

RESTRICTIONS ON THE PHYSICAL PRESCRIPTION FOR THE VISCOSITY IN ADVECTION-DOMINATED ACCRETION DISKS

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ABSTRACT

It has recently been demonstrated that the Shakura-Sunyaev prescription for the kinematic viscosity in an advection-dominated accretion disk yields physically reasonable solutions for the structure of the inflow close to the event horizon. In particular, no violations of relativistic causality occur at the horizon. This is somewhat surprising considering the diffusive nature of the angular momentum transport in the Shakura-Sunyaev scenario, and it is therefore natural to ask whether one can also obtain acceptable solutions for the disk structure based on the various alternative models for the viscosity that have been proposed, including the “deterministic” forms. In this paper we perform a rigorous asymptotic analysis of the structure of an advection-dominated accretion disk close to the

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event horizon of a nonrotating black hole based on three of the alternative prescriptions for the viscosity that have been suggested in the literature.

We constrain the physical disk model by stipulating that the stress must vanish at the horizon, which is the fundamental inner boundary condition imposed by general relativity. Surprisingly, we find that none of the three alternative viscosity prescriptions yield physically acceptable disk structures close to the horizon when the zero-torque condition is applied, whether the flow is in vertical hydrostatic equilibrium or free-fall. Hence we conclude that the original Shakura-Sunyaev prescription is the only one proposed so far that is physically consistent close to the event horizon. We argue that, somewhat ironically, it is in fact the diffusive nature of the Shakura-Sunyaev form that is the reason for its success in this application. Our focus here is on advection-dominated accretion disks, but we expect that our results will also apply to generalized disks provided that losses of matter and energy become negligible as the gas approaches the event horizon.

Subject headings: accretion disks — hydrodynamics — black holes — general relativity

1. INTRODUCTION

The advection-dominated accretion flow (ADAF) model remains a popular scenario for the physical structure of the accretion disks in X-ray underluminous, radio-loud active galactic nuclei (AGNs), as first proposed by Narayan & Yi (1994, 1995). The sub-Eddington accretion rates in these systems cause the plasma to be rather tenuous, which greatly reduces the efficiency of two-body radiative processes such as thermal bremsstrahlung. Consequently, the ratio of the X-ray luminosity divided by the accretion rate is much lower than that associated with luminous AGNs, which presumably have near-Eddington accretion rates. In the standard ADAF scenario, the ions absorb most of the energy dissipated via viscosity and achieve a nearly virial temperature ($T_i \sim 10^{12}$ K), far in excess of the electron temperature ($T_e \sim 10^9$ K). The energy per unit mass in the ions is therefore comparable to the gravitational binding energy, and consequently most of the energy deposited in the disk by viscous dissipation is advected across the event horizon into the black hole, unless outflows of relativistic particles are able to significantly cool the disk (Becker, Subramanian, & Kazanas 2001; Blandford & Begelman 1999).

1.1. Causality and Stress in the Inner Region

One of the most intriguing unresolved questions related to the structure of ADAF disks, and indeed, to accretion disks in general, concerns the behavior of the torque in the inner region, where the material begins to plunge into the black hole. This issue is of central importance in the development of computational models because the boundary conditions applied in the inner region can influence the structure of the entire disk. A variety of approaches have been applied towards the modeling of the variation of the torque in the inner regions of accretion disks. For example, in the context of standard thin-disk accretion, it is usually supposed that the stress vanishes (and the disk truncates) at the marginally stable orbit (e.g., Frank, King, & Raine 1985), although this suggestion has been contradicted by Watarai & Mineshige (2003) based on the results of hydrodynamical simulations. Furthermore, several authors have argued that the stress must actually vanish at the sonic point (e.g., Kato 1994; Popham & Narayan 1992). These models are based on what one might term the “hydrodynamic” causality scenario, in which the stress is transmitted by subsonic turbulence, and therefore no torque can exist in the inner, supersonic region of the flow.

The situation in an actual accretion disk is far from clear, and the angular momentum may actually be transported by some combination of fluid turbulence, particles, and electromagnetic fields. The velocity of the viscous transport associated with particles and fields is not limited to the sound speed, and therefore it is plausible that torques can be generated even in the supersonic region between the horizon and the sonic point (e.g., Zimmerman et al. 2004; Reynolds & Armitage 2001; Hawley & Krolik 2001; Agol & Krolik 2000; Gammie 1999). In these scenarios, the “hydrodynamic” causality restriction must be replaced with the more fundamental “relativistic” causality constraint, which states that *no signal of any kind can propagate faster than the speed of light*. We shall refer to flows satisfying this requirement as “deterministic.” The associated relativistic restriction on the torque is that *it must vanish at the event horizon*, since the horizon itself cannot support a shear stress (Weinberg 1972). Taken together, these two related constraints comprise the most conservative and model-independent statements one can make about the causal and viscous structure of any accretion disk, and we shall therefore adopt them as the basis for the analysis presented here.

1.2. Angular Momentum Transport

The specific results obtained for the structure of an accretion disk depend on the detailed prescription employed for the kinematic viscosity, ν , which establishes the basic connection between the angular velocity Ω , the torque \mathcal{G} , and the stress (force per unit area) Σ through

the expression

$$\Sigma = -\rho \nu r \frac{d\Omega}{dr} = \frac{\mathcal{G}}{4\pi r^2 H}, \quad (1)$$

where ρ is the mass density, H is the half-thickness of the disk, and the disk is rotating differentially so that $d\Omega/dr < 0$. Various prescriptions for the physical form of ν have been proposed over the years, starting with the “diffusive” approach employed by Shakura & Sunyaev (1973). In this scenario, the angular momentum transport is governed by a one-dimensional diffusion equation, which technically leads to the propagation of an infinitesimal amount of signal to infinite distances in zero time (Pringle 1981). The apparent violation of relativistic causality associated with the Shakura-Sunyaev formulation has stimulated other workers to consider a variety of alternative, “deterministic” forms for the viscosity (e.g., Narayan, Kato, & Honma 1997; Yuan et al. 2000). These alternative forms are important in the context of accretion onto compact objects with solid surfaces, such as neutron stars and white dwarfs, because in these environments the diffusive Shakura-Sunyaev approach can result in a non-causal disk structure.

However, the results are quite different when one considers accretion onto a black hole. In this case, Becker & Le (2003) demonstrated conclusively that despite the diffusive nature of the Shakura-Sunyaev scenario, there are in fact no relativistic causality violations in the region close to the event horizon. This is because the propagation of signals near the horizon is dominated by *inward advection* at the speed of light, which overwhelms the outward, diffusive propagation. Furthermore, Becker & Le used the standard conservation equations to confirm that the Shakura-Sunyaev prescription yields physically reasonable global solutions for the structure of ADAF disks, whether the disk is in vertical hydrostatic equilibrium near the horizon or in free-fall. The fact that the original Shakura-Sunyaev prescription yields an acceptable disk structure close to the black hole naturally causes one to ask whether this is also true for the alternative viscosity formulations suggested by subsequent authors. Our goal in this paper is to apply the same asymptotic analysis technique employed by Becker & Le (2003) in the vicinity of the event horizon to answer the “existence question” for the two specific deterministic viscosity prescriptions considered by Narayan, Kato, & Honma (1997) and Yuan et al. (2000). We shall also include the model proposed by Richard & Zahn (1999) which, although not deterministic, is nonetheless interesting to examine from a conceptual viewpoint.

We shall focus mainly on ADAF disks in the present paper because they appear to be correlated with the presence of outflows in systems possessing high radio luminosities and low X-ray luminosities. The low density plasmas in the hot ADAF disks seem to provide ideal environments for the shock acceleration of relativistic particles that can escape to power the outflows (Le & Becker 2004). The apparent association of ADAF disks with outflows

and jets suggests that a complete understanding of the disk structure is essential in order to make further progress in the development of global models that simultaneously account for the accretion of the gas as well as the production of the outflows in a self-consistent manner. ADAF disks are also particularly amenable to mathematical modeling because the energy transport is simplified by the fact that radiative losses are negligible throughout the flow. Furthermore, it is expected that the escape of matter and energy due to the acceleration of relativistic particles in the disk occurs outside the radius of marginal stability, and therefore the variation of the internal energy density is essentially adiabatic in the vicinity of the horizon, where the viscous dissipation becomes negligible (Le & Becker 2004; Becker et al. 2001). Although our discussion here centers on ADAF disks around nonrotating black holes, we will argue that the constraints obtained on the form for the viscosity in the vicinity of the event horizon also apply to more general flows, provided they do not radiate strongly near the horizon.

In the standard Shakura-Sunyaev formulation of the viscosity variation in the disk, the transport of angular momentum is a diffusive process and the time evolution of the angular velocity Ω is therefore governed by a second-order differential equation, requiring two boundary conditions. One of these conditions is provided by imposing that the viscous stress must vanish at the event horizon of the black hole, which is a mandatory requirement of general relativity (Weinberg 1972). However, two of the alternative viscosity prescriptions examined here are “deterministic” in the sense that they result in first-order differential equations for the time evolution of Ω . In these cases, obviously one less boundary condition is required in order to specify the global flow solution. We argue that regardless of which formulation is adopted for the viscosity, the fundamental zero-stress boundary condition at the event horizon must be retained for consistency with general relativity. Further discussion of this point is provided in § 5.1.

The remainder of the paper is organized as follows. In § 2 we briefly review the fundamental equations governing the structure of one-dimensional ADAF disks. In § 3 the causal structure of the viscous transport in the disk is discussed in the context of the four viscosity prescriptions of interest here. The associated existence conditions are derived in § 4 under the assumption of either vertical hydrostatic equilibrium or free-fall in the inner region of the disk. The implications of our results for the physical variation of the viscosity in advection-dominated black-hole accretion disks are discussed in § 5.

2. FUNDAMENTAL EQUATIONS

The approach to the modeling of the disk structure is simplified considerably if the effects of general relativity are incorporated in an approximate manner by expressing the gravitational potential per unit mass using the pseudo-Newtonian form (Paczynski & Wiita 1980)

$$\Phi(r) \equiv \frac{-GM}{r - r_s}, \quad (2)$$

where $r_s = 2GM/c^2$ is the Schwarzschild radius for a black hole of mass M . Becker & Le (2003) confirmed that this potential reproduces perfectly the motions of particles falling freely in the Schwarzschild metric close to the event horizon. It also correctly predicts the location of the event horizon, the radius of marginal stability, and the radius of the marginally bound orbit (for a complete discussion, see Paczynski & Wiita 1980). This potential, while providing a good approximation of the effects of general relativity, is also rather convenient mathematically because it facilitates a semi-classical approach to the problem that simplifies the analysis considerably. Due to these advantages, the pseudo-Newtonian potential has been utilized by many authors in their studies of the accretion of gas onto Schwarzschild black holes (e.g., Matsumoto et al. 1984; Abramowicz et al. 1988; Chen et al. 1997; Narayan, Kato, & Honma 1997; Hawley & Krolik 2001, 2002; Yuan 1999; Yuan et al. 2000; Reynolds & Armitage 2001).

2.1. Energy and Momentum Conservation

Since ADAF disks are radiatively inefficient, the vertical height can become comparable to the radius and therefore it is appropriate to utilize the vertically-averaged “slim disk” equations first discussed by Abramowicz et al. (1988). These equations were adopted by Narayan & Yi (1994) in their study of self-similar ADAF solutions, and also by Narayan, Kato, & Honma (1997) in their simulations of transonic flows. In the slim-disk approximation, the inertial ($v dv/dr$) and pressure gradient ($\rho^{-1} dP/dr$) terms are retained in the radial momentum equation, so that in a steady state the radial acceleration rate in the frame of the accreting gas is given by

$$\frac{Dv}{Dt} \equiv -v \frac{dv}{dr} = \frac{1}{\rho} \frac{dP}{dr} + \frac{d\Phi}{dr} - r \Omega^2, \quad (3)$$

where the radial velocity v is defined to be positive for inflow. The escape of energy from the disk is assumed to be negligible in the ADAF approximation, and therefore the rate of

change of the internal energy density U in the frame of the gas can be written as

$$\frac{DU}{Dt} \equiv -v \frac{dU}{dr} = -\gamma \frac{Uv}{\rho} \frac{d\rho}{dr} + \dot{U}_{\text{viscous}} , \quad (4)$$

where

$$\dot{U}_{\text{viscous}} = -\frac{\mathcal{G}}{4\pi r H} \frac{d\Omega}{dr} = -r \Sigma \frac{d\Omega}{dr} \quad (5)$$

is the viscous energy dissipation rate per unit volume and γ denotes the specific heats ratio. In a steady-state situation, the radial variation of the angular velocity Ω is determined by (see eq. [1])

$$\frac{d\Omega}{dr} = -\frac{\mathcal{G}}{4\pi r^3 H \rho \nu} . \quad (6)$$

2.2. Transport Rates

In one-dimensional, stationary, advection-dominated disks, three integrals of the flow can be identified. The first is the accretion rate,

$$\dot{M} = 4\pi r H \rho v = \text{constant} , \quad (7)$$

and the second is the angular momentum transport rate,

$$\dot{J} = \dot{M} r^2 \Omega - \mathcal{G} = \text{constant} . \quad (8)$$

The third conserved quantity is the energy transport rate,

$$\dot{E} = -\mathcal{G} \Omega + \dot{M} \left(\frac{1}{2} v^2 + \frac{1}{2} w^2 + \frac{P+U}{\rho} + \Phi \right) = \text{constant} , \quad (9)$$

where $U = P/(\gamma - 1)$ is the internal energy density and $w = r \Omega$ is the azimuthal velocity. The constancy of \dot{E} can be demonstrated explicitly by combining equations (3), (4), (5), (7), and (8). Despite the classical appearance of the conservation equations for \dot{M} , \dot{J} , and \dot{E} , it is important to bear in mind that close to the horizon, v and w are actually more correctly interpreted as the radially and azimuthal components of the four-velocity, respectively (see Becker & Le 2003).

The vanishing of the stress and the torque at the horizon, combined with equation (8), together imply that

$$\lim_{r \rightarrow r_s} \Omega(r) = \frac{\dot{J}}{\dot{M} r_s^2} \equiv \Omega_0 , \quad (10)$$

and therefore Ω achieves a finite value at the horizon. By eliminating the torque \mathcal{G} between equations (8) and (9), we can reexpress the energy transport rate as

$$\dot{E} = \dot{J} \Omega + \dot{M} \left(\frac{1}{2} v^2 - \frac{1}{2} r^2 \Omega^2 + \frac{a^2}{\gamma - 1} + \Phi \right) , \quad (11)$$

where

$$a \equiv \left(\frac{\gamma P}{\rho} \right)^{1/2} \quad (12)$$

denotes the adiabatic sound speed.

Since the angular velocity Ω approaches a finite value at the horizon and the flow must be supersonic there, it follows from equation (11) that

$$v(r) \rightarrow v_{\text{ff}}(r) \equiv \left(\frac{GM}{r - r_{\text{s}}} \right)^{1/2} , \quad r \rightarrow r_{\text{s}} , \quad (13)$$

where v_{ff} denotes the free-fall velocity in the pseudo-Newtonian potential. Note that the divergence of v_{ff} implies that it is not a Newtonian velocity, but rather a four-velocity, as established by Becker & Le (2003). Equation (13) clearly indicates that the inflow approaches free-fall close to the event horizon, and consequently the radial velocity $v^{\hat{r}}$ approaches the speed of light (Shapiro & Teukolsky 1983).

2.3. Specific Angular Momentum

Next we shall focus on the asymptotic variation of the angular velocity Ω and the specific angular momentum $\ell \equiv r^2 \Omega$ in the limit $r \rightarrow r_{\text{s}}$. Equation (8) can be rewritten in terms of ℓ as

$$\ell - \ell_0 = \frac{\mathcal{G}}{\dot{M}} , \quad (14)$$

where $\ell_0 \equiv \dot{J}/\dot{M} = r_{\text{s}}^2 \Omega_0$ is the accreted specific angular momentum. Since the torque \mathcal{G} is positive for $r > r_{\text{s}}$ and it vanishes at the horizon ($r = r_{\text{s}}$), it follows from equation (14) that

$$\lim_{r \rightarrow r_{\text{s}}} \frac{d\ell}{dr} \geq 0 . \quad (15)$$

Based on the relationship between Ω and ℓ , we find that

$$\frac{d\ell}{dr} = 2 r \Omega + r^2 \frac{d\Omega}{dr} . \quad (16)$$

We can express the variation of $\Omega(r)$ in the neighborhood of the event horizon using the general form

$$\Omega(r) \doteq \Omega_0 - A(r - r_s)^q , \quad (17)$$

where A and q are positive constants and the symbol “ \doteq ” will be used to denote asymptotic equality at the horizon. This is the simplest form satisfying the conditions $\Omega(r) \rightarrow \Omega_0$ and $d\Omega/dr \leq 0$ as $r \rightarrow r_s$. From a slightly more technical point of view, the right-hand side of equation (17) represents the first two terms in the Frobenius expansion of $\Omega(r)$ around $r = r_s$, in which case q is the exponent of the solution (Boyce & DiPrima 1977). The values of the constants A and q are determined through analysis of the conservation equations.

Combining equations (16) and (17) yields for the asymptotic behavior of the specific angular momentum

$$\frac{d\ell}{dr} \doteq 2r\Omega_0 - 2Ar(r - r_s)^q - qAr^2(r - r_s)^{q-1} . \quad (18)$$

This result implies that we must have $q \geq 1$ in order to avoid divergence of $d\ell/dr$ to negative infinity at the horizon, which is unacceptable according to equation (15). The restriction on q in turn implies that $d\ell/dr$ must be equal to zero or a finite positive value at the horizon. Following Becker & Le (2003), we shall express the variation of the specific angular momentum in the vicinity of the event horizon using the general form

$$\ell(r) \doteq \ell_0 + B(r - r_s)^\beta , \quad \beta \geq 1 , \quad (19)$$

where B is a positive constant. This equation represents the first two terms of the Frobenius expansion for $\ell(r)$ about $r = r_s$, with β denoting the exponent of the solution (cf. equation 17). The restriction $\beta \geq 1$ is required in order to ensure that $d\ell/dr$ is equal to zero or a finite positive quantity at the event horizon, as established above. Different values of β will be obtained depending on the viscosity model employed, as discussed in § 4. It should be emphasized that any viscosity model that yields $\beta < 1$ fails to satisfy the basic existence condition, and must be rejected. While Paczyński & Wiita (1980) imposed equation (19) as an ad hoc expression for the global variation of the specific angular momentum $\ell(r)$, we shall utilize it only in the asymptotic limit, where it is fully consistent with the conservation equations.

Equations (16) and (19) together imply that the radial derivative of Ω at the horizon is given by

$$\left. \frac{d\Omega}{dr} \right|_{r=r_s} = \begin{cases} -2\ell_0/r_s^3 , & \beta > 1 , \\ (Br_s - 2\ell_0)/r_s^3 , & \beta = 1 . \end{cases} \quad (20)$$

Hence $d\Omega/dr$ vanishes at the horizon only in the special case $\beta = 1$ and $B = 2 \ell_0/r_s$. These conditions are not satisfied in the situations of interest here, and therefore $d\Omega/dr$ has a finite (negative) value at the horizon. By combining equations (1), (14), and (19), we obtain the fundamental asymptotic relation

$$B(r - r_s)^\beta \doteq - \frac{r^2 \nu}{v} \frac{d\Omega}{dr} . \quad (21)$$

This expression will prove useful in establishing the existence conditions for inflows subject to a variety of prescriptions for the kinematic viscosity ν . The procedure will be to compute the value of β for each viscosity model by balancing the powers of $(r - r_s)$ on the two sides of equation (21), and then to compare the result with the existence condition $\beta \geq 1$. Satisfaction of this condition ensures that the variation of the specific angular momentum and its derivative are physically acceptable close to the event horizon, and also that the torque vanishes there as required.

2.4. Entropy Function

Following Becker & Le (2003), we will adopt the “perfect ADAF” approximation, and consequently the escape of energy from the disk will be ignored for the moment. Since the stress Σ vanishes as $r \rightarrow r_s$, it follows that the flow approaches a purely adiabatic behavior ($U \propto \rho^\gamma$) in the vicinity of the event horizon (see eq. [5]). Furthermore, if the gas is in local thermodynamic equilibrium, then the viscous heating is a quasi-static process, and in this case the flow is isentropic wherever the dissipation vanishes.

