Institute for Advanced Studies in Basic Sciences

Reconstructing Features of the Inflationary Scattering Potential from Primordial Power Spectra

Research Project Report

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Abstract

It is widely believed that the large scale structures in our present Universe have originated from quantum mechanical fluctuations of a scalar field during an inflationary epoch in the early Universe. According to the inflationary scenario, the phenomenon of super-adiabatic amplification is responsible for generating these large scale structures. This phenomenon can be viewed as a quantum mechanical scattering by a potential barrier in the early Universe, which we call the inflationary scattering potential.

There has been a lot of work on reconstructing the scalar inflaton potential using the observed spectrum of initial perturbations. However, this is the first attempt at recovering the related initial scattering potential which is model independent characterization of the inflationary scenario in terms of super-adiabatic amplification phenomenon. We are not presently in a position where there is a well established fundamental inflationary theory. In the absence of a single distinguished inflationary model it may be wiser to characterize and constrain inflation in terms of the model independent scattering potential.

We begin with a quick review of the standard cosmology and motivations of inflation. We analyze various problems the standard model is facing with and explain how the inflationary scenario resolves the problems of the Hot Big Bang Universe.

Then we study in more details the dynamics of the inflaton field, which fills the Universe in the inflationary epoch. This will lead us to study of the relics from inflation and evolution of scales. We also will derive the results giving the form of perturbations produced by inflation.

Then we discuss briefly the density power spectrum of the cosmic microwave background (CMB) and the shapes of the spectrum one can observe from the CMB. We explain in detail the Starobinsky's method for generating steps in the power spectrum by considering a singular inflaton potential. This method can be considered as the framework of this project.

We present a method for recovering features of the scattering potential from broken scale-invariant power spectrum. We show the recovery of the effective mass features for toy power spectra. In the future, we plan to reconstruct the inflationary scattering potential features for the real power spectrum determined from WMAP.

Contents

"What I am really interested in is whether God could have created the world differently."

A.Einstein

1 The Hot Big Bang Universe

In his Scientific American essay [7], Guth notices several assumptions for the Big Bang model. First assumption is that the fundamental laws of Physics do not change with time and the effects of gravitation are described by Einstein's General Relativity. It is also assumed that the early Universe was filled with an almost uniform, expanding hot gas of elementary particles in thermal equilibrium. The gas filled all of space and they both expanded together at the same rate. It is also assumed that any changes in the state of the matter and the radiation have been so smooth that they have had a negligible effect on the thermodynamic history of the Universe. The violation of the last assumption is a key to the Inflationary Universe Scenario.

1.1 An Overview of the Standard Cosmology

The central premise of modern cosmology is that, at least on large scales the Universe is homogeneous and isotropic. This admits a family of homogeneous and isotropic spatial hyper-surfaces labeled by a time-like parameter, t.[1] For a set of 'fundamental' observers who see the same universe at the same cosmic time, the Friedmann-Robertson-Walker (FRW) metric

$$ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2\theta\ d\phi^2)\right] \tag{1}$$

provides an adequate geometrical description of the Universe.The curvature parameter K determines the topology of the spatial section. It takes the values +1,0,-1 for three possible realizations of the maximally symmetric three-space, -closed, flat and open respectively. The dynamical variable in the FRW model is the universal scale factor $a(t)$.[2] If the Universe is homogeneous and isotropic, the distance between any two *comoving points*[3] is proportional to $a(t)$. The scale factor obeys the Friedmann equations:[4]

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{k}{a^2} = \frac{8\pi G}{3}\rho \tag{2}$$

[1]Since GR is time reparametrization invariant, a different time variable $\eta = \int \frac{dt}{a(t)}$ known as *conformal time* is also used. In space-conformal time diagram, the light cone structure is the same as in Minkowski space.

[2]In a spatially flat universe, one can set the scale factor equal to 1 at the present time without loss of generality.

[3]A comoving coordinate is one that moving with the expansion of the Universe.

[4]Throughout this report the superscript 'dot' refers to derivative with respect to t and 'prime' refers to derivative with respect to ϕ.

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p)\,. \tag{3}$$

The above equations emerge from the Einstein equations for the FRW metric with a perfect fluid stress tensor $T^{\mu}_{\nu} = diag[\rho, -p, -p, -p]$ where ρ is the mass density and p is the pressure. See [13]. We assume that the matter content of the Universe is a perfect fluid with an equation of state $p = \omega\rho$, ω is equal to $\frac{1}{3}$ when the Universe is radiation dominated, equal to 0 when it is matter dominated and equal to -1 when we consider the vacuum energy. From this equation one can show that for normal fluids (fluids with $\omega \geq -\frac{1}{3}$), there would exist a singularity in the past when the classical equations of general relativity break down [14]. Finally combining (2),(3) we obtain an equation for the conservation of the matter in the Universe:

$$\ddot{\rho} + 3H(\rho + p) = 0 \tag{4}$$

Where H is the Hubble parameter with $H = \frac{\dot{a}}{a}$. The conservation equation leads us to the following relations for the density and the scale parameter:

$$\rho = \rho_{\circ}\left(\frac{a}{a_{\circ}}\right)^{-3(1+\omega)}, \tag{5}$$

$$a(t) = \left(\frac{t}{t_{\circ}}\right)^{\frac{-2}{1+3\omega}}. \tag{6}$$

From (6) it is clear that the condition $\omega \geq -\frac{1}{3}$ will lead to a decelerating universe, and $\omega \leq -\frac{1}{3}$ will lead to an accelerating universe. For further reading see [8].

1.2 Problems of the Standard Cosmology

Standard Big Bang Cosmology based on the FRW model has been quite successful in describing many of the properties of our universe. The three classic observational pillars of Standard Cosmology are Hubble's law, the existence and black body nature of the nearly isotropic Cosmic Microwave Background (CMB) and the abundances of light elements. These successes will not be reviewed here.[5] However, There are some difficult problems which have no solution within standard theories and are the main motivation for Inflationary Cosmology. The most important of these are listed below with a related question by Linde [2]:

1. The homogeneity problem:
Why, this problem asks, did all the causally disconnected regions of the Universe start their expansions simultaneously (at t=0)?

2. The flatness problem:
Why is our Universe so flat, its geometry almost exactly Euclidean?

3. The inhomogeneity problem:
Observations show that our Universe on the very large scale is extremely homogeneous, but on much smaller scales contains stars, galaxies and clusters of galaxies and is not homogeneous at all. Just as it is difficult to explain homogeneity over

[5] For a brief description see [6].

causally disconnected domains, it is a problem to explain why the tiny primordial inhomogeneity should have correlation on scales larger than the causal horizon. This problem asks: what is the origin of these local inhomogeneities in the Universe?

These problems do not reflect any conflict within standard cosmology; rather, they require extreme fine tuning of initial conditions in Hot Big Bang Universe. Following is a short description of the above problems. For further reading see [5], [6], [12] and [14].

The first problem comes from the finite age of the classical FRW universe. It arises because the comoving region $l_p(t_{rec})$ over which the CMB is observed to be homogeneous to better than 1 part in 10^4 is much larger than the comoving forward light cone $l_f(t_{rec})$ at time of recombination, which is the maximal distance over which micro-physical forces could have caused homogeneity:

$$l_p(t_{rec}) = \int_{t_{rec}}^{t_\circ} dt\ a^{-1}(t) \simeq 3t_\circ \left(1 - \left(\frac{t_{rec}}{t_\circ}\right)^{\frac{1}{3}}\right) \tag{7}$$

$$l_f(t_{rec}) = \int_0^{t_{rec}} dt\ a^{-1}(t) \simeq 3t_\circ^{2/3}\ t_{rec}^{1/3} \tag{8}$$

This means that the Cosmic Microwave Background photons coming from regions separated more than the horizon scale at last scattering, can not have interacted before decoupling. The Big Bang model therefore offers no prospect of explaining why the temperature seen in different places of the sky is so accurately the same; the homogeneity must form part of the initial conditions.

The flatness problem is a restatement of the fact that the age of our universe is unnaturally large when compared to the Planck time($t_\circ \simeq 10^{60} t_p$). The standard Big Bang model cannot explain why the Universe began so finely tuned to the flat $K = 0$ model as to avoid either a re-collapse within a few Planck times (for closed models), or a cold death due to curvature dominated expansion(for open universes). To be more quantitative, we define the density parameter $\Omega = \frac{\rho}{\rho_{cr}}$, where $\rho_{cr} = \frac{3H^2}{8\pi G}$. The open, flat and closed models correspond to Ω <,=,> 1. For a constant state parameter ω, the Ω-parameter evolves as

$$\mid \Omega - 1 \mid = \mid \Omega_\circ - 1 \mid \left(\frac{t}{t_\circ}\right)^{\frac{2(\omega+1/3)}{1+\omega}} . \tag{9}$$

For $\omega \geq -\frac{1}{3}$, the observed value of $\mid \Omega_\circ - 1 \mid$ is of order unity. This fact implies that our universe must have started with a value of $\mid \Omega_p - 1 \mid \simeq 10^{-60}$ at the Planck epoch, which would require extreme fine tuning of initial conditions. What is the origin of this fine tuned initial condition? This is the flatness problem of standard cosmology.

The third problem is the difficulty of explaining the origin of primordial perturbations required to form the structures in the Hot Big Bang model. This is closely related to the homogeneity problem. Consider any astrophysical scale with wave length, $\lambda(t)$, with respect to the Hubble radius H^{-1} at an earlier epoch t is given by:

$$\frac{\lambda(t)H(t)}{\lambda(t)H_\circ} = \left(\frac{t}{t_\circ}\right)^{\frac{-2(1+3\omega)}{(1+\omega)}} \tag{10}$$

If $\omega > -\frac{1}{3}$, astrophysical scales which are smaller than the Hubble radius at present $(\lambda(t_\circ)H_\circ < 1)$ will be much larger than the Hubble radius at a sufficiently early time $(\lambda(t)H(t) \gg 1)$. Hence, in order to generate the present large scale structures[6], the primordial fluctuations need to be correlated on scales much larger than the Hubble radius. Since the Hubble radius sets the scale beyond which astrophysical processes are ineffective in the standard Hot Big Bang model, any mechanism invoked to generate these primordial perturbations has to be *acausal.*

The Hot Big bang theory is unable to explain the problems listed above. It yet may be able to explain the generation of inhomogeneities [6], but as we will see, the most attractive way to resolve all the problems, is to go beyond the Standard Hot Big Bang model.

[6]Notice that the scale of large scale structures is less than the scale of present Hubble radius.

2 The Inflationary Universe

The Inflationary Scenario resolves most of the problems that the Hot Big Bang Universe is faced with. Inflation is not a replacement for the Hot Big Bang model, but rather an add-on that occurs at very early times without disturbing any of the Hot Big Bang successes. Inflationary theories have gradually gained the status of a paradigm for modern cosmology.

2.1 Definition of the Inflation

The precise definition of Inflation is simply a sufficiently long period during which the scale factor of the Universe is accelerating:

$$Inflation \iff \ddot{a} > 0 \tag{11}$$

It assumes that there is a time interval during which the Universe is exponentially expanding, i.e.

$$a(t) \sim e^{Ht} \ , \qquad t \in [t_i, t_f] \tag{12}$$

With a roughly constant Hubble expansion parameter H.[7] the "Inflationary" universe is locally, resembles the "de Sitter" space-time. The duration of this phase has to allow the Universe to inflate by at least 60 e-folds (i.e. $Ln(\frac{a_f}{a_i}) = 60$ where a_i and a_f are the values of the scale factor at the beginning and end of the inflationary phase.)
The condition for inflation also can be rewritten as a requirement on the material deriving the expansion. Directly from (11) we find that during inflation the Universe was dominated by a form of matter with an effective equation of state parameter, $\omega < -\frac{1}{3}$ (or $\rho + 3p < 0$). It implies that for that kind of matter, p is necessarily negative, Since we always assume ρ to be positive.

2.2 Inflationary Solution to the Hot Big Bang Problems

Inflationary Universe Scenario has resolved the problem of homogeneity and flatness. Figure 1 gives a schematic illustration of the inflationary solution to the homogeneity problem. During inflation the forward light cone $l_f(t)$ is expanded exponentially when measured in physical coordinates. Hence, it does not require many e-folding of inflation in order that $l_f(t)$ becomes larger than the past light cone at the time of last scattering.

Inflation also can solve the flatness problem. The key point is that in inflationary scenario the entropy density is no longer constant [12]. By definition, during inflation Ω is driven toward 1, and with sufficient inflation it will finish so close by the time inflation ends that in all the subsequent evolution up to the present it remains indistinguishably close. Only in the distant future will it move away again. See

[7] For all inflationary models the Hubble parameter is not strictly constant during inflation. *Quasi-exponential inflation* when the Hubble parameter varies very slowly on Hubble time scales, i.e. $|\dot{H}/h^2| \ll 1$, is satisfied by many inflationary scenarios.

figure 35.

Inflation also provides a 'causal' mechanism for generating the large scale structures. In the framework of inflation, the large scale structure that we observe today is believed to have originated from quantum fluctuations of a scalar field. We will discuss this later in section 3.

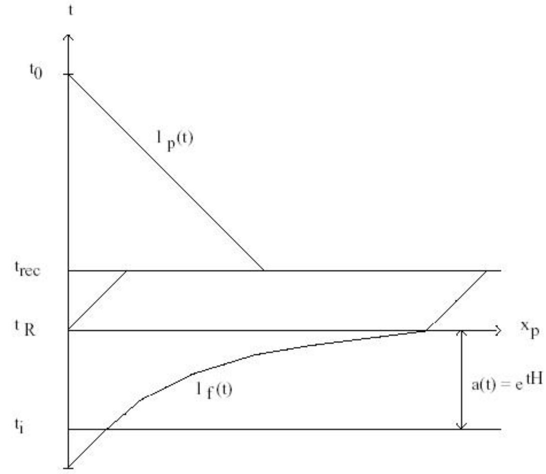

Figure 1: Sketch of how a period of inflation can solve the homogeneity problem. During inflation, the forward light cone increases exponentially compared to a model without inflation, whereas the past light cone is not affected for $t \geq t_R$. (Fig. taken from [12])

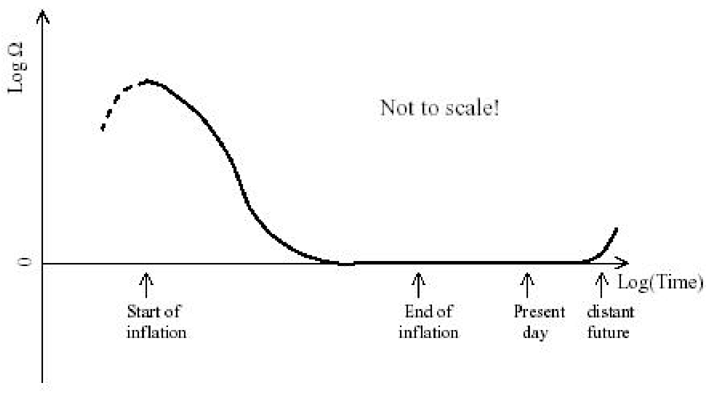

Figure 2: A schematic illustration of the inflationary solution to the flatness problem. Inflation drives the Universe so close to flatness that only in the extremely distant future is it possible for the Universe to deviate significantly. (Fig. taken from [4])

2.3 Inflaton Field Dynamics

As we have seen, the inflationary paradigm postulates that there existed a period of accelerated expansion when the Universe was dominated by a material with the unusual property of negative pressure. Such a material is a *scalar field*[8], describing scalar (spin-0) particles.[9] The simplest realization of inflation is perhaps the *chaotic model of inflation*, in which the Universe is assumed to be dominated by a single scalar field(inflaton), ϕ, with a self-interaction potential $V(\phi)$ (The inflaton potential). The picture here is that around the Planckian epoch, random initial conditions (initial chaos) will ensure that the field will be found displaced from its ground state (minimum of its potential V) in some regions of the Universe. These regions will then undergo a phase of rapid accelerated expansion and will "inflate" to a physical size large enough to accommodate the region corresponding to the observable universe.

The standard way to specify a particle theory is via its Lagrangian, from which the equations of motion can be derived [13]. From the Lagrangian we can deduce the energy-momentum tensor of the field which leads to two expressions for the density and pressure of the homogeneous scalar field.[10]

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi)\,, \tag{13}$$

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi)\,. \tag{14}$$

One can think of the first term in each as a kinetic energy, and the second as a potential energy which measures how much internal energy is associated with a particular field value.[11] Substituting above equations into the Friedmann and conservation equations(2),(3) and (4), we obtain an equation of motion for the inflaton field during inflationary epoch[12]

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0\,. \tag{15}$$

This equation resembles that of particle moving under a potential gradient in a viscous medium. The motion of the scalar field is damped by expansion of the Universe, characterized by the Hubble parameter. Using the Friedmann equations (2),(3) and the conservation of the energy (4), we can also find another two expressions for the Hubble parameter during inflation:

$$H^2 = \frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}[\frac{1}{2}\dot{\phi}^2 + V(\phi)] \tag{16}$$

[8]The scalar field responsible for inflation is often called *inflaton*.

[9]Although, as yet, there has been no direct observation of a fundamental scalar particle (such as the Higgs particle), such particles proliferate in modern particle theories. They play a crucial rule in bringing about symmetry breaking between the fundamental forces. See [7].

[10]See Chapter 14 of [6].

[11]Since $\ddot{a} > 0 \Leftrightarrow p < -\frac{\rho}{3} \Leftrightarrow \dot{\phi} < V(\phi)$ we will have inflation whenever the potential energy dominates.

[12]In the absence of inflaton perturbations there is an add-on term $\nabla^2\phi$ on the left hand side of the equation of motion. See [6] and § 3.1.

$$\dot{H} = -4\pi G(\rho + p) = -4\pi G\dot{\phi}^2 \,. \tag{17}$$

If the inflaton field is initially at a large value of the potential, equation (16) implies that the Hubble parameter is large and consequently, the field *rolls down slowly* ($\frac{\dot{\phi}}{H} \ll 1$), since the viscous term dominates in (15). It is clear that the state parameter is less than $-\frac{1}{3}$ as long as $\frac{\dot{\phi}}{V} < 1$, and the Universe enters a phase of accelerated expansion.[13] One way of quantifying the slow roll of the field is to introduce the dimensionless slow roll parameters ϵ and δ and study inflaton dynamics in the $\epsilon - \delta$ plane. See [14] and § 2.4.

In a given inflationary theory, there would be a specific form for the potential $V(\phi)$, at least up to some parameters. However, we are not presently in a position where there is a well established fundamental theory that one can use. So in the absence of such a theory, inflation workers tend to regard $V(\phi)$ as a function to be chosen arbitrarily, with different choices corresponding to different models of inflation.

2.4 Slow-Roll Approximation(SRA)

Many inflationary scenarios lead to Quasi-exponential inflation where a standard strategy for solving the equation of motion of inflaton is the slow-roll approximation (SRA). SRA implies that the acceleration term in eqn. (15) can be neglected ($\ddot{\phi}/3H\dot{\phi} \ll 1$) and the kinetic energy of the scalar field in each of eqn. (16) may be neglected. This leads to the simpler set

$$H^2 \simeq \frac{8\pi G}{3} V \,. \tag{18}$$

$$3H\dot{\phi} \simeq -V' \tag{19}$$

If we define dimensionless *slow-roll parameters*

$$\epsilon(\phi) = \frac{1}{16\pi G}\left(\frac{V'}{V}\right)^2 = \frac{1}{3H}\left|\frac{\ddot{\phi}}{\dot{\phi}}\right| \tag{20}$$

$$\delta(\phi) = \frac{1}{8\pi G}\left(\frac{V''}{V}\right) = \frac{\dot{\phi}^2}{2V(\phi)} \tag{21}$$

where the first measures the slope of the potential and the second the curvature, then *necessary* conditions for the slow-roll approximation to hold are

$$\epsilon \ll 1 \, ; \qquad |\, \eta \,| \ll 1 \,. \tag{22}$$

[13] As the field rolls down to its minimum, the value of the potential steadily decreases and the Universe exits out of inflationary phase. The inflaton then undergoes coherent oscillations around the minimum of $V(\phi)$ and decays by radiating out excitations/quanta of the other matter fields that are weakly coupled to it, thus reheating the Universe to a high temperature. This process is another feature of the inflation with which the inflationary expansion gives way to the standard Hot Big Bang evolution.

Unfortunately, although these are necessary conditions for the slow-roll approximation to hold, they are not sufficient, since even if the potential is very flat the scalar field may have a large velocity [5]. However, the slow-roll conditions is a sufficient condition for inflation. To see this, rewrite the condition for inflation as

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 > 0$$

This is obviously satisfied if $\dot{H}$ is positive. Otherwise, we require

$$-\frac{\dot{H}}{H^2} < 1$$

Meanwhile, substitution of the slow-roll equations yields

$$-\frac{\dot{H}}{H^2} \simeq \frac{1}{2G}\left(\frac{V'}{V}\right)^2 = \epsilon$$

Consequently, if the slow-roll approximation is valid ($\epsilon \ll 1$), then inflation is guaranteed.

3 Relic Fluctuations from Inflation

It is widely accepted that the large scale structures in the present Universe grew out of small initial perturbations in the inflaton field. Inflationary models also predict the form of the power spectrum of primordial scalar fluctuations. This section presents a short sketch of the theory of cosmological perturbations.

3.1 Quantum Fluctuations of the inflaton field

In chapter 14 of their book [6], Liddle and Lyth show that in the absence of cosmological perturbations, the scalar field equation is

$$\ddot{\phi} + 3H\dot{\phi} + \nabla^2\phi + \frac{dV}{d\phi} = 0\,. \tag{23}$$

This equation includes, through $V(\phi)$, the effect of any interaction of ϕ with itself or other scalar fields. We now split the field into an unperturbed part and a perturbation,

$$\phi(\mathbf{x},\eta) = \phi(\eta) + \delta\phi(\mathbf{x},\eta)\,. \tag{24}$$

Substituting this equation into (23) and linearizing, we can obtain the equation of motion for $\delta\phi$ on an inflationary background

$$\delta\ddot{\phi} + 3H\delta\dot{\phi} + \frac{1}{a^2}\nabla^2\delta\phi + m^2_{eff}\delta\phi = 0\,, \tag{25}$$

where m^2_{eff} is the effective mass of the free, minimally coupled scalar field, given by

$$m^2_{eff} = \frac{d^2V}{d\phi^2} + 8\pi G\frac{\dot{\phi}}{H}\frac{dV}{d\phi} + H\frac{d}{dt}\left(\frac{\dot{H}}{H^2}\right) \tag{26}$$

The effective mass of the inflaton fluctuations determines the shape of the inflaton perturbation spectrum and is also an important quantity for describing the dynamics of the inflaton field. We can expand inflaton perturbations in the Fourier space and deduce the equation of motion for each mode

$$\delta\ddot{\phi}_k + 3H\delta\dot{\phi}_k + \left[\left(\frac{k}{a}\right)^2 + m^2_{eff}\right]\delta\phi_k = 0\,, \tag{27}$$

where $a(\eta)$ is the scale factor. As long as inflaton field has negligible mass, the temporal modes of the inflaton field perturbations can be shown to obey an equation of the form

$$\nu''_k + \left[k^2 - \frac{\theta''}{\theta}\right]\nu_k = 0 \tag{28}$$

where $\nu_k \equiv a\delta\phi_k$ and $\theta \equiv a^2\frac{\phi'}{a'}$. This equation resembles the Schrodinger equation in Quantum Mechanics. It can be interpreted as a quantum mechanical scattering (in time) by a potential barrier, where the role of potential barrier is played by $V_s(\eta) \equiv \frac{\theta''}{\theta}$. We call V_s the *inflationary scattering potential.* The equation (28) is very useful in analyzing the evolution of scales in the inflationary epoch. See § 3.4.

3.2 Primordial Density Power Spectrum

The departure from homogeneity and isotropy due to the inflaton perturbation $\delta\phi(\mathbf{x},\eta)$ leads to perturbation in the energy density $\delta\rho(\mathbf{x},\eta)$ and the metric of space-time. Hence after inflation there will be inherited perturbations $\delta\rho_i(\mathbf{x},\eta)$ in the densities of each individual particle species.[14] However, all perturbations are determined by $\delta\phi_k(\eta)$'s which are the modes of inflaton perturbations in the Fourier space. A generic perturbation function $f(\mathbf{x},\eta)$ can be expanded in a Fourier integral

$$\delta f(\mathbf{x},\eta) = \frac{1}{(2\pi)^{3/2}} \int \delta f(\mathbf{k},\eta) \exp(i\mathbf{k}.\mathbf{x}) d^3k \tag{29}$$

The *Power Spectrum of* $f(\mathbf{x},\eta)$, P_f, is defined by

$$< \delta f^{\star}(\mathbf{k},\eta)\delta f(\mathbf{k}',\eta) >= \delta^3(\mathbf{k}-\mathbf{k}')\frac{2\pi^2}{k^3}P_f(\mathbf{k},\eta) \tag{30}$$

If we use adiabatic density perturbations $\delta = \frac{\delta\rho}{\rho}$ for $f(\mathbf{x},\eta)$, equation (30) leads to an *adiabatic density perturbation spectrum.* The initial spectrum of adiabatic density perturbations dictates the nature of large scale structures and is reflected in the CMB anisotropy.

3.3 Evolution of Scales

The ability of inflation to generate perturbations on large scales comes from the unusual behavior of the Hubble length during inflation. In the Big Bang Universe the comoving Hubble length is always increasing and so all scales are initially much larger than it, and hence unable to be affected by causal Physics. However, inflation reverses this behavior, as is sketched in figure 3.

In the inflationary scenario, a given comoving scale has a more complicated history. Early on in inflation, the scale could be well inside the Hubble length and hence causal Physics can act. Some time before inflation ends, the scale crosses outside the Hubble radius and causal Physics becomes ineffective. So any perturbation generated before inflation, becomes imprinted or in the usual terminology, “frozen in”. Long after inflation is over, the scales cross inside the Hubble radius again. We will discuss this concepts more quantitatively in the next section.

3.4 Scattering of Scales During Inflationary Epoch

As already noted, during inflation a physical scale with wave number k can be large or small with respect to the value of the Hubble radius. Figure 4 is another pictorial representation of the phenomenon of evolution of scales. A mode with comoving wave number $k = \frac{2\pi a}{\lambda}$ is shown to leave the Hubble radius during inflation (t_1) and re-enter it during matter domination (t_2).

[14] There will be more complicated perturbations too, such as the perturbation $\Theta(\mathbf{x},\mathbf{n},\eta)$ in the function specifying the number of photons at position $\mathbf{x}$, with momentum in the direction $\mathbf{n}$. At our position and at the present epoch, this is the *CMB anisotropy.*

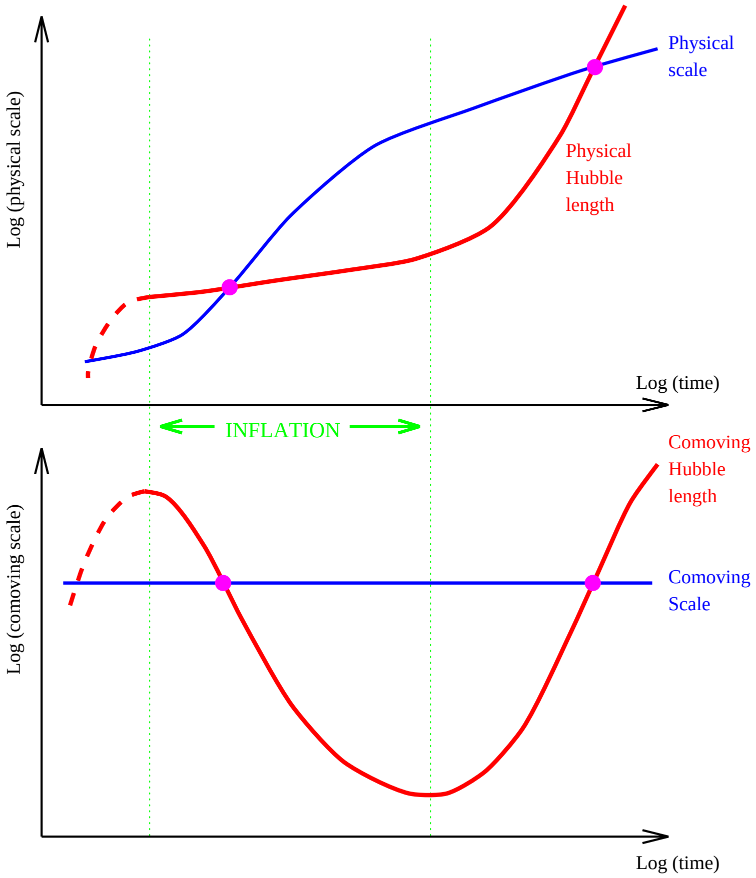

Figure 3: Two equivalent views of how scales evolves during inflation, as compared to the Hubble length which sets the scale of causality. The comoving picture is more transparent, when one recalls that inflation is defined as an epoch where the comoving Hubble length is decreasing. Scales of interest to us today begin much smaller than the Hubble length, where quantum fluctuations are acquired, and by the end of inflation have become much larger than the Hubble length. (Fig. taken from [4])

It can be seen from equation (28) that in the small wavelength limit ($k\eta \gg 1$ or, more precisely, $k^2 \gg \frac{\theta''}{\theta}$) the inflaton perturbation is damped

$$\delta\phi_k \simeq \frac{1}{a(\eta)} e^{ik\eta} \tag{31}$$

whereas in the long wavelength limit ($k\eta \ll 1$ *or* $k^2 \ll \frac{\theta''}{\theta}$)

$$\frac{H}{\dot{\phi}}\delta\phi_k = S_1(k) + S_2(k) \int \frac{d\big(\eta/\eta_\circ\big)}{a^2} \tag{32}$$

the quantity $\frac{H}{\dot{\phi}}\delta\phi_k$ freezes to the scale dependent constant $S_1(k)$ on scales larger than the Hubble radius.

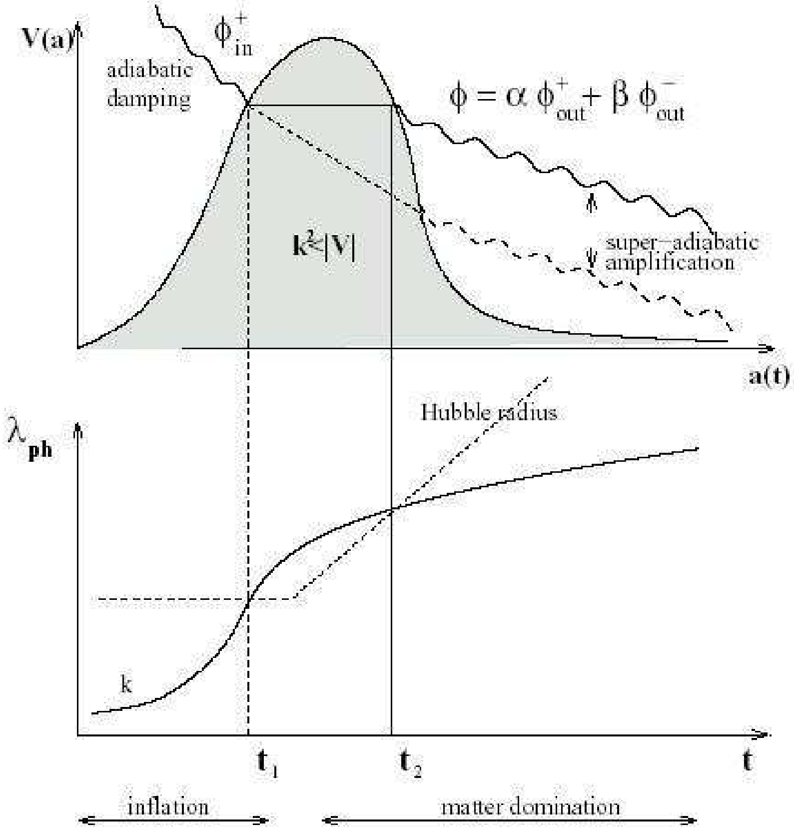

Figure 4: The super-adiabatic amplification of inflationary fluctuations is illustrated pictorially. The field ϕ could represent the inflaton field fluctuation $\delta\phi$ and the "scattering potential", V, could be viewed as V_S. A mode with comoving wave number k is shown to leave the Hubble radius during inflation and re-enter it during matter domination. (Fig. taken from [10])

As is sketched in figure 4, at early times during the inflationary epoch, the 'in'-state of the inflaton field perturbations corresponds to its vacuum state $\delta\phi_k^{(+)}$. The amplitude of these modes is damped with the expansion of the Universe as long as their wavelength is small relative to the Hubble radius. Once a wave enters the region in which its wavelength is larger than the Hubble radius, its amplitude freezes in time. At late times the wave will re-enter the Hubble radius during the epoch of matter or radiation domination. During this epoch the 'out'-state of the field will, in general, be described by a linear superposition of positive and negative frequency solutions[15]:

$$\delta\phi_k(\eta) = a_s(k)\delta\phi_k^{(+)} + \beta_s(k)\delta\phi_k^{(-)} . \tag{33}$$

The large wavelength limit of the 'out'-state of $\frac{H}{\dot\phi}\delta\phi_k$ specifies the initial conditions for the spectrum of adiabatic density perturbations. The amplitude of the density perturbations will appear to be greater than what they would have been if they

[15]The superscript (+) refers to a wave going toward large t's and (-) refers to a wave going toward small t's.

have never been greater than the Hubble radius. The phenomenon of super-adiabatic amplifications can be viewed as a quantum mechanical scattering by potential barrier V.

4 The Shape of the Density Power Spectrum

In most inflationary models, the density fluctuations generated by inflation are often supposed to be scale-invariant. However, a number of recent observations tend to give certain evidence for considering a non-scale invariant initial density spectrum. These observations are consistent with the large scale structures data. Therefore it is worth discussing how such features could have appeared in the power spectrum.

4.1 Generation of Non-flat Perturbation Spectra

To obtain a non-flat spectrum from inflation one can allow complicated inflaton potentials. Another way is to consider isothermal fluctuations in the model which includes two or more scalar fields. For example, in their paper on kinds of perturbation spectra to which inflation can lead [15], Mukhanov and Zelnikov introduced the possibility of considering two scalar fields with a power-law potential[16] in the inflationary model which leads to a varying effective mass. It can be seen that whenever m_{eff}^2 is initially positive and small, then increases, decreases, becomes negative and finally becomes small positive again, some sharp features (known as mountains in Mukhanov's paper) arise in the density perturbation spectrum. See figure 5.

In the same paper, they also proved that certain conditions between the effective mass of the field and the Hubble parameter can produce some sinusoidal fluctuations in the spectrum. This effect is shown in the figure 6.

Another way for generating a non-flat spectrum is introduced in a paper by Adams, Cresswell and Easther [9]. They show that a sharp step in the inflaton potential generates k-dependent oscillations in the spectrum of primordial density perturbations. The amplitude and extent in wave number of these oscillations depend on both the magnitude and the gradient of the step in the inflaton potential.

Another way to generate such non-flat perturbation spectrum is illustrated in a paper by Starobinsky [1]. In his paper, Starobinsky introduces a method in which some singularities in the inflaton potential lead to a non-flat potential. For example, an abrupt change in the derivative of V will lead to apparition of a step with some other modulations in the shape of the density spectrum. In our study, we will be interested in reconstructing the scattering potential using the Starobinsky's method. Utilizing the fact that every step in the primordial power spectrum (obtained from say, the CMB anisotropy power spectrum) resembles a rounded slope change in the inflaton potential, we make a sequence of scatterings through a number of Dirac delta functions in the effective mass of inflaton, which leads us to recovering the scattering potential features. So from now on we will focus our study on the Starobinsky method which can be considered as the framework of this project.

[16] In a power-law inflation model, the inflaton potential is given by $V(\phi) = V_\circ \exp(\alpha\phi)$ where α and $V_\circ$ are constant. This model does not satisfy the condition of a minimum in which inflation ends; it permits inflation to continue forever. However, As Liddle explains in his paper [5], power-law inflation gives more satisfactory predictions for inflation with respect to other models.

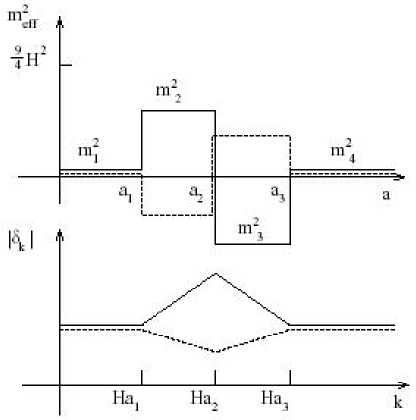

Figure 5: The spectra of the perturbations which are produced by the inflation in the models which include two scalar fields are represented for two different cases. The spectrum with the mountain type peculiarity which is depicted by the solid line corresponds to the time dependence of the mass squared represented by the solid line. When the changes of the sign of the squared mass is opposite to the previous case (dashed line) the spectrum has a 'well'-type peculiarity. (Fig. taken from [15])

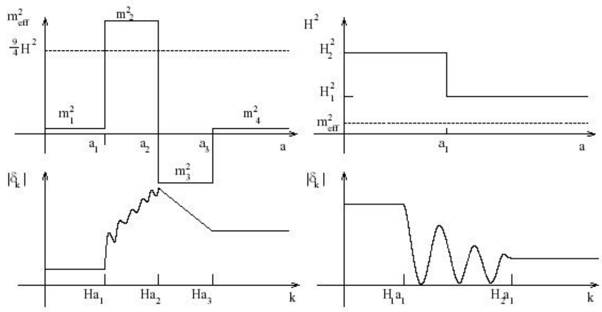

Figure 6: In the left figure we have partial modulation of the spectrum for $Ha_1 \ll k \ll Ha_2$ and the long wave ($k \ll Ha_1$) part of the spectrum can be suppressed in comparison with the short wave one ($k \gg Ha_3$). In the right there is a full spectrum modulation in some range of the wave numbers. (Fig. taken from [15])

4.2 Generating Steps in the Power Spectrum

This section is dedicated to the representation of the Starobinsky's method [1] for generating a step in the power spectrum. We show that how a singularity in the inflaton potential can lead to a step in the density power spectrum.

As we noted before, in the inflationary stage, the inflaton field is in a slow-roll, with

$$|\ddot{\phi}| \ll 3H |\dot{\phi}| \ , \qquad \dot{\phi}^2 \ll 2V(\phi) \tag{34}$$

On the other hand, it can be shown that for an approximately flat spectrum the condition

$$\left| \frac{d\ln(k^3\phi^2(k))}{d\ln k} \right| \ll 1 \tag{35}$$

is established. Under the above inequality (34),(35) and with a small spatial curvature in the Universe, we deduce that after the beginning of the inflationary stage, we have $H^2 \simeq \frac{8}{3}\pi GV$ and $\dot{\phi} \simeq -\frac{V'}{3H}$. Therefore the condition (34) can be rewritten in terms of limitations on the derivatives of V:

$$|V''| \ll 24\pi GV \ , \qquad V'^2 \ll 48\pi GV^2 \tag{36}$$

Here we are interested in obtaining a broken scale-invariant spectrum of adiabatic perturbations, while the statistical nature of the spectrum does not change. One way to do this is to abandon the conditions of slow-roll, i.e. (34). For a discussion about this see [1]. In order to do that, it is reasonable to assume that inequalities (34) and (36) hold everywhere, except in a narrow region $\phi \simeq \phi_\circ$. The spectrum then becomes flat far from the point $k_\circ = a(t_\circ)H_\circ$, where we have $H_\circ^2 = \frac{8}{3}\pi GV(\phi)$, where $t_\circ$ is the time at which the equality $\phi = \phi_\circ$ holds. (For simplicity, we shift the origin of the time scale to satisfy $t_\circ = 0$.) We set $x = \phi - \phi_\circ$. If $V(x)$ has a singularity in the form of a rounded jump in the second derivative or a weaker singularity near the point $x = 0$, then it follows from the satisfaction of conditions (34) before and after passage through the point $x = 0$ that the perturbation spectrum remains flat. The minimal local singularity in $V(x)$ sufficient to give rise to a non-flat spectrum is a rounded slope change:

$$V(x) = V_\circ + v(x) \ , \qquad v(x) \simeq \begin{cases} A_+ x & \text{if } x \gg x_\circ, \\ A_- x & \text{if } x < 0, \quad |x| \gg x_\circ \end{cases}$$

$$v(0) = 0 \ , \quad A_+ > 0 \ , \quad A_- > 0 \ , \quad A_+ \neq A_-. \tag{37}$$

Where $x_\circ$ is a scale width of the transition region. It can be shown that if the slow roll is to be disrupted during transition, we must have $|A_+ - A_-| \geq H_\circ^2 x$ which is equivalent to conditions $x_\circ \ll G^{-\frac{1}{2}}$ and $max(A_+, A_-).x_\circ \ll V_\circ$. Thus in calculating $a(t)$, we can ignore the contribution of v to the potential, and we have $a(t) = a(0)\exp(H_0 t)$ near the Horizon.

