TOWARDS A PATH INTEGRAL FOR PURE SPIN CONNECTION FORMULATION OF GRAVITY

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Abstract

A proposal for the path-integral of pure-spin-connection formulation of gravity is described, based on the two-form formulation of Capovilla et. al. It is shown that the resulting effective-action for the spin-connection, upon functional integration of the two-form field Σ and the auxiliary matrix field ψ is *non-polynomial*, even for the case of vanishing cosmological constant and absence of any matter couplings. Further, a diagramatic evaluation is proposed for the contribution of the matrix-field to the pure spin connection action.

PACS numbers :- 04.20.Cv, 04.20.Fy, 2.40.+m

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In the past few years, it has become increasingly evident that the description of gravity in terms of connection variables instead of the metric, originally due to Ashtekar [1] is well-tailored for the discussion of quantum aspects of the theory. This has been attributed to the close parallel between this description and Yang-Mills theories, topological solutions thereof as well as the invention of loop variables.

Motivation to obtain a natural covariantization of Ashtekar theory, led Capovilla et al to introduce a classical action for gravity (and a one-parameter family of generally covariant gauge-theories) purely in terms of a spin-connection [2]. This action is obtained by solving the classical equation of motion for the 'metric-variable' Σ, from the self-dual two form action for a $SL(2, C)$ connection A, a non-dynamical matrix field ψ and two-forms Σ [3]. The equivalence of the pure-connection theory to that of Ashtekar can be shown by a $3 + 1$ decomposition [4], as well as by comparing the constraints arising due to diffeomorphism and gauge invariances of the theory in the two formulations [5]. In an interesting alternative approach, Peldán [6] performed inverse Legendre transform on the Hamiltonian comprising purely of constraints (characteristic of diffeomorphism invariant theories) and obtained a pure-spin connection action. The apparent discrepancy between the actions of Peldán and Capovilla et al, can be removed by rewriting the tracelessness condition on ψ in the pure-spin connection action [7].

With an overall agreement on the consistency of the pure-spin-connection formulation of gravity at the classical level at hand, it is only natural now to start exploring the quantum properties of it. In a recent paper, Smolin [8] has furnished a path integral for Euclidean case, starting from the Hamiltonian of the 'googly' theory. For eliminating the Gauss law and diffeomorphism constraints from the integrand, he uses the 'time' component of the gauge field as a Lagrange multiplier and solves the diffeomorphism constraints explicitly using the *classical* solution of Capovilla et al [2]. He further proposes to choose gauge-fixing conditions for the A field as linear expressions, so that the gauge field action remains at most quadratic (this happens only in the limit $G_N \to 0$, and hence for the 'googly' theory alone) and the path integral can, be evaluated exactly, producing an 'effective action' for the matrix field and the ghost fields introduced by the Faddeev-Popov determinant. Although motivated in part by Smolin's paper, we wish to make a different proposal for the path integral. We begin with the two-form action for the metric variable Σ, coupled to the gauge-field A and the auxiliary matrix field ψ, as in reference [3]. As in any quantum theory of gravity, the path integral must include fluctuations of the metric, we functionally integrate over Σ first to obtain the effective action for the gauge-field A and ψ. Throughout the discussion the integral over A is only in a formal sense, since we do not display the gauge-fixing terms and the $F-P$ determinant, those will be discussed in a future publication as work is still in progress on these issues.

Consider then the following formal definition of the Euclidean path integral :

$$Z = \int DAD\psi D\Sigma \; e^{-\int \Sigma^a \wedge F^a + \frac{1}{2}\int \psi_{ab}\Sigma^a \wedge \Sigma^b - S_{gf} - S_{FP}} \delta(tr\psi) \tag{1}$$

where S_{gf} and S_{FP} are the gauge-fixing and Faddeev-Popov terms in the action needed for the path integral over the gauge field A^a_μ. The matrix ψ_{ab} is symmetric and $\delta(tr\psi)$ denotes the constraint in the path integral that ψ should be traceless.

To perform the path integral over Σ, we follow the standard procedure and write

$$\Sigma^a = \bar{\Sigma}^a + \sigma^a \tag{2}$$

where σ^a is the fluctuation part and $\bar{\Sigma}^a$ satisfies the classical equation of motion

$$F_a = \psi_{ab}\Sigma^b \tag{3}$$

Substituting in the path integral we get

$$Z = \int DAD\psi D\sigma \; e^{-S}.\delta(tr\psi) \tag{4}$$

where the action S is given by

$$S = \frac{1}{2}\int \psi^{-1ab} F_a \wedge F_b - \frac{1}{2}\int \psi_{ab}\sigma^a \wedge \sigma^b + S_{gf} + S_{FP} \tag{5}$$

The integration over σ^a then produces the factor $\frac{1}{(det\ \psi)}$ in the path integral (in Euclidean signature, Σ's are hermitian) and we write

$$Z = \int DAD\psi e^{-\int \frac{1}{2}\psi^{-1ab}F_a\wedge F_b - S_{gf} - S_{FP}}\frac{1}{(det\ \psi)}.\delta(tr\psi) \tag{6}$$

In order to perform next the integration over the auxiliary field ψ, it is convenient to perform a change of variables from $\psi \rightarrow \phi = \psi^{-1}$. (Invertibility of ψ is anyway assumed in the pure spin connection formulation and thus existence of this transformation is no additional assumption.) The functional integration measure changes accordingly as

$$D\psi \rightarrow D\phi \left| det\ \frac{\partial\psi}{\partial\phi}\right|$$

where the jacobian of the transformation can easily be seen to be

$det(\psi^2) = (det\psi)^2 = \frac{1}{(det\phi)^2}$.