In our analysis of the flow structure close to the horizon, we shall find it convenient to introduce the “entropy function,”

$$K(r) \equiv r H v a^{2/(\gamma-1)} . \quad (22)$$

To understand the physical significance of K , we can combine equations (7), (12), and (22) to show that

$$K^{\gamma-1} \propto \frac{U}{\rho^\gamma} . \quad (23)$$

This result establishes that K has a constant value near the event horizon because the viscous dissipation rate vanishes there and the flow becomes adiabatic. It is interesting to note that if the gas is in local thermodynamic equilibrium, then we can use equation (23) to show that the value of K is connected to the entropy per particle S by (Reif 1965)

$$S = k \ln K + c_0 , \quad (24)$$

where c_0 is a constant that is independent of the state of the gas, but may depend upon its composition. The constancy of K near the horizon will allow us to obtain a convenient expression for the asymptotic variation of the adiabatic sound speed a in terms of r and v for disks in hydrostatic equilibrium or free-fall, as discussed in § 4.

3. VISCOUS TRANSPORT AND CAUSALITY

The various prescriptions for the kinematic viscosity ν considered by Shakura & Sunyaev (1973), Narayan, Kato, & Honma (1997), Yuan et al. (2000), and Richard & Zahn (1999) have very different implications for the (relativistic) causal structure of the angular momentum transport occurring in an accretion disk. The distinctions between the different approaches can be understood clearly by analyzing the equation describing the time-dependent transport of angular momentum in the disk. Following Becker & Le (2003) and Blandford & Begelman (1999), we write

$$\frac{\partial}{\partial t} (\mu r^2 \Omega) = \frac{\partial}{\partial r} (\mu r^2 \Omega v - \mathcal{G}) \ , \quad (25)$$

where

$$\mu \equiv 4\pi r H \rho = \frac{\dot{M}}{v} \quad (26)$$

represents the mass per unit radius in the disk. The corresponding conservation equation for μ is given by

$$\frac{\partial \mu}{\partial t} = \frac{\partial}{\partial r} (\mu v) \ . \quad (27)$$

By combining equations (1) and (26), we find that the torque \mathcal{G} can be expressed in terms of μ , ν , and Ω as

$$\mathcal{G} = -\mu r^2 \nu \frac{\partial \Omega}{\partial r} \ . \quad (28)$$

Utilizing equations (25), (27), and (28), we can show that the time derivative of Ω is given by

$$\frac{\partial \Omega}{\partial t} = \frac{1}{\mu r^2} \frac{\partial}{\partial r} \left(\mu r^2 \Omega v + \mu r^2 \nu \frac{\partial \Omega}{\partial r} \right) - \frac{\Omega}{\mu} \frac{\partial}{\partial r} (\mu v) \ . \quad (29)$$

The nature of this equation depends on the functional form assumed for the variation of ν , which determines whether equation (29) is first- or second-order in Ω , and whether it is linear or nonlinear. We shall discuss below the implications of each of the four viscosity prescriptions of interest here.

3.1. Shakura & Sunyaev

Shakura & Sunyaev (1973) were the first to suggest that the variation of the kinematic viscosity in an accretion disk can be approximated using the form

$$\nu = \alpha a H , \quad (30)$$

with the corresponding shear stress (see eq. [1])

$$\Sigma = -\alpha a H \rho r \frac{d\Omega}{dr} . \quad (31)$$

where α is a positive constant of order unity. The formulation is based on the idea that the viscosity may be expected to be roughly proportional to the product of the turbulent velocity (usually some fraction of the sound speed) and the largest scale of the turbulent eddies (the disk height H). This fundamental prescription should hold whether the disk is hydrostatic or in free-fall. Combining equations (29) and (30) yields in the case of the Shakura-Sunyaev viscosity

$$\frac{\partial\Omega}{\partial t} = \frac{1}{\mu r^2} \frac{\partial}{\partial r} \left(\mu r^2 \Omega v + \alpha a H \mu r^2 \frac{\partial\Omega}{\partial r} \right) - \frac{\Omega}{\mu} \frac{\partial}{\partial r} (\mu v) . \quad (32)$$

This second-order equation in Ω has a diffusive character, because the angular momentum is always transported in the direction opposed to the radial gradient of Ω (Pringle 1981). We can obtain additional insight on this point by focusing on the time evolution of an initially localized component of the angular momentum distribution, represented by a δ -function at some arbitrary radius $r = r_0$ at an arbitrary time $t = t_0$. As time advances, the distribution will initially spread in radius in an approximately Gaussian manner, implying the propagation of an infinitesimal portion of the angular momentum to infinite distance in a finite time.

The diffusive nature of the transport in this case has led a number of authors to point out the potential for violations of relativistic causality in accretion disks (e.g., Kato 1994; Narayan 1992). However, as discussed by Becker & Le (2003), this phenomenon has a negligible effect on the structure of an accretion disk in the outer, subsonic region, because the *mean* transport velocity for the angular momentum remains small despite the fact that an infinitesimal amount of angular momentum is transported to an infinite distance in zero time. Moreover, in the context of accretion onto a nonrotating black hole, they demonstrated through an explicit derivation of the relevant Fokker-Planck coefficients that there are no relativistic causality violations near the event horizon associated with the Shakura-Sunyaev viscosity prescription. This is due to the fact that signals originating near the horizon are simply advected into the black hole at the speed of light, in agreement with general relativity. In a broader context, however, it is important to note that the causality issue

remains a central consideration in situations involving the accretion of matter onto the solid surfaces of neutron stars and white dwarfs.

3.2. Narayan, Kato, & Honma

The relativistic causality problem associated with the original Shakura-Sunyaev prescription for the variation of ν has motivated several authors to consider a variety of “deterministic” alternatives to equation (30). These all involve the replacement of equation (31) for the stress with a new expression that does not contain $d\Omega/dr$, thereby rendering equation (29) first-order in Ω . Since the transport of angular momentum is no longer a diffusive process, these alternatives satisfy the relativistic causality constraint. We note, however, that they may still fail the “hydrodynamic” causality test discussed in § 1.1, although this is not necessarily a problem if the stress is transmitted by particles and/or fields rather than by fluid turbulence. For example, Narayan, Kato, & Honma (1997) considered the alternative form for the viscous stress

$$\Sigma = -\alpha P \frac{d \ln \Omega_K}{d \ln r} , \quad (33)$$

with the associated kinematic viscosity given by

$$\nu = \frac{\alpha a^2}{\gamma \Omega_K} \left(\frac{d\Omega_K}{dr} \right) \left(\frac{d\Omega}{dr} \right)^{-1} , \quad (34)$$

where $\Omega_K(r)$ represents the Keplerian angular velocity of matter in a circular orbit at radius r . In the case of the pseudo-Newtonian potential given by equation (2), we obtain

$$\Omega_K^2(r) \equiv \frac{GM}{r(r-r_s)^2} = \frac{1}{r} \frac{d\Phi}{dr} . \quad (35)$$

Substituting equation (34) into equation (29) yields

$$\frac{\partial \Omega}{\partial t} = \frac{1}{\mu r^2} \frac{\partial}{\partial r} \left(\mu r^2 \Omega v + \frac{\alpha a^2 \mu r^2}{\gamma \Omega_K} \frac{d\Omega_K}{dr} \right) - \frac{\Omega}{\mu} \frac{\partial}{\partial r} (\mu v) . \quad (36)$$

This first-order equation for Ω has no diffusive character, and therefore the propagation of perturbations in the angular momentum distribution will occur in a deterministic manner. However, despite this apparent advantage, we shall see in § 4 that the prescription for ν given by equation (34) implies an unphysical structure in the inner region of an accretion flow around a nonrotating black hole.

3.3. Yuan et al.

Another “deterministic” form for the viscosity was suggested by Yuan et al. (2000; see also Matsumoto et al. 1984 and Shakura & Sunyaev 1973), who proposed that

$$\Sigma = \alpha P , \quad (37)$$

which yields for the viscosity

$$\nu = - \frac{\alpha a^2}{\gamma r} \left(\frac{d\Omega}{dr} \right)^{-1} . \quad (38)$$

Substitution into equation (29) now yields

$$\frac{\partial\Omega}{\partial t} = \frac{1}{\mu r^2} \frac{\partial}{\partial r} (\mu r^2 \Omega v - \alpha \gamma^{-1} a^2 \mu r) - \frac{\Omega}{\mu} \frac{\partial}{\partial r} (\mu v) . \quad (39)$$

As in the previous example, we again obtain a first-order equation in Ω that does not have any diffusive character. While this form for ν does result in angular momentum transport that satisfies the relativistic causality constraint, we will nonetheless find that equation (38) produces an unphysical structure when applied in the inner region of an accretion flow.

3.4. Richard & Zahn

Based on laboratory studies of differentially rotating flows in the Couette-Taylor experiment, Richard & Zahn (1999) proposed that the kinematic viscosity in accretion disks scales as

$$\nu = -\alpha r^3 \frac{d\Omega}{dr} , \quad (40)$$

with the associated stress

$$\Sigma = \alpha r^4 \rho \left(\frac{d\Omega}{dr} \right)^2 . \quad (41)$$

Combining equations (29) and (40) yields for the Richard & Zahn scenario

$$\frac{\partial\Omega}{\partial t} = \frac{1}{\mu r^2} \frac{\partial}{\partial r} \left[\mu r^2 \Omega v - \alpha \mu r^5 \left(\frac{\partial\Omega}{\partial r} \right)^2 \right] - \frac{\Omega}{\mu} \frac{\partial}{\partial r} (\mu v) . \quad (42)$$

This expression for $\partial\Omega/\partial t$ is quite different from those obtained for the other three viscosity prescriptions, because it is a nonlinear diffusion equation. From a physical point of view, the diffusive character of equation (42) implies a violation of the relativistic causality requirement. Furthermore, the nonlinear nature of equation (42) results in the propagation of angular momentum in the *outward* direction *regardless* of the sign of the gradient $d\Omega/dr$.

Despite this unusual property, it is possible to model accretion disks using equation (40) for ν because $d\Omega/dr$ is generally negative in disks, which is consistent with angular momentum transport in the outward direction. However, we shall see that the structure of the innermost region of an accretion flow computed using equation (40) is unphysical.

4. ASYMPTOTIC BEHAVIOR IN ADVECTION-DOMINATED DISKS

Many of the theoretical models for advection-dominated accretion disks appearing in the literature have been based on the assumption of vertical hydrostatic equilibrium. While this assumption is valid in the outer, subsonic region, it is clear that close to the horizon, where the inflow becomes supersonic, sound waves do not have enough time to maintain hydrostatic equilibrium before the gas enters the black hole. One therefore expects that in the inner region, the disk will have a free-fall structure, with the disk half-thickness H proportional to the radius r . Nonetheless, the hydrostatic assumption has been used to model the structure of accretion flows all the way in to the event horizon. The popularity of the hydrostatic model, combined with the physical plausibility of the free-fall model, motivates us to consider both possibilities here. Consequently, our goal in this section is to determine whether or not the “existence condition” $\beta \geq 1$ (see equation 19) is satisfied for each of the four viscosity prescriptions of interest, in both hydrostatic and free-fall disks. In each of our calculations we shall make use of the asymptotic relations (see equations 13 and 35)

$$v \propto (r - r_s)^{-1/2}, \quad \Omega_K \propto (r - r_s)^{-1}, \quad r \rightarrow r_s, \quad (43)$$

which are valid in general.

4.1. Asymptotic Behavior in Hydrostatic Disks

The results discussed in §§ 2 and 3 apply to all advection-dominated disks in the pseudo-Newtonian potential. In this section, we shall specialize to the case of ADAF disks that maintain vertical hydrostatic equilibrium all the way in to the event horizon. While this is not necessarily justified on physical grounds, it nonetheless provides a useful basis for comparison with several published models based on this assumption.

For accretion disks in vertical hydrostatic equilibrium, the vertical half-thickness of the disk H is given by the standard relation (Abramowicz et al. 1988)

$$H(r) = \frac{b_0 a}{\Omega_K}, \quad (44)$$

where b_0 is a dimensionless constant of order unity that depends on the details of the vertical averaging, and $\Omega_K(r)$ is given by equation (35). By combining equations (22) and (44), we find that the entropy function in a hydrostatic disk is given by

$$K = K_{\text{eq}} \equiv \frac{b_0 r v}{\Omega_K} a^{\frac{\gamma+1}{\gamma-1}} . \quad (45)$$

Close to the event horizon, where dissipation becomes unimportant and K is essentially constant, the variation of the adiabatic sound speed a can therefore be expressed as

$$a \propto \left(\frac{\Omega_K}{v} \right)^{\frac{\gamma-1}{\gamma+1}} , \quad r \rightarrow r_s . \quad (46)$$

By combining this result with equations (43), we find that the explicit asymptotic radial dependence of the sound speed in a hydrostatic disk is given by

$$a^2 \propto (r - r_s)^{\frac{1-\gamma}{1+\gamma}} , \quad r \rightarrow r_s . \quad (47)$$

Note that $a \rightarrow \infty$ as $r \rightarrow r_s$ due to the effect of adiabatic compression as the gas flows towards the horizon. This expression will prove useful when we analyze the existence conditions for hydrostatic accretion disks in the vicinity of the event horizon.