The equations for inhomogeneous perturbations of the field ϕ can be reduced to a single equation for $\zeta = \delta\phi$. As we already derived, the Fourier component of ζ obeys equation (27). We can rewrite this equation in terms of the derivatives of ζ with respect to η.[17]

$$\eta^2 \frac{d^2\zeta_k}{d\eta^2} - 2\eta \frac{d\zeta_k}{d\eta} + \left(k^2\eta^2 + \frac{m_{eff}^2}{H_\circ^2}\right)\zeta_k = 0 \tag{38}$$

Where m_{eff}^2 is given by equation (26):

$$m_{eff}^2 = \frac{d^2V}{d\phi^2} + 8\pi G \frac{\dot{\phi}}{H}\frac{dV}{d\phi} + H\frac{d}{dt}\left(\frac{\dot{H}}{H^2}\right)$$

By virtue of equation (37), it is apparent that $\frac{d^2V}{d\phi^2}$ makes the basic contribution in the effective mass, since we have $\dot{H} = 4\pi G\dot{\phi}^2$. Considering the rounded slope in the potential, we can replace the quantity m_{eff}^2 in (38) by $[3H_\circ(A_+ - A_-)/A_+]\delta(t)$ under some approximations. Since in our assumptions t can take any value between $-\infty$ to $+\infty$, the cosmic time η varies between $-\infty$ to 0; and at $t_\circ$, η is equal to $\eta_\circ \equiv -\frac{1}{k_\circ}$, where $k_\circ$ is the wave number of the mode which reaches the Horizon at the origin of time t.
Equation (38) can be modified in the form of a Bessel equation, by solving which we can find ζ_k for $\eta \neq \eta_\circ$, which is

$$\zeta_k = \begin{cases} Ae^{-ik\eta}(-\eta + \frac{i}{k}) + Be^{ik\eta}(-\eta - \frac{i}{k}) \ , & \eta < \eta_\circ \\ Ce^{-ik\eta}(-\eta + \frac{i}{k}) + De^{ik\eta}(-\eta - \frac{i}{k}) \ , & \eta > \eta_\circ \end{cases} \tag{39}$$

In anticipation of future use, we are interested in finding the 'out'-state of the wave with respect to its 'in'-state before horizon crossing. To do that we should find C and D in terms of A and B. Under the quantum mechanical scattering theory, we know that the perturbation component ζ_k remains continuous at $\eta = \eta_\circ$. This leads us to a useful equation

$$C = A - \left(\frac{k - ik_\circ}{k + ik_\circ}\right)e^{2ik\eta_\circ}(D - B)\,. \tag{40}$$

On the other hand, integrating equation (38) by part over the interval $[\eta_\circ - \epsilon, \eta_\circ + \epsilon]$, where ϵ is a small number, gives another equation

$$i\frac{k^2}{k_\circ^2}\left(De^{ik\eta_\circ} - Ce^{-ik\eta_\circ} - Be^{ik\eta_\circ} + Ae^{-ik\eta_\circ}\right) + \frac{3}{H_\circ}\left(1 - \frac{A_-}{A_+}\right)\left(Ae^{-ik\eta_\circ}(k + ik_\circ) + Be^{ik\eta_\circ}(k - ik_\circ)\right) = 0 \tag{41}$$

Combining equations (40) and (41), one can directly obtain two expressions for C and D in terms of the incoming parameters:

$$\begin{aligned} C &= \alpha(k)A + \beta^\star(k)B, \\ D &= \beta(k)A + \alpha^\star(k)B \end{aligned} \tag{42}$$

[17]Notice that η is related to t by $\eta = \int \frac{dt}{a(t)} = \int dt\ a_{(0)}\exp(H_\circ t) = -(H_\circ a(t))^{-1}$.

Where

$$\alpha(k) = 1 + \frac{3i}{2}\Big(\frac{A_-}{A_+} - 1\Big)\frac{k_\circ}{k}\Big(1 + \frac{k_\circ^2}{k^2}\Big)a_\circ \tag{43}$$

$$\beta(k) = -\frac{3i}{2}\Big(\frac{A_-}{A_+} - 1\Big)\exp(2i\frac{k}{k_\circ})\frac{k_\circ}{k}\Big(1 + \frac{ik_\circ}{k}\Big)^2 a_\circ \tag{44}$$

and the star superscript refers to the complex conjugate of each parameter. The expressions (42) are useful in § 5.2, where we study the effect of a large number of singularities in the inflaton potential on the shape of the power spectrum.

Going back to the solution of equation of motion of a Fourier component of ζ, equation (39) is not a correct solution for ζ_k. To obtain an exact solution, it must be normalized and the initial conditions should be satisfied. The initial conditions come from the state of the perturbations before $\eta = \eta_\circ$. Since the perturbation ζ_k can only be an 'in' vacuum fluctuation, the coefficient of the component of the wave which is going toward negative values of t, must be zero, i.e. $B = 0$, since $\eta < 0$. By assuming this fact and normalizing the waves, we find the correctly normalized solution for ζ_k which is

$$\zeta_k = \frac{H_\circ}{\sqrt{2k}}\Bigg\{\Big(e^{-ik\eta}(-\eta + \frac{i}{k})\Big)\Theta(\eta - \eta_\circ) + \Big(\alpha(k)e^{-ik\eta}(-\eta + \frac{i}{k}) + \beta(k)e^{ik\eta}(-\eta - \frac{i}{k})\Big)\Theta(\eta_\circ - \eta)\Bigg\} \tag{45}$$

$$\text{where} \qquad \Theta(\eta - \eta_\circ) = \begin{cases} 1\,, & \text{if} \quad \eta > \eta_\circ \\ 0\,, & \text{if} \quad \eta < \eta_\circ \end{cases}$$

In case we choose $a_\circ$ equal to unity, there is a relation between $\alpha(k)$ and $\beta(k)$, given by

$$\mid \alpha(k) \mid^2 + \mid \beta(k) \mid^2 = 1 \tag{46}$$

which resembles a conservation equation. In quantum mechanics view, $\mid \alpha(k) \mid^2$ and $\mid \beta(k) \mid^2$ are the transmission and the reflection coefficients of the scattering problem, respectively. In inflation view, the quantity $\mid \beta(k) \mid^2$ also can be interpreted as the number of pairs of scalar particles with momenta $\mathbf{k}$ and $-\mathbf{k}$ which are created because of the rapid variation in ϕ. However, we are interested in that part of the effect which contributes to a growing model of the perturbations. This part is determined by the asymptotic behavior of ζ_k as $t \to \infty$ ($\eta \to 0$): $\zeta_k(\infty) = i\frac{H_\circ}{\sqrt{2k^3}}(\alpha - \beta)$. From § 3.2 we know that the density power spectrum is proportional to $k^3 \mid \zeta_k \mid^2$. So in the present stage of the Universe that primordial perturbations of the early Universe can be considered as the limit of ζ_k at $t \to \infty$, the present power spectrum of perturbations and consequently the CMB power spectrum is proportional to $D^2(y)$, where:

$$P_\zeta(k) \propto D^2 = \mid \alpha - \beta \mid^2 = 1 - 3\Big(\frac{A_-}{A_+} - 1\Big)\frac{1}{y}\Bigg(\Big(1 - \frac{1}{y^2}\Big)\sin 2y + \frac{2}{y}\cos 2y\Bigg) + \frac{9}{2}\Big(\frac{A_-}{A_+} - 1\Big)^2\frac{1}{y^2}\Big(1 + \frac{1}{y^2}\Big)\Bigg(1 + \frac{1}{y^2} + \Big(1 - \frac{1}{y^2}\Big)\cos 2y - \frac{2}{y}\sin 2y\Bigg)$$

$$y = \frac{k}{k_\circ}\,, \qquad D(0) = \frac{A_-}{A_+}\,, \qquad D(\infty) = 1. \tag{47}$$

We see that for a given y the slope of the function depends on only $\frac{A_-}{A_+}$, which is the ratio of the slopes of the inflaton potential at the singularity.
The function $D(y)$ as is sketched in Fig. 7 with respect to y, is a step (with some superimposed modulations) with an increase toward the large scale values if $A_- > A_+$ and toward small scale values in the opposite case.

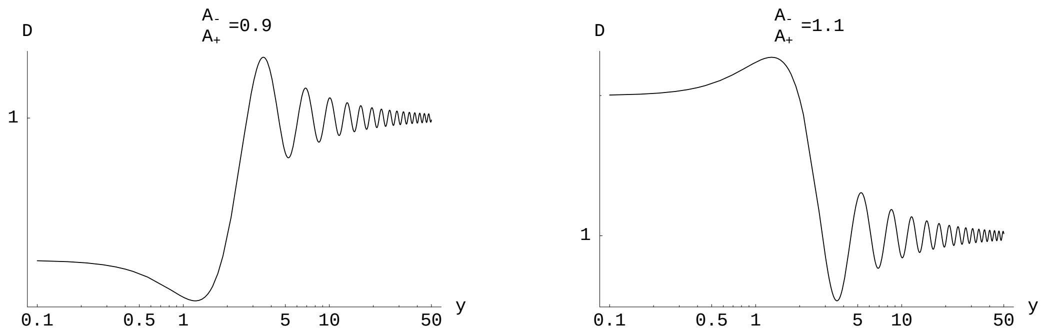

Figure 7: Plot of function D with respect to y for two different cases $A_- < A_+$ and $A_- > A_+$.

This function has two other interesting properties sketched in Fig. 8:
(1) In the case $A_- \gg A_+$, the function $D(y)$ falls off slowly toward large values of y. (This property was once thought useful to explain some problems in the explaining the large scale structure in the Universe.)
(2) In the case $A_- \ll A_+$, the function $D(y)$ falls off $\propto\ y^2$ toward small values of y only as long as the condition $D \gg D(0)$ holds. Then $D(y)$ has a deep minimum at $y = \sqrt{\frac{5A_-}{2A_+}}$. This property is interesting in some numerical calculations for polynomial potentials.

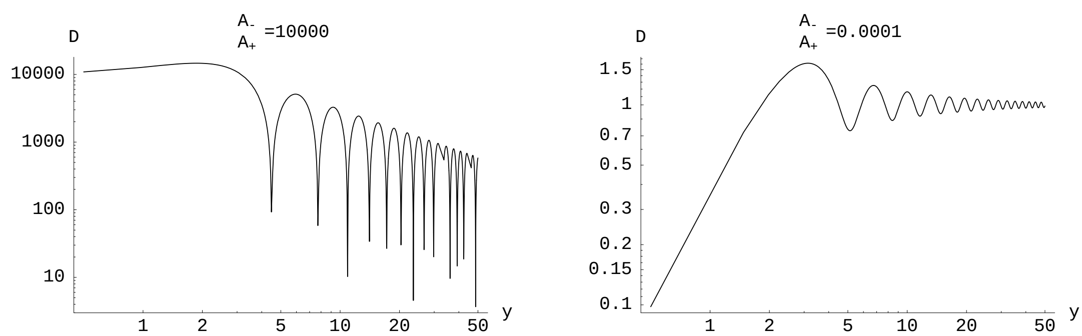

Figure 8: Plot of function D with respect to y for two different cases $A_- \gg A_+$ and $A_- \ll A_+$.

We have seen that how a singular inflaton potential can lead to generating a step in the density power spectrum. There are some other conditions on the inflaton potential which lead to some particular spectra. For example, if the singularity in the potential is instead a rounded jump:

$$V(x) = V_\circ + Ax + v(x)\ , \qquad v(x) \simeq \begin{cases} \frac{\Delta}{2} & \text{if } x \gg x_\circ, \\ -\frac{\Delta}{2} & \text{if } x < 0, \quad |\, x\,| \gg x_\circ \end{cases}$$

$$x = \phi - \phi_\circ\ , \quad v(0) = 0\ , \quad A > 0\ , \quad \Delta > 0. \tag{48}$$

As before the term $\frac{d^2V}{d\phi^2}$ makes the basic contribution in m^2_{eff}. However, in this case the quantity m^2_{eff} in (38) has a singularity stronger than a δ-function in the formal limit $x_\circ \to 0$. The result for the spectrum is thus non-universal, i.e. it depends on the shape of $v(x)$ and on the value of $x_\circ$.

These cases are not interesting in our study whatsoever. The interesting thing for us is that we can construct steps in the power spectrum by assuming some singularities in the inflaton potential. In the next section, we use the Starobinsky's method for reconstructing singularities of the inflaton potential from steps of toy power spectra and consequently generating some Dirac delta functions in the effective mass.

5 Recovering Features of the Scattering Potential

As we have seen in figure 7 , if the potential of the scalar field which controls the inflationary epoch in the early Universe has a singularity, a step of a Universal form arises in the spectrum of density perturbations. Along with this step, there are some superimposed modulations, which are not interesting since they arise solely due to the singular change in slope. In any realistic model, the singularity is only an approximation to a rounded change in slope.

In this section we use this fact to recover features of the scattering potential of inflation.

First we explain how we can use singularities in the inflaton potential for recovering the scattering potential. Then we construct some simple power spectra from some simple effective mass patterns consisting of a number of Dirac delta functions. This discussion will be useful for us to get some ideas about the basic problem which is the derivation of features of the effective mass from the primordial power spectra. We are interested in using some toy power spectra in order to find the effective mass of the inflaton and consequently features of the initial scattering potential of inflation. We present our results of finding features of the effective mass for different cases. The method we use has not been published yet and the results are original.

5.1 Our Approach to Recovery of the Scattering Potential

We have noted before that the field equation for the modes of the inflaton perturbation, $\delta\phi_k$, can be obtained from the equation of motion of the inflaton field, and is given by equation (28)

$$\nu_k'' + \Big[k^2 - \frac{\theta''}{\theta}\Big]\nu_k = 0 \tag{49}$$

where $\delta\phi_k = \frac{\nu_k}{a}$, and $\theta = a^2\frac{\phi'}{\theta}$ is called the inflationary scattering potential. It can be shown that there is a relation between the effective mass of inflaton and V_s, that is

$$a^2 m_{eff}^2 = \frac{a''}{a} - V_s \tag{50}$$

With this equation we can recover the inflationary scattering potential whenever the behavior of $\frac{a''}{a}$ and $a^2 m_{eff}^2$ is known. In every model of inflation, the behavior of $a(t)$ and consequently $\frac{a''}{a}$ is straightforward. However, the behavior of the effective mass, as is given by equation (26) is rather complicated.

Our method for finding m_{eff}^2 is based on the Starobinsky's method described in § 4.2. Recent results WMAP measurement of the power spectrum of the CMB anisotropy suggest ds that the primordial spectrum consists of at least one big step. Moreover, if someone zooms into the power spectrum, he/she can find more and more steps with different heights and step-sizes.[18] This can be interpreted as some singularities (sharp slope changes) in the inflaton potential.[19]

[18]If we skip the high-frequency modes of the spectrum in the Fourier space, these small steps become precisely apparent.

[19]This is the property of every experimental data that it always consists of a discontinuous set of numbers. So even in each point of data we can easily find a discontinuity in the slope unless the

Let us proceed with the notion that there are some step-like features in the power spectrum. Now suppose that in some modes of the perturbation, say k_i, the power spectrum has large steps. Suppose again that each of these modes crosses the horizon at time t_i, where $k_i = a(t_i)H_\circ$. Considering the Starobinsky's method, at each t_i the slope change of the inflaton potential is large, since it is assumed that at each k_i the step size in the power spectrum is significant. Since each singularity in the inflaton potential leads to generating a Dirac delta function in the m^2_{eff} plot, we can simulate these assumptions by introducing a sequence of delta functions in the effective mass of the inflaton, corresponding to the set of singularities in the inflaton potential. On the other hand, to each delta function a constant coefficient γ_i can be assigned, which gives the total information about the amount of change of the slope of the inflaton potential at its corresponding singularity, since $\gamma_i = 3H_\circ\Big(1 - (\frac{A_-}{A_+})_i\Big)$. Now consider a perturbation coming from the Prue-inflationary epoch, when the Universe was dominated by a phase of vacuum-state. In § 3.4 we have discussed that before crossing the horizon the amplitude of each mode of perturbation is damped and after horizon crossing it freezes. However, the interesting thing happens on the horizon crossing. We assume a mode of the perturbation with wave number k, which is going through an inflaton potential that has some singularities at t_i's. In this case, in each singularity we have a quantum mechanical scattering by a Dirac delta function potential, which can be easily solved by the method presented in § 4.2.
It is very interesting that the amount of scattering by one delta function at k_i may not be equal to that of another delta function at k_j $(i \neq j)$. It means that each delta function has its own transmission and reflection coefficients dependent on its own γ_i. In this case for each mode of the perturbation we can find two expressions for $\alpha_T(k)$ and $\beta_T(k)$, the norm squared of which correspond to total transmission coefficient and total reflection coefficient, respectively. Now if the values of γ_i are given, we can find the corresponding power spectrum by plotting $\mid \alpha_T - \beta_T \mid^2$ for each mode k. In the reverse case, if the values of γ_i are unknown but we have the power spectrum instead, we can find a set of equations for the parameters γ_i and solve them to find the corresponding coefficient of each delta function and consequently the slope change of the inflaton potential at each singularity.

The method discussed above is essentially about reconstructing the power spectrum from the effective mass or recovering the inflaton potential from the scattering potential. However, once we can understand it well and could use it for constructing some simple power spectra from simple effective masses, we can easily utilize it in the reverse case, to derive features of the effective mass from the density power spectrum.

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data values are equal.

5.2 Designing Some Simple Power Spectra

We know that for a given set of delta function scatterings, we can always find a transmission and reflection coefficient, which both depend on the wave number of the mode. Having this coefficients one can easily construct the power spectrum. Suppose that our inflaton potential has n singularities at times t_i. For i-th scattering corresponding to the i-th singularity, it is possible to write the expressions (42) in the matrix notation:

$$\begin{pmatrix} \phi_{out}^{i,(+)} \\ \phi_{out}^{i,(-)} \end{pmatrix} = \begin{pmatrix} \alpha_i(k) & \beta_i^\star(k) \\ \beta_i(k) & \alpha_i^\star(k) \end{pmatrix} \begin{pmatrix} \phi_{in}^{i,(+)} \\ \phi_{in}^{i,(-)} \end{pmatrix} \tag{51}$$

Here the subscripts 'in' and 'out' refer to the state of the mode, before and after i-th scattering, respectively. The superscripts $(+)$ and $(-)$ as before denote the component of the wave which is heading toward positive or negative t's. We call the 2×2 matrix of equation (51), the *scattering matrix* of the i-th singularity. With this notation, it is very easy to construct the power spectrum for a given effective mass which consists of a sequence of delta functions. To do this we just multiply the scattering matrices to obtain the total scattering matrix $\mathbf{S_T}$. The final 'out' state of the perturbation mode is thus related to its initial 'in' state by

$$\begin{pmatrix} \phi_{out}^{(+)} \\ \phi_{out}^{(-)} \end{pmatrix} = \mathbf{S_T} \begin{pmatrix} \phi_{in}^{(+)} \\ \phi_{in}^{(-)} \end{pmatrix},$$

$$\mathbf{S_T} = \begin{pmatrix} \alpha_n(k) & \beta_n^\star(k) \\ \beta_n(k) & \alpha_n^\star(k) \end{pmatrix} \begin{pmatrix} \alpha_{n-1}(k) & \beta_{n-1}^\star(k) \\ \beta_{n-1}(k) & \alpha_{n-1}^\star(k) \end{pmatrix} \cdots \begin{pmatrix} \alpha_1(k) & \beta_1^\star(k) \\ \beta_1(k) & \alpha_1^\star(k) \end{pmatrix} \tag{52}$$

It is very interesting that the form of $\mathbf{S_T}$ is the same as the form of each scattering matrix; i.e.

$$\mathbf{S_T} = \begin{pmatrix} \alpha_T(k) & \beta_T^\star(k) \\ \beta_T(k) & \alpha_T^\star(k) \end{pmatrix} \tag{53}$$

It can be easily shown that by equation (30) and (45) the density power spectrum is given by

$$P_\zeta(k) = \frac{H_\circ^2}{4\pi^2} \mid \alpha_T(k) - \beta_T(k) \mid^2 \tag{54}$$

The power spectrum derived by the above equation has got some interesting features. One of these features is that at every singularity that $A_- > A_+$, the coefficient of the corresponding delta function, say $\gamma_i = 1 - \left(\frac{A_-}{A_+}\right)_i$, is negative. In this case there is a step-down toward large wave numbers in the spectrum at k_i. In the reverse case, when we have $A_- < A_+$ at i-th singularity, γ_i is positive and it can be seen that there is a step-up in the power spectrum at k_i (toward large wave numbers).

In the next few pages we present our results of constructing some power spectra from toy effective masses. For each result we discuss properties of the effective mass and steps of the spectrum, to which the effective mass leads. Then in the next section we reconstruct features of the effective mass of inflaton for one of the spectrums which we design in this section.

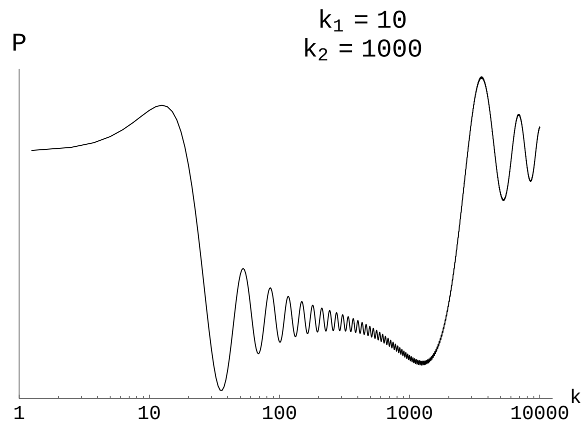

Figure 9: Power spectrum of perturbations for an inflaton potential which consists of two singularities at k_1 and k_2 with $\gamma_1 = -0.0001$ and $\gamma_2 = 0.0001$. The first singularity represents a negative delta function, and the second represents a positive delta function in the effective mass. The step-down and step-up in the spectrum are apparent corresponding to negative and positive delta, respectively.

