions we stated above will hold for the smoothed F_M . A discussion of the coarse-graining procedure can be found in Takahashi [22] and in Habib [8] (also see [9]). Recently, Paz and Sinha [20] have discussed the role of decoherence (as an alternative to coarse-graining) in obtaining semiclassical Einstein equations from the Wigner function. They also find adiabaticity as a necessary condition for the validity of these equations.

c. *The Born–Oppenheimer Approximation*

The Born–Oppenheimer approximation (BOA) was first developed in molecular physics [2, 1] to study the motion of electrons in the field of nearly static nuclei. Recently, there has been an interest in applying BOA in the context of semiclassical gravity (see, for example, [15]). In this section we start by giving a brief summary of standard BOA. We then describe two variations of BOA which adapt it to the case of the Lagrangian (1) and compare them with the two semiclassical methods described above.

In standard BOA one starts by writing the stationary Schrödinger equation for the electron q (having energy ε) in a generalized, external nuclear coordinate C . Next, this equation is substituted in the full Schrödinger equation for q and C (having energy E) to deduce the equation of motion for the nuclei C . It is first assumed that the nuclear coordinate is fixed at one value \bar{C} and that the electron's Schrödinger equation (11) (with the oscillator potential now replaced by a general potential $u(q, C)$) has stationary states $\psi(q, \bar{C})$ as solutions, with energy $\mathcal{E}(\bar{C})$. Next, one obtains corrections to $\psi(q, \bar{C})$ by allowing for small deviations of C from \bar{C} . To do this a new nuclear coordinate w is defined, as

$$\kappa w(C) = C - \bar{C}, \quad \kappa = \left(\frac{m}{M}\right)^{1/n}, \quad (30)$$

where m and M are respectively the electronic and nuclear mass. The index n is to be fixed on physical grounds and is reasoned to be four for the nuclear–electronic system. The Hamiltonian h for q , the state $\psi(q, C)$ and the energy \mathcal{E} are expanded about their mean values (corresponding to $C = \bar{C}$) by using κ as an expansion parameter. Using this expansion in the electron's Schrödinger equation yields corrections to ψ at various orders.

One then moves over to the Schrödinger equation (27) for the full q, C system and carries out a perturbation expansion in κ for the full Hamiltonian H , the wavefunction f , and the energy E . It is then shown that the wave-function $f(q, C)$ can be written in the form

$$f(q, C) = \chi(C)\psi(q, C) \quad (31)$$

and the nuclear wave-function $\chi(C)$ satisfies the equation

$$\left[-\frac{1}{2M} \frac{\partial^2}{\partial w^2} + MV_2 + \mathcal{E}_2 - ME_2 \right] \chi(w) = 0, \quad (32)$$

where w has been defined in Eq. (30). The subscript 2 on V , E , and \mathcal{E} indicates the second term in a perturbation expansion in κ . Thus there is a back-reaction \mathcal{E}_2 of the electron on the nucleus.

There is a hidden assumption here. It was assumed that the nucleus is nearly fixed at \bar{C} . This means that the corresponding wave-function $\chi(C)$ has to be peaked around $C = \bar{C}$ or around $w = 0$. So we cannot choose any solution of (32), but must

choose a WKB-solution for consistency. In that case the mean value of the momentum will be determined by the effective potential

$$V_{\text{eff}}(q, C) = MV_2 + \mathcal{E}_2(\bar{C}). \quad (33)$$

The above result would be useful in the present context but for the fact that it depends on choosing the index n as four. If, for example, we choose $n = 1$ we do not get such a back-reaction equation. It appears that BOA in its original form (expansion in powers of κ) may not be useful for other systems like gravity. A variation of this procedure turns out to be more useful [15]. While we retain a general potential $u(q, C)$ the following discussion in particular applies to the time-dependent oscillator discussed above. If we start by writing f in a different way than before, as $f(q, C) = \chi(C)\psi(q, C)$, the Schrödinger equation (27) becomes

$$E = \left[-\frac{1}{2M} \frac{\chi''}{\chi} + V(C) \right] + \left[-\frac{1}{2} \frac{\psi_{qq}}{\psi} + u(q, C) \right] + \left[-\frac{1}{2M} \left(\frac{\psi''}{\psi} + 2 \frac{\chi'}{\chi} \frac{\psi'}{\psi} \right) \right]. \quad (34)$$

In an attempt to obtain a semiclassical equation for χ which will include the effect of $u(q, C)$, we proceed as follows: Ignore (34) for the time-being and assume that the C -mode represents a *classical, static* particle (nearly fixed at position C_0) and that the Schrödinger equation for the q -mode has stationary solutions,

$$\mathcal{E}(C_0)\psi = -\frac{1}{2} \frac{\partial^2 \psi}{\partial q^2} + u(q, C_0)\psi. \quad (35)$$

The energy \mathcal{E} explicitly depends on C_0 .