4.2. Asymptotic Behavior in Freely-Falling Disks

ADAF disks are likely to maintain vertical hydrostatic structure in the outer, subsonic region. However, close to the horizon, the inflow must become supersonic and therefore sound waves will not be able to maintain hydrostatic equilibrium (e.g., Matsumoto et al. 1984). The transition between the hydrostatic, subsonic outer region and the freely-falling, supersonic inner region can be modeled using the prescription suggested by Abramowicz, Lanza, & Percival (1997). However, in the present paper, we are interested only in the asymptotic behavior near the event horizon. Hence, based on equation (18) from Abramowicz et al. (1997), we can express the variation of the disk half-thickness H using

$$H(r) = d_0 r , \quad (48)$$

where d_0 is a dimensionless constant of order unity. By utilizing equations (22) and (48), we conclude that the entropy function in a freely-falling disk is given by

$$K = K_{\text{ff}} \equiv d_0 r^2 v a^{\frac{2}{\gamma-1}} . \quad (49)$$

The variation of the sound speed close to the event horizon, where K is essentially constant, can be expressed in this case as

$$a \propto v^{(1-\gamma)/2}, \quad r \rightarrow r_s. \quad (50)$$

The corresponding asymptotic radial dependence of the sound speed close to the horizon in a freely-falling disk is obtained by combining equations (43) and (50), which yields

$$a^2 \propto (r - r_s)^{(\gamma-1)/2}, \quad r \rightarrow r_s. \quad (51)$$

Note that $a \rightarrow 0$ as $r \rightarrow r_s$, in contrast with the hydrostatic case, where we found that the sound speed diverges at the horizon (cf. equation 47). It is interesting to consider how this alternative central inflow condition will affect our conclusions regarding the physical applicability of the various viscosity prescriptions in the region close to the event horizon. We shall treat each of the viscosity prescriptions separately below, for both freely-falling and hydrostatic disks.

4.3. Shakura & Sunyaev

In a Shakura-Sunyaev (1973) disk the viscosity and stress are given by

$$\nu = \alpha a H, \quad \Sigma = -\alpha a H \rho r \frac{d\Omega}{dr}, \quad (52)$$

which can be combined with equation (21) to obtain the asymptotic relation

$$B (r - r_s)^\beta \doteq - \frac{\alpha a H r^2}{v} \frac{d\Omega}{dr}. \quad (53)$$

If the disk is in vertical hydrostatic equilibrium, then we can substitute for H using equation (44), which yields

$$B (r - r_s)^\beta \doteq - \frac{\alpha b_0 a^2 r^2}{v \Omega_K} \frac{d\Omega}{dr}. \quad (54)$$

Utilizing the general asymptotic relations given by equations (43) along with equation (47), we conclude that in order to balance the exponents of $(r - r_s)$ on both sides of equation (54) in a *hydrostatic* Shakura-Sunyaev disk, we must have

$$\beta = \frac{\gamma + 5}{2(\gamma + 1)} \quad (\text{hydrostatic}), \quad (55)$$

The asymptotic expression given by equation (53) is valid in both hydrostatic and freely-falling disks subject to the Shakura-Sunyaev viscosity prescription. Hence by combining

equations (43), (48), and (53) and balancing powers of $(r - r_s)$ in the resulting expression, we find that in a *freely-falling* Shakura-Sunyaev disk the value of β is given by

$$\beta = \frac{\gamma + 1}{4} \quad (\text{free - fall}) . \quad (56)$$

This result is slightly different from equation (70) in Becker & Le (2003), who found that $\beta = 1 + \gamma/2$. The difference between the two results arises because we have expressed the viscosity in the freely-falling disk using $\nu \propto aH$, whereas Becker & Le (2003) used $\nu \propto a^2/\Omega_K$. It is not obvious which form is the most appropriate to use in the supersonic region, because either one can be viewed as the fundamental definition of the viscosity following the phenomenological arguments given by Shakura & Sunyaev (1973). In any event, the difference between the two forms is negligible for the present considerations since in either case we find that $\beta > 1$.

We therefore conclude that $\beta > 1$ for all values of γ in both equations (55) and (56), and consequently it follows that the Shakura-Sunyaev prescription for ν satisfies the existence condition $\beta \geq 1$ required by equation (19) *in both freely-falling and hydrostatic disks*. Moreover, Becker & Le have also established conclusively that the Shakura-Sunyaev formulation for the viscosity is consistent with general relativistic causality requirements close to the event horizon.

4.4. Narayan, Kato, & Honma

Next we examine the implications of the “deterministic” viscosity prescription considered by Narayan, Kato, & Honma (1997), for which we have

$$\nu = \frac{\alpha a^2}{\gamma \Omega_K} \left(\frac{d\Omega_K}{dr} \right) \left(\frac{d\Omega}{dr} \right)^{-1}, \quad \Sigma = -\alpha P \frac{d \ln \Omega_K}{d \ln r} . \quad (57)$$

The angular momentum transport in this case is non-diffusive, and therefore the relativistic causality constraint is satisfied. Combining equations (21) and (57), we conclude that

$$B (r - r_s)^\beta \doteq - \frac{\alpha r a^2}{\gamma v} \frac{d \ln \Omega_K}{d \ln r} . \quad (58)$$

Incorporating the asymptotic behaviors for v , Ω_K , and a close to the horizon given by equations (43) and (47), we now find that the in a *hydrostatic* disk described by the Narayan et al. viscosity, the exponents of $(r - r_s)$ on the two sides of equation (58) balance if

$$\beta = \frac{1 - 3\gamma}{2(\gamma + 1)} \quad (\text{hydrostatic}) . \quad (59)$$

Since this quantity is negative for all physically reasonable values of γ , it follows that this prescription fails to satisfy the existence condition $\beta \geq 1$. Hence we conclude that the alternative, “deterministic,” prescription for the viscosity considered by Narayan, Kato, & Honma (1997) is unphysical close to the event horizon in a hydrostatic disk, in apparent contradiction to the global solutions for the disk structure presented in their paper. This probably reflects the fact that Narayan et al. did not use their equation (2.19) when treating this prescription for the viscosity variation, which stipulates that the torque vanishes at the horizon. However, we argue that in fact one does not have the freedom to ignore this essential physical boundary condition, as discussed in § 1.1 and § 1.2.

Equation (58) also applies in the free-fall case, and we can therefore combine it with equations (43) and (51) to conclude that in a *freely-falling* Narayan et al. disk, the counterpart of equation (59) is given by

$$\beta = \frac{\gamma}{2} - 1 \quad (\text{free - fall}) . \quad (60)$$

Hence we once again find that $\beta < 0$ for $4/3 < \gamma < 5/3$, and therefore we conclude that the deterministic prescription for the viscosity examined by Narayan, Kato & Honma (1997) is unphysical close to the event horizon for both freely-falling and hydrostatic disks.