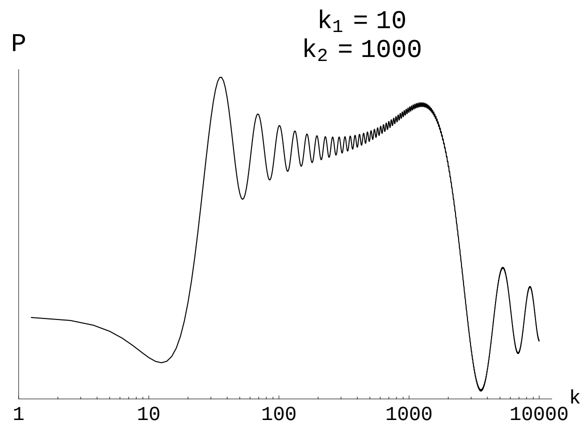

Figure 10: Here, the effective mass consists of two Dirac delta functions at k_1 and k_2 with $\gamma_1 = 0.0001$ and $\gamma_2 = -0.0001$. This leads to apparition of two steps in the power spectrum, one step-up and one step-down as sketched above. The superimposed modulations are not interesting for us.

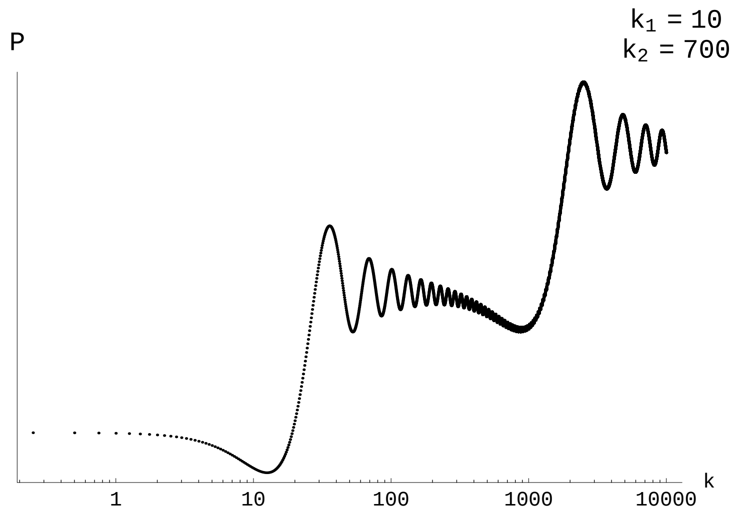

Figure 11: Introducing two positive delta functions in the effective mass gives rise to two step-up's in the power spectrum. ($\gamma_1 = \gamma_2 = 0.0001$)

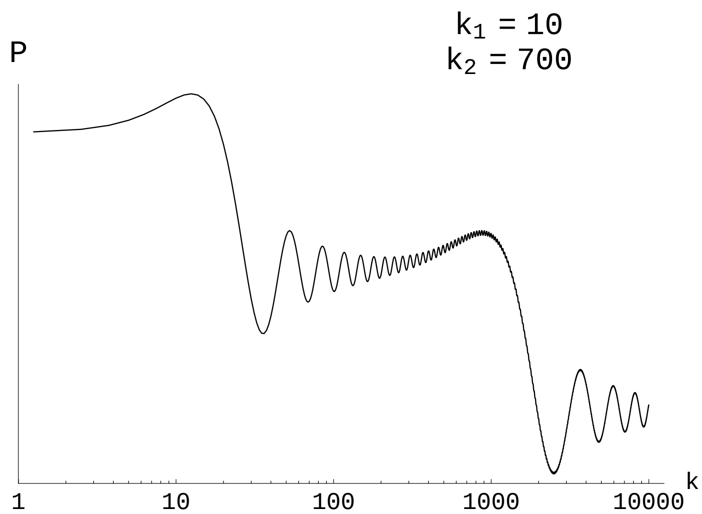

Figure 12: Introducing two negative delta functions in the effective mass gives rise to two step-down's in the power spectrum. ($\gamma_1 = \gamma_2 = -0.0001$)

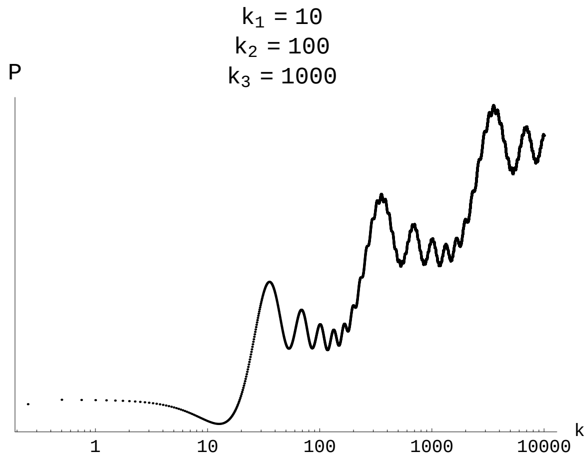

Figure 13: This figure shows three step-up's in the power spectrum. These steps have been appeared because of introducing three positive delta functions in the effective mass of inflaton. ($\gamma_1 = \gamma_2 = \gamma_3 = 0.0001$)

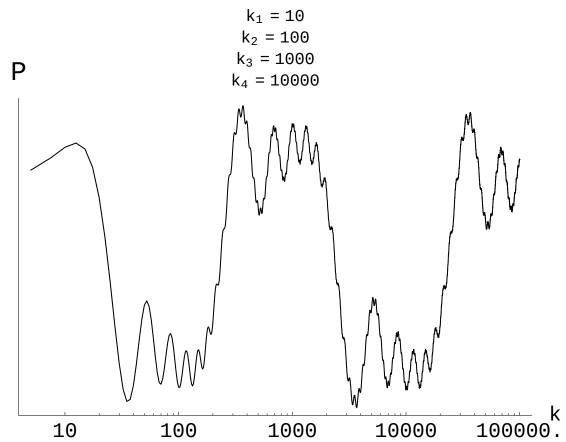

Figure 14: Power spectrum of perturbations for an effective mass which consists of four alternating positive-negative delta functions. As before the step-up's come from positive deltas and the step-down's from negative ones. ($\gamma_1 = \gamma_3 = -0.0001$, $\gamma_2 = \gamma_4 = 0.0001$)

5.3 Deriving Features of the Effective Mass of Inflaton

We have seen that a Dirac delta function in the effective mass of inflaton gives rise to apparition of a step in the power spectrum of perturbations. If the delta function is a positive delta -whose peak is toward positive values of m_{eff}^2- the step in the spectrum is a step-up. On the other hand, if the delta function is negative, a step-down appears in the spectrum.

In this section we are interested in using these facts to find features of the effective mass for one of the spectra derived in § 5.2. The power spectrum which we are using is sketched in figure 15. It consists of two steps, one step-down at $k_1 = 10$ and one step-up at $k_2 = 1000$. The spectrum is constructed by the method of § 5.2 for an effective mass which consists of two delta functions at k_1 and k_2 with $\gamma_1 = -0.0001$ and $\gamma_2 = 0.0001$. Here, we assume that the coefficients of delta functions, γ_1 and γ_2, have not been given and we try to derive them from power spectrum.

The key to the problem is very easy by remembering § 5.2. We know that for the power spectrum of figure 15, number of the delta functions in the effective mass of inflaton is n=2. Then by equations (52) and (53), $\alpha_T(k)$ and $\beta_T(k)$ can be written in terms of α_1, β_1, α_2 and β_2, and by equations (43) and (44) and considering $\gamma_i = 1 - \left(\frac{A_-}{A_+}\right)_i$ we can write $\alpha_T(k)$ and $\beta_T(k)$ in terms of γ_1 and γ_2 and the wave number of the mode k. Substituting α_T and β_T into the equation (54), we can express power spectrum of perturbations in terms on γ_1 and γ_2 and k:

$$P_\zeta = P_\zeta(\gamma_1, \gamma_2, k) \tag{55}$$

Now, we can use power spectrum of figure 15 for deriving values for γ_1 and γ_2. From equation 55, it is clear that for each k there is an equation for γ_1 and γ_2; i.e. for the spectrum of figure 15 which consists of 10000 points, we can find 10000 equations dependent on γ_1 and γ_2. We solve each two consecutive equations together[20] to find γ_1 and γ_2; i.e. we solve sets of equations consist of equation (55) for two wave numbers k and $k + \Delta k$[21] and find γ_1 and γ_2 for each k.

By sketching results of solving sets of equations, we find two plots for γ_1 and γ_2 with respect to k, as depicted in figures 17 and 19. From these figures it is apparent that the accuracy of results are very good, since most of the points are set on a line near the exact value. However, there are some large fluctuations which lead to a mean value different from the exact value of γ_1 and γ_2. By omitting these large fluctuations, the distribution of results are better seen in figures 16 and 18 and the mean values of γ_1's or γ_2's are modified to be equal to their exact values to an accuracy of 10^{-5}. In the next pages the results for recovering γ_1 and γ_2 from power spectrum of figure 15 are presented.

[20] We can solve any two equations together without any preferences in their order. However, as we will see in the next pages, solving consecutive equations leads to obtain a plot for results of γ_1 and γ_2 with respect to the wave number k.

[21] Δk is the difference between wave numbers of each two consecutive modes in figure 15.