Rewriting the reciprocal determinant arising from Σ integral in terms of $det\phi$, we get

$$Z = \int DAD\phi\frac{1}{(det\ \phi)}e^{-\int \frac{1}{2}\phi^{ab}F_a\wedge F_b - S_{gf} - S_{FP}}.\delta(tr\psi) \tag{7}$$

The constraint that ψ is traceless is equivalent to the constraint on its inverse, viz.

$$(tr\phi)^2 - tr\phi^2 = 0 \tag{8}$$

which follows from the characteristic equation satisfied by a non-degenerate 3 × 3 matrix. We use this equivalent form also because it is this form which leads to the agreement between the actions of Capovilla et al and Peldán [5,7]. The delta function imposing this constraint can be promoted to the action by using its functional representation via the introduction of a (complex) auxiliary field μ [9], as

$$Z = \int DAD\phi D\mu e^{-\int \frac{1}{2}\phi^{ab}F_a\wedge F_b - \int \mu[(tr\phi)^2 - tr\phi^2] - S_{gf} - S_{FP}}.\frac{1}{det\ \phi} \tag{9}$$

The inverse determinant factor can also be promoted to the action as

$$\frac{1}{det\ \phi} = e^{-\delta^{(4)}(0)\int tr\ ln\phi} \tag{10}$$

where the zero-momentum delta function is just the volume of (Euclidean) four space and is to be understood in a regularized sense. We suppress this factor henceforth [10].

The path integral can then finally be written over the gauge field, ψ and μ of the 'effective-action'

$$S_{eff} = \int \frac{1}{2}\phi^{ab} F_a \wedge F_b + \int \mu[(tr\phi)^2 - tr\phi^2] + tr\ ln\phi + S_{gf} + S_{FP} \tag{11}$$

Several remarks are in order at this point. In order to compute the 'effective-action' for the spin-connection, one now needs to integrate over ϕ and μ besides the ghost fields. The action for ϕ is however no more quadratic and so the integral can at best be evaluated by methods of standard perturbation theory. Even treating this as the *classical action* $S_{eff}[A, \phi, \mu]$ for ϕ, the equation of motion

$$\frac{1}{2}\epsilon^{\mu\nu\rho\lambda} F_{a\mu\nu} F_{b\rho\lambda} + 2\mu[(tr\phi)\delta_{ab} - \phi_{ab}] + \phi^{-1}_{ab} = 0 \tag{12}$$

can not be solved in a closed form for ϕ in terms of the curvature of the spin-connection. The upshot of all this is that even for the vanishing cosmological constant case, the $S_{eff}[A, \phi, \mu]$ is *non-polynomial* in F as opposed to the quartic action obtained by solving the equation of motion for the two-form Σ á la Capovilla et al in [3]. Thus the effect of including the fluctuations in Σ is to render the spin-connection action non-polynomial in curvature.

So, to obtain the effective action for the spin-connection we now set up the perturbation theory and obtain the associated Feynman rules for the field ϕ. For this purpose it is convenient to make one more change of variables, viz. $\phi \to \tilde{\phi} = \phi - 1$. The path integral then becomes

$$Z = \int DAD\tilde{\phi}D\mu e^{-S[A,\tilde{\phi}]} \tag{13}$$

where

$$S[A,\tilde{\phi}] = \int \frac{1}{2}\tilde{\phi}^{ab} F_a \wedge F_b + \int \frac{1}{2} trF \wedge F + \int tr\ ln(1+\tilde{\phi}) \\ - \int \frac{1}{2}\mu[(tr\tilde{\phi})^2 - tr\tilde{\phi}^2] - \int \mu[3 + 2tr\tilde{\phi}] \qquad (14)$$

It is interesting to note the presence of the toplogical term $trF \wedge F$ in this action. We can now read off the Feynman rules from the ϕ-part of the action :

$$S[\tilde{\phi}] = \frac{1}{2}\int \tilde{\phi}^{ab} M_{ab,cd} \tilde{\phi}^{cd} + \frac{1}{2}\int J_{ab}\tilde{\phi}^{ab} - \int \sum_n \frac{(-1)^n}{n} tr\tilde{\phi}^n \qquad (15)$$

where

$$M_{ab,cd} = \mu(\delta_{ab}\delta_{cd} - \delta_{ac}\delta_{bd}) \qquad (16)$$

the inverse of which defines the (non-dynamical) ‘propagator’ for the field ϕ and

$$J_{ab} = \epsilon^{\mu\nu\rho\lambda} F_{a\mu\nu} F_{b\rho\lambda} + 4\mu\delta_{ab} \qquad (17)$$

is the ‘source’ coupling linearly to ϕ.

The last term implies arbitrary-order self-interactions of the field ϕ. It is now clear that the effective action for the spin-connection will involve arbitrary powers of $F \wedge F$, obtained by summation of the diagrams arising from the ϕ^n vertices. In the absence of these vertices, one would obtain precisely the action of reference [2] for zero cosmological constant. They produce terms in the action, proportional to the zero-momentum delta function.

Work is in progress on evaluating these contributions to the effective action for the spin-connection as well as choice of gauge conditions and computation of the $F - P$ determinant and will be dealt with in a future publication.

To summarize, we have explicitly demonstrated that the effect of quantum fluctuations of the ‘metric-variable’ Σ is to render the pure-spin connection action non polynomial. Needless to say, it remains so even for non-zero cosmological constant as well as couplings to other fields. We have further set up Feynman rules for the diagramatic evaluation of the effective-action. A byproduct of the perturbation expansion for ϕ is the generation of the topological term for the gauge field representing spin-connection.

It is a pleasure to acknowledge useful conversations with Abhay Ashtekar and Naresh Dadhich and correspondence with the Maryland Relativity Group.

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