4.5. Yuan et al.

The deterministic prescription suggested by Yuan et al. (2000) gives the results

$$\nu = -\frac{\alpha a^2}{\gamma r} \left(\frac{d\Omega}{dr} \right)^{-1} , \quad \Sigma = \alpha P . \quad (61)$$

In this case, the angular momentum transport once again satisfies the relativistic causality condition since it is non-diffusive in nature. Combining equations (21) and (61), we now obtain the asymptotic relation

$$B (r - r_s)^\beta \doteq \frac{\alpha r a^2}{\gamma v} . \quad (62)$$

Utilizing equations (43) and (47) to describe the asymptotic behaviors of v and a close to the horizon, we find that in a *hydrostatic* disk described by the Yuan et al. viscosity, the exponents of $(r - r_s)$ on the two sides of equation (62) balance if

$$\beta = \frac{3 - \gamma}{2(\gamma + 1)} \quad (\text{hydrostatic}) . \quad (63)$$

Note that $\beta < 1$ for $4/3 < \gamma < 5/3$, which fails to satisfy the existence condition $\beta \geq 1$. Therefore this viscosity prescription does not yield a physically acceptable structure in the inner region of a hydrostatic disk.

Since equation (62) is also valid in the case of a freely-falling inflow, we can combine it with equations (43) and (51) to conclude that in a *freely-falling* disk governed by the Yuan et al. viscosity, the counterpart of equation (63) is given by

$$\beta = \frac{\gamma}{2} \quad (\text{free - fall}) . \quad (64)$$

We again find that $\beta < 1$ for $4/3 < \gamma < 5/3$, and therefore the viscosity prescription proposed by Yuan et al. (2000) is unphysical close to the horizon whether the disk has a free-fall or hydrostatic structure.

4.6. Richard & Zahn

The prescription proposed by Richard & Zahn (1999) gives for the viscosity and the shear stress

$$\nu = -\alpha r^3 \frac{d\Omega}{dr} , \quad \Sigma = \alpha r^4 \rho \left(\frac{d\Omega}{dr} \right)^2 . \quad (65)$$

In this scenario, the angular momentum transport is based on a nonlinear diffusion equation. Equations (21) and (65) can be combined to obtain

$$B (r - r_s)^\beta \doteq \frac{\alpha r^5}{v} \left(\frac{d\Omega}{dr} \right)^2 . \quad (66)$$

Incorporating the asymptotic behavior of v near the event horizon (equation 43), and requiring that the powers of $(r - r_s)$ balance on the two sides gives for β the result

$$\beta = \frac{1}{2} \quad (\text{hydrostatic}) . \quad (67)$$

This result fails to satisfy the existence condition $\beta \geq 1$, and therefore we must conclude that the Richard-Zahn prescription for the viscosity is unphysical in the region close to the event horizon if the disk is in hydrostatic equilibrium.

Equation (66) also applies in the case of an accretion disk with a free-fall inner region. Since the sound speed a does not appear in this expression, we find that the same result is obtained for β in a freely-falling disk subject to the Richard & Zahn viscosity, namely

$$\beta = \frac{1}{2} \quad (\text{free - fall}) . \quad (68)$$

It therefore follows that the Richard & Zahn prescription for the variation of the viscosity ν yields unphysical structure in the inner region of an accretion disk, whether the disk is freely falling or in vertical hydrostatic equilibrium.

5. CONCLUSIONS

In this paper we have explored the implications of several alternative prescriptions for the kinematic viscosity ν on the structure of an advection-dominated accretion disk close to the event horizon of a nonrotating black hole. Our investigation has been motivated primarily by the results obtained by Becker & Le (2003), who found that the original Shakura-Sunyaev (1973) prescription yields acceptable disk structure, despite its “diffusive” nature, which has caused other authors to propose “deterministic” alternatives. Our approach has been to employ rigorous asymptotic analysis to determine which of the various prescriptions for ν considered here satisfy the fundamental existence conditions for the inflow. In order to make our comparison in the most model-independent way possible, we have stipulated that the stress must vanish exactly at the event horizon, which is a fundamental requirement of general relativity (Weinberg 1972). The specific scenarios for the viscosity variation we have examined are those considered by Narayan et al. (1997), Richard & Zahn (1999), and Yuan et al. (2000), as well as the Shakura-Sunyaev form. The disk was assumed to be in either vertical hydrostatic equilibrium or free-fall, and the pseudo-Newtonian potential was utilized.

We have analyzed the structure of the accretion disk near the event horizon associated with the Shakura-Sunyaev viscosity prescription in § 4 and compared the results with those obtained using the three alternative formulations of interest here. Interestingly, we find that none of the three alternatives yields a physically consistent structure for the accretion disk close to the event horizon. This includes both the deterministic forms considered by Narayan et al. (1997) and Yuan et al. (2000), as well as the nonlinear prescription suggested by Richard & Zahn (1999). From a physical point of view, these alternative models fail to satisfy the zero-stress boundary condition at the event horizon of the black hole. Hence we conclude that the only acceptable form for the viscosity is the “diffusive” prescription originally proposed by Shakura & Sunyaev (1973). This conclusion holds both for disks in hydrostatic equilibrium, as well as for those experiencing free-fall in the inner region. One implication of our results is that the radial derivative of the torque vanishes at the horizon, since this is a property of the Shakura-Sunyaev viscosity formulation (see Becker & Le 2003).

5.1. Inner Boundary Condition

The deterministic viscosity prescriptions considered by Narayan, Kato & Honma (1997) and Yuan et al. (2000) result in first-order equations for the time evolution of Ω , as pointed out in § 3.2 and § 3.3. A first-order differential equation only needs one boundary condition; one is therefore free to choose either the inner or outer boundary condition. Since the flow is typically supersonic near the black hole, Narayan, Kato & Honma (1997) argue that information cannot propagate upstream from the region near the event horizon. According to this line of reasoning, the inner boundary condition is not relevant, and one is justified in using the outer boundary condition (which corresponds to conditions far away from the black hole). However, as discussed in § 1.1, angular momentum in accretion flows around black holes might well be transported by magnetic stresses, or by torques arising from some combination of magnetic fields and particles. In particular, a number of recent papers dealing with MHD simulations of black hole accretion flows have emphasized the role of magnetic fields in producing viscous stresses (Reynolds & Armitage 2001; Hawley & Krolik 2001; Agol & Krolik 2000; Gammie 1999; Menou 2003; Krolik & Hawley 2002).

If magnetic fields and/or particles play a significant role in transmitting torques, then the viscous stress can propagate at the Alfvén speed, or at some other velocity that could well exceed the local sound speed. Furthermore, several of the papers mentioned above find that viscous stresses can exist in the supersonic region of the flow well below the radius of marginal stability, where hydrodynamical effects are no longer important. It follows that in such situations, the hydrodynamical sonic point (where the fluid flow velocity equals the local sound speed) does not represent the fundamental boundary beyond which mechanical stresses cannot propagate upstream. We therefore argue that the only essential inner boundary condition for the torque is that *the viscous stress must vanish exactly at the event horizon*, as mandated by general relativity. The first-order equation for the time evolution of Ω associated with the “deterministic” viscosity prescriptions studied by Narayan, Kato & Honma (1997) and Yuan et al. (2000) requires one fewer boundary conditions than the second-order equation for Ω obtained in the “diffusive” scenarios (such as the one originally proposed by Shakura & Sunyaev). However, even in these cases, the zero-stress condition at the event horizon is not optional, and it must be given preference over less fundamental conditions applied at a large distance from the black hole.

We suggest that the diffusive nature of the Shakura-Sunyaev form is the reason it is able to satisfy the zero-stress inner boundary condition. Hence, rather than representing a drawback, this feature actually enables the development of a physically consistent flow structure in the vicinity of the horizon. Furthermore, as demonstrated by Becker & Le (2003), the diffusive character of the Shakura-Sunyaev prescription does not lead to relativistic

causality violations close to the event horizon because signals propagating in that region are advected into the black hole at the speed of light, as required. In our view, the success of the original Shakura-Sunyaev formulation reflects the simple fact that the viscous transport of angular momentum in accretion disks is indeed a diffusive physical process (Pringle 1981).

5.2. Kerr Black Holes

It is interesting to consider how the picture presented here would be modified if the black hole possessed finite angular momentum, rather than being nonrotating as we have assumed. While a definitive answer to this question is beyond the scope of the present paper, we shall nonetheless make a few general observations and a conjecture. The location of the sonic point (or points if the disk contains a shock) is quite sensitive to the angular momentum of the black hole, as discussed by Sponholz & Molteni (1994) and more recently by Barai, Das, & Wiita (2004). The values of the various disk structure variables such as the pressure, density, etc. will therefore depend on both the angular momentum of the black hole, and also on whether the gas is orbiting in the prograde or retrograde directions.

In the Schwarzschild metric, the static limit and the event horizon are both located at radius $r_s = 2GM/c^2$. However, when the black hole possesses finite angular momentum, frame dragging causes the values of these two radii to bifurcate (Bardeen, Press, & Teukolsky 1972). This complicates efforts to model accretion flows around rotating black holes using pseudopotentials, but despite this there have been several attempts to do so (e.g., Sponholz & Molteni 1994; Mukhopadhyay 2003; Chakrabarti & Khanna 1992). For example, the Kerr pseudopotential analyzed by Chakrabarti & Khanna (1992) can be written in the form

$$\Phi_{\text{Kerr}}(r) = c^2 - \frac{GM}{r - r_*} + \frac{1}{2} \omega^2 r^2 + \frac{A_* r_s a_* \ell}{r^3} + \frac{(1 - r_s/r)\ell^2}{2r^2}, \quad (69)$$

where a_* is the angular momentum per unit mass of the black hole, A_* is a dimensionless spin-orbit coupling constant, and ω is the angular velocity of the metric rotation (see, e.g., Bardeen, Press, & Teukolsky 1972). The singularity radius, r_* , is set in an ad hoc manner in order to maximize the agreement with the fully relativistic calculations. For rapidly-rotating black holes, Chakrabarti & Khanna (1992) use $r_* = \pm GM/c^2$, where the plus and minus signs refer to prograde and retrograde orbits, respectively. In the prograde case, the potential diverges at the event horizon (Bardeen, Press, & Teukolsky 1972), but this is not so in the retrograde case. Hence the accretion dynamics in these two situations will be quite different.

For disks orbiting in the prograde sense, the Kerr pseudopotential given by equation (69) has the same divergent behavior near the horizon as the nonrotating potential used here if

we replace r_s in equation (2) with the actual horizon radius (GM/c^2 for a rapidly rotating hole). When combined with the Newtonian energy equation employed by Chakrabarti & Khanna (1992), we therefore conclude that the radial component of the four-velocity has the same asymptotic behavior as v in our equation (13) if the flow is prograde. Furthermore, we point out that our equation (14) for the specific angular momentum ℓ can be combined with equations (1) and (7) to obtain

$$\dot{M} \ell + 4\pi r^2 H \Sigma = \dot{M} \ell_0 , \quad (70)$$

which is identical to equation (52) in the general relativistic treatment of Gammie & Popham (1998), aside from a sign difference in the definition of the shear stress. Based on these conceptual similarities, we expect that the asymptotic structure of a prograde accretion disk (close to the event horizon) in the Kerr case is likely to be similar to that derived here for nonrotating black holes. If this turns out to be true, then the conclusions we have reached regarding the “existence conditions” for the various viscosity prescriptions will also hold for prograde disks around Kerr black holes. This conjecture clearly needs to be checked in the future using a detailed quantitative calculation. On the other hand, the pseudopotential in the retrograde case has a completely different behavior near the horizon (i.e., the potential is not divergent), and therefore one cannot draw a direct analogy with the work we have presented here for nonrotating holes. We therefore defer further discussion of this case to a subsequent paper.

5.3. Discussion

Our results have direct relevance for the computational modeling of the structure of accretion disks around black holes, since the prescriptions for the viscosity that fail to satisfy the existence conditions developed here cannot be used as the basis for self-consistent accretion models close to the event horizon. Because our approach is based on a careful consideration of the fundamental boundary conditions for the accretion flow near the event horizon, our results provide insight into how these conditions constrain the global structure of the accretion flow. This is particularly important when one examines the relation between inflow (accretion) and the powerful outflows (winds and jets) commonly observed to emanate from radio-loud systems containing black holes. These outflows may be powered by particle acceleration occurring at a standing, centrifugally-supported shock in the underlying accretion disk (Le & Becker 2004; Yuan et al. 2002; Chakrabarti 1989; Abramowicz & Chakrabarti 1990; Chakrabarti & Das 2004; Lu, Gu, & Yuan 1999; Chakrabarti 1990). The location of the shock is determined by the conservation equations and the shock jump conditions, along with the boundary conditions. Hence the location of the shock (and therefore its

Mach number, compression ratio, etc.) depends on the behavior of the accretion flow close to the event horizon, since this determines the inner boundary conditions. It follows that a firm understanding of this behavior is essential in order to develop self-consistent global models for the disk/shock/outflow system.

In this work we have utilized the pseudo-Newtonian potential in lieu of a full treatment of general relativity, in contrast to the work of Das (2004) and Barai et al. (2004). However, many studies in the literature have confirmed that this potential provides remarkably good agreement with the predictions of full general relativity, even very close to the event horizon (e.g., Becker & Le 2003; Paczyński & Wiita 1980). In particular, the dynamics of free particles close to the horizon in the pseudo-Newtonian potential agrees exactly with the relativistic results. Hence we are confident that the conditions derived here are fully applicable in the relativistic case. By utilizing the most conservative possible boundary condition for the stress (that it vanish at the horizon), we have obtained general results that facilitate the critical evaluation of the various forms for the viscosity that have been proposed in the literature. Our focus in this paper has been on the asymptotic behavior close to the event horizon of “perfect” advection-dominated disks, which lose no matter or energy. However, we argue that our conclusions will also apply to disks that lose matter and energy (whether ADAF or not), provided the losses do not occur very close to the horizon.

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