Return now to Eq. (34) and assume that the wave-function $\psi(q, C)$ is sharply peaked at C_0 , so that (35) and the corresponding stationary states make sense. If the q -mode is in a stationary state of (35), the full Schrödinger equation reduces to

$$E = \left[-\frac{1}{2M} \frac{\chi''}{\chi} + V(C) + \mathcal{E}(C_0) \right] + \left[-\frac{1}{2M} \left(\frac{\psi''}{\psi} + \frac{2\chi'}{\chi} \frac{\psi'}{\psi} \right) \right]. \quad (36)$$

Further, if the terms in the second bracket are negligible, we obtain a Schrödinger equation for $\chi(C)$, with back-reaction,

$$E = -\frac{1}{2M} \frac{\chi''}{\chi} + V(C) + \mathcal{E}(C_0). \quad (37)$$

In this equation we must choose a solution $\chi(C)$ peaked at C_0 (if such a solution exists at all) so as to be consistent with the earlier assumption. As is known, the initial position for C must be specified along with the WKB wave-function, to select a specific trajectory.

The full Schrödinger equation can thus admit a solution such that C is nearly fixed at a point C_0 and the q -mode is in a state of definite energy $\mathcal{E}(C_0)$. This is a meaningful solution for the motion of an electron in the field of nuclei, for which

BOA was designed. However, it is still not relevant to the semiclassical theory, except in those cases when C is static.

What is relevant is a modification of (35). We assume that the wave function $f(C, q) = \chi(C)\psi(q, C)$ is peaked, not about a point C_0 , but about a trajectory $C(t)$. The mean value $\overline{C(t)}$ is assumed to evolve slowly in time, so that one can still solve (35) for *quasi-stationary states* $\psi(q, \overline{C(t)})$ in an adiabatically varying potential $u(q, \overline{C(t)})$. Each of these states can be associated with an energy $\mathcal{E}(\overline{C(t)})$. If the q -mode is in one of these states, one can do a calculation similar to the earlier one to obtain

$$i \frac{\partial}{\partial t} \chi(C, t) = -\frac{1}{2M} \chi'' + (V(C) + \mathcal{E}[C(t)]) \chi. \quad (38)$$

This is interesting because we can choose a consistent WKB solution, which will be peaked on the set of classical trajectories determined by

$$\ddot{C} = -\frac{dV(\overline{C})}{d\overline{C}} - \frac{\partial}{\partial \overline{C}} \langle \psi | h(q, \overline{C}) | \psi \rangle. \quad (39)$$

This is the same as the back-reaction equation (24). There is a self-consistent solution to the full Schrödinger equation such that $C(t)$ obeys the classical equation (39) and q is in a quasi-stationary state obtained by solving for $\psi(q, \overline{C})$,

$$\mathcal{E}(Q)\psi = -\frac{1}{2} \frac{\partial^2 \psi}{\partial q^2} + u(q, \overline{C})\psi. \quad (40)$$

However, this solution is valid only under stringent conditions: the wave-function $f(q, C, t) = \chi(C, t)\psi(q, C)$ is peaked about a family of classical trajectories $C(t)$ which are obtained by solving (39) and which evolve sufficiently slowly to allow the definition of quasi-stationary states in (40). Above all, the terms in the second bracket in (37) have to be small.

A comparison of BOA with the above two procedures shows BOA to be of limited use; in its present form it describes only adiabatic evolution, where it gives the same back-reaction as the effective action and the average energy equation. It does not take into account level-transitions in the q -mode. It would be of interest to obtain a generalisation of BOA which is applicable when the evolution of C is non-adiabatic.

3. CONCLUSION

Out of the three procedures discussed here for defining back-reaction, the average energy equation (24) provides the most accurate description of the approximations involved and is the natural prescription for back-reaction. The overall picture regarding the back-reaction is the following: An attempt to obtain the semiclassical

equation (24) as an approximation to the full quantum mechanical theory shows that the semiclassical equation holds only if quasi-stationary states can be defined for q and the background $C(t)$ is varying adiabatically. We also find that in the adiabatic limit, the effective action is real and there is no particle production; only level shifts occur in the adiabatic approximation. Together, these conclusions imply that Eq. (24) cannot be legitimately used to find the back-reaction of produced particles. They further suggest that if the effective action acquires an imaginary part comparable to the real part, the background picture breaks down, and one must resort to the full quantum theory. This, of course, goes against the belief that semiclassical Einstein equations should be used to find the back-reaction due to particle creation. It remains to be seen if a way can be found around the above conclusions which will enable us to treat particle creation as a semiclassical process.

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