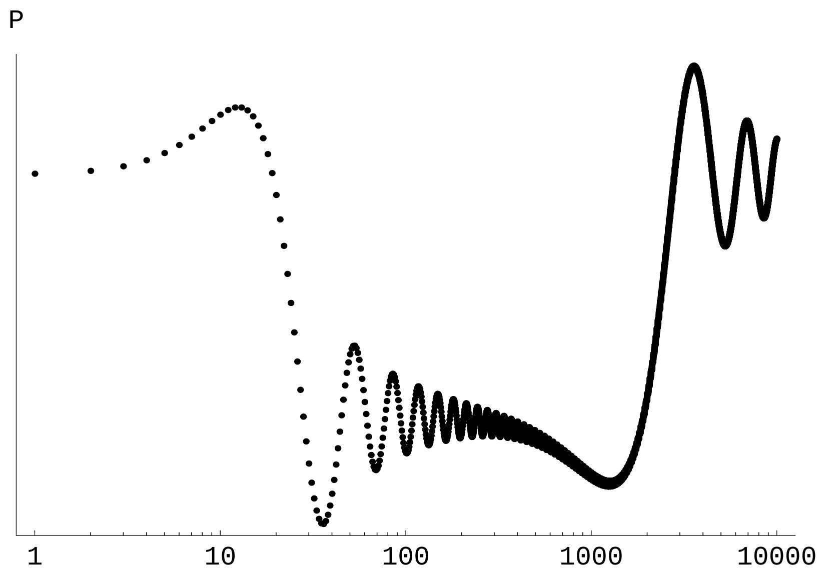

Figure 15: This figure shows one of the power spectra which have been constructed by method of § 5.2. The effective mass which gives rise to this spectrum consists of one negative delta function at $k_1 = 10$ with $\gamma_1 = -0.0001$ and one positive delta function at $k_2 = 1000$ with $\gamma_2 = 0.0001$. The spectrum is sketched for 10000 different modes of the perturbation. We assume that the values of γ_1 and γ_2 are unknown and try to derive them from the power spectrum. The results are shown in the next few pages.

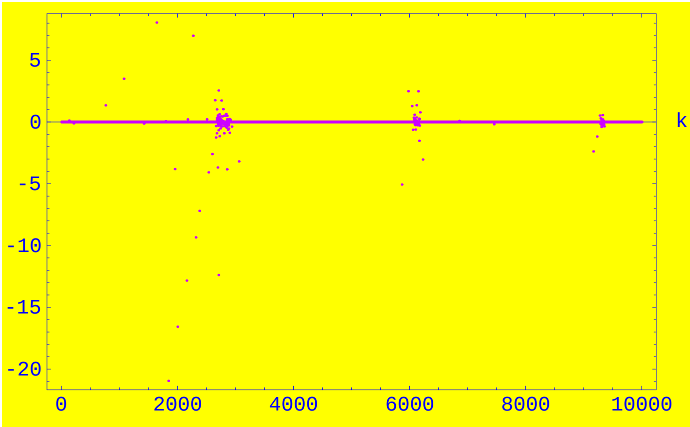

Figure 16: The distribution of the results of deriving γ_1 for power spectrum of figure 15 is depicted. We can find one equation for each mode k which depends on γ_1 and γ_2. By solving every two consecutive equations, we find a plot for γ_1 versus k, as sketched above.

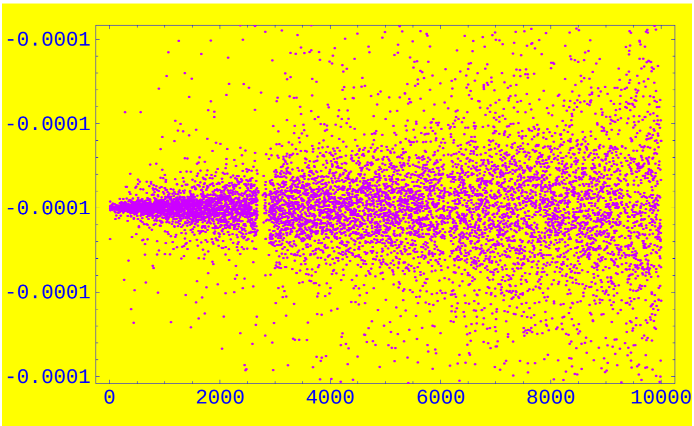

Figure 17: This figure is a close view of figure 16, as we zoom into it. We can interpret this figure as the final results for γ_1 when the large fluctuations of figure 16 are emitted. It is clear that the mean value of γ_1 is equal to -0.0001. The accuracy of results is up to 10^{-5}.

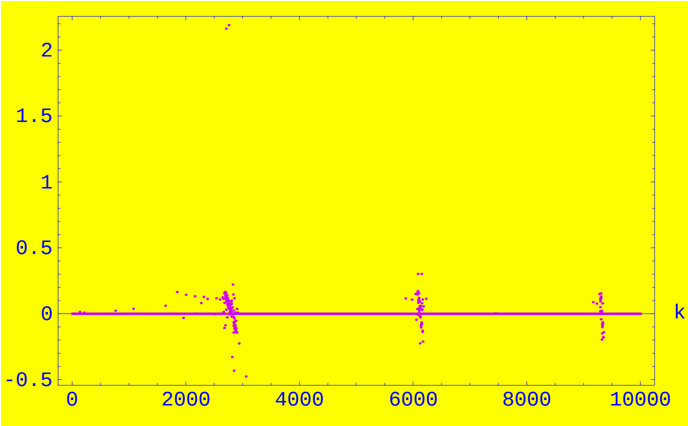

Figure 18: The distribution of the results of deriving γ_2 for power spectrum of figure 15 is depicted. We can find one equation for each mode k which depends on γ_1 and γ_2. By solving every two consecutive equations, we find a plot for γ_2 versus k, as sketched above.

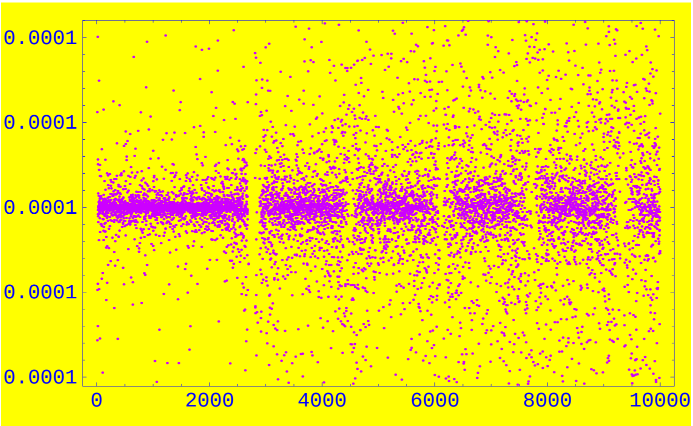

Figure 19: This figure is a close view of figure 18, as we zoom into it. We can interpret this figure as the final results for γ_2 when the large fluctuations of figure 18 are emitted. It is clear that the mean value of γ_2 is equal to 0.0001. The accuracy of results is up to 10^{-5}.

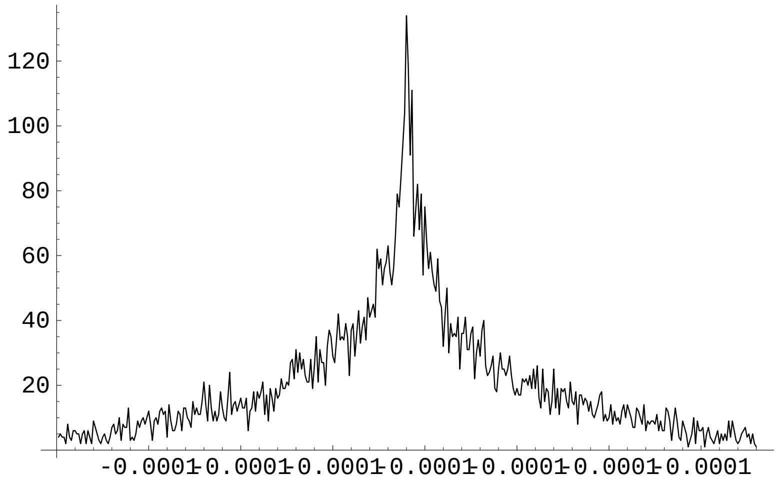

Figure 20: This figure represents the frequency of γ_1 in the figure 17. The gaussian distribution with a peak at -0.0001 is rather surprising.

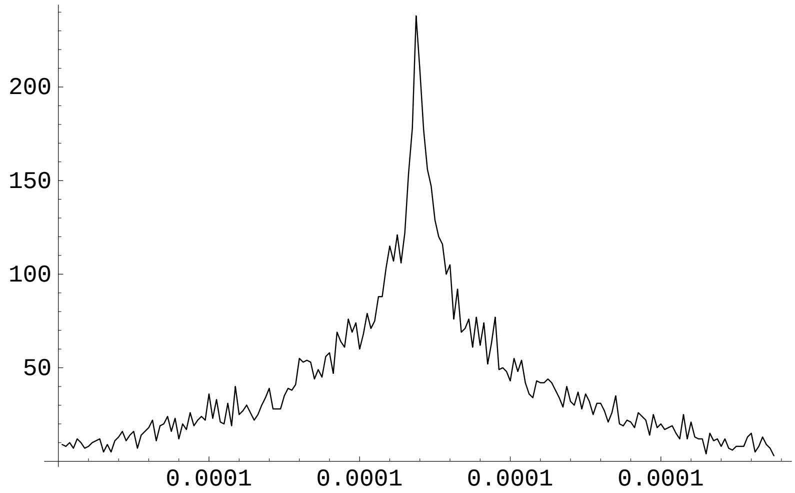

Figure 21: This figure represents the frequency of γ_2 in the figure 19. The gaussian distribution with a peak at 0.0001 is rather surprising.

6 Discussion and Future Prospects

In this report we presented a method for determining features of the inflationary scattering potential. According to the Starobinsky's method every step in the CMB power spectrum can be considered as a singularity in the inflaton potential. It can be shown that under some assumptions, the existence of a singularity in the inflaton potential is equivalent to the existence of a Dirac delta function in the effective mass of inflaton. In § 5.2 we constructed some power spectra from some toy effective masses consisting of delta functions. Then in § 5.3 we solved the reverse problem, i.e. we considered one of the spectra which we derived in § 5.2 and tried to recover the initial effective mass. We assumed that at some modes for which we have large steps in the power spectrum, there are some Dirac delta functions in the plot of the effective mass. We guessed some places on which Dirac delta functions peaked in the effective mass and obtained some values as coefficients of that delta functions. The remarkable thing is that the results we obtained are very close to the correct values to a very good accuracy, as sketched in figure 20 and 21. The reason of getting such high procession results is that we took advantage of an idealized assumption, according to which the places of the guessed modes were chosen exactly equal to that of the modes on which the delta functions are really peaked in the initial effective mass plot.

For this method to be used for the real CMB power spectrum, the situation is not as easy as for the one we solved here. We have to make some modifications and idealizations to guess places of the delta functions precisely. It is clear that the better the places of the delta functions are chosen, the more accurate results this method leads to.

However, this method has worked well for simple toy effective masses. We are hopeful that it would also work for the real power spectrum of CMB. In the future, we hope to apply this method for reconstructing features of the real inflationary scattering potential from the recent WMAP data.

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