Gravitational Dynamics in an Expanding Universe

By T.PADMANABHAN

Inter-University Centre for Astronomy and Astrophysics, Post Bag 4, Ganeshkhind, Pune - 411 007, INDIA.
email: paddy@iucaa.ernet.in.

The dynamical evolution of collisionless particles in an expanding background is described. After discussing qualitatively the key features, the gravitational clustering of collisionless particles in an expanding universe is modelled using some simple physical ideas. I show that it is indeed possible to understand the nonlinear clustering in terms of three well defined regimes: (1) linear regime (2) quasilinear regime which is dominated by scale-invariant radial infall and (3) nonlinear regime dominated by nonradial motions and mergers. Modelling each of these regimes separately I show how the nonlinear two point correlation function can be related to the linear correlation function in hierarchical models. This analysis leads to results which are in good agreement with numerical simulations thereby providing an explanation for numerical results. The ideas presented here will also serve as a powerful analytical tool to investigate nonlinear clustering in different models. Several implications of the result are discussed.

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1. Introduction

Consider a collection N point particles, interacting with each other by the Newtonian gravity, in an expanding background characterized by a scale factor $a(t)$. What can we say about the time evolution of such a system?

This problem is of considerable interest for several reasons. To begin with, the behaviour of large number of particles interacting via Newtonian gravity poses a formidable challenge to the usual methods of statistical mechanics (T. Padmanabhan, 1990). An isolated system, made of such particles, possesses no equilibrium state in the conventional thermodynamic sense. But – as we shall see – the expanding background works in the direction opposite to the gravitational clustering and changes the behaviour of the system quantitatively. So, purely from an academic point of view, this seems to be a challenging but solvable problem.

Secondly, this problem might even have some practical interest. There is considerable evidence that the universe is dominated by collisionless non-baryonic dark matter particles. In that case, they will play a key role in the formation of large scale structures. If the length scales of interest are (i) small compared to Hubble radius but (ii) large compared to the scales at which non-gravitational processes are significant, then the system of dark matter particles constitutes an example in which the question raised in the first paragraph becomes relevant. In fact, most of the work in this subject has been inspired by considerations of structure formation.

A brute force method for solving this problem relies on numerical simulations. In such an approach one starts with large number of particles distributed nearly uniformly and calculates the future trajectories by a suitable numerical algorithm. Given a sufficiently powerful computer, this procedure will lead to the positions and velocities of the particles at any later epoch. All questions related to this physical system can be answered using the output of the numerical simulations and one may be tempted to declare the problem as solved.

I suspect Donald Lynden-Bell will not quite like the above approach to the solution of the problem. The "Lynden-Bell tradition" consists of discerning the essence of the problem, modelling it analytically and obtaining a solution which contains all the relevant features. In this talk, I shall outline how one can make analytic progress in the problem of gravitational clustering and thereby reproduce the key features of numerical simulations. There are, of course, some details which an analytic approach cannot bring out but the analytic treatment has the advantage of providing genuine physical understanding.

In the next section I shall describe some general features of the gravitational clustering in an expanding background. Section 3 will introduce the analytic model which is capable of reproducing the key features of numerical simulations. The last section describes possible extensions of this approach and conclusions.

2. Gravitational clustering in an expanding background

Let us begin by considering the evolution of a system of particles under self-gravity in the absence of expansion. Such a system has no stable thermodynamic equilibrium. It is possible to increase the phase volume available for the system without bound by separating the particles into a "core" and "halo" with the core becoming more and more tightly bound and the halo dispersing to larger and larger radii. The "final" configuration for such a system will consist of a few tightly bound binaries with the rest of the particles dispersed to large distances with positive energy.

This situation changes drastically when we introduce an expanding background. Consider a system of particles distributed homogeneously, on the average, with a mean density $\bar{\rho}(t)$. This uniform density will cause an expansion of the universe and the proper distance $\mathbf{r} = a(t)\mathbf{x}$ between particles will increase with time. If the distribution was not strictly uniform, then the perturbations in the density will act as local centres of clustering. A region with overdensity will accrete matter around it while an underdense region will repel matter in its surroundings. As a result, perturbations in density will tend to grow and when the density contrast is of order unity, these cluster centres will exert significant influence on the evolution. Particles in a highly overdense regions will evolve essentially under their own self-gravity and will tend to form gravitationally bound systems. Further evolution will crucially depend on a competition between the influence of these clusters on each other compared to the effects of expansion. The clusters themselves will attract each other and can merge gravitationally to form still larger objects thereby leading to a picture of hierarchical clustering. But the overall effect of expansion will be to pull the cluster centres apart thereby reducing the effectiveness of mergers.

In an underdense universe with $\Omega < 1$ the exapansion will win over merging at late times and we will be left with a bunch of stable, virialized clusters flying apart from each other with the overall expansion. Each of these individual systems will follow a core-halo evolution leading to a tightly bound core and a dispersed halo. The high energy halo will form a hot background, punctuated by compact gravitationally bound clusters. The evolution, in some sense, "freezes out" in such an $\Omega < 1$ model.

If $\Omega = 1$, mergers of individual cluster centres can actively compete with the effect of expansion. In such a case, it is far from clear whether stable virialized clusters can exist over a Hubble timescale. The evolution can go on forever hierarchically with mergers and expansion delicately balanced.

In the above description we have tacitly assumed that the initial perturbation has power at all scales and - in particular - at small scales. This allows clustering to proceed from small scales to large scales. A somewhat different picture emerges if the initial perturbations have all their power concentrated on a narrow band of width $\triangle L$ around

some scale, say L. As the system evolves we will first form objects with a characteristic scale L. At this stage, the universe will be (essentially) made of shells with radius L and thickness $\triangle L$. But as time progresses, power from large scales is transferred to small scales. This small scale power grows fairly fast and at late stages one has to take into account both the initial power at large scales and the newly generated power at small scales.

The above picture illustrates several interesting features of gravitational clustering which any analytical model should take cognizance of. To begin with, the transfer of power in gravitational clustering is from large scales to small scales. This can be seen most celarly from the following feature. Suppose, at some stage during the evolution of clustering, we lump together sets of nearby particles and declare them as particles of larger mass. We then continue the evolution taking into account the gravitational interaction of these new kind of particles. Such a procedure is equivalent to averaging over smallscale power and it has very little effect on large scales. [A more figurative way of saying this would be that highly nonlinear density contrast inside a galaxy or a star has no effect on the large scale dynamics of the universe]. But as the evolution procees, one certainly generates small scale power from large scales due to the breaking of long waves. In this sense gravitational clustering is similar to fluid turbulence in which also we have cascading of power from larger to smaller scales.

The second point which is clear from the above discussion is that we will most probably require two different kinds of approximations, one to deal with hierarchical clustering and another one for models with very little small scale power. In this talk I shall concentrate on models for hierarchical clustering and will make some comments about other possibilities in the last section.

Thirdly, the qualitative picture of gravitational clustering shows that one has to discover a suitable "unit" for the description of the nonlinear phase. In the linear epoch one can study the evolution very effectively using the Fourier components of the density contrast. In other words, each of these Fourier modes is an independent unit unaffected by the rest. In the highly nonlinear epoch, the coupling between the modes cannot be ignored if one uses standard Fourier transform techniques. It is, however, quite conceivable that the nonlinear phase can be described by a superposition of a different set of "units"which evolve reasonably independent of each other. For models with little small scale power, spherical shells with different radii and thickness seem to fit this bill adequately. But a similar description for the case of hierarchical clustering is hard to come by. We shall say more about this in section 4.

3. Model for nonlinear clustering

When these density perturbations are small, it is possible to study their evolution using linear theory. But once the density contrast becomes comparable to unity, linear perturbation theory breaks down and one must use N-body simulations to study the growth of perturbations. While these simulations are of some value in making concrete predictions for specific models, they do not provide clear physical insight into the process of non-linear gravitational dynamics. To obtain such an insight into this complex problem, it is necessary to model the gravitational clustering of collisionless particles using simple physical concepts. I shall develop one such model in this section, which - in spite of extreme simplicity - reproduces the simulation results for hierarchical models fairly accurately. Further, this model also provides insight into the clustering process and can be modified to take into account more complicated situations.

The paradigm for understanding the clustering is based on the well known behaviour

of a spherically symmetric overdense region in the universe. In the behaviour of such a region, one can identify three different regimes of interest: (1) In the early stages of the evolution, when the density contrast is small, the evolution is described by linear theory. (2) Each of the spherical shells with an initial radius x_i can be parametersed by a mass contained inside the shell, $M(x_i)$, and the energy, $E(x_i)$ for the particular shell. Each shell will expand to a maximum radius $x_{max} \propto M/|E|$ and then turn around and collapse. Such a spherical collapse and resulting evolution allows a self similar description (Filmore & Goldreich, 1984; Bertshinger, 1985) in which each shell acts as though it has an effective radius proportional to x_{max}. This will be the quasilinear phase. (3) The spherical evolution will break down during the later stages due to several reasons. First of all, non radial motions will arise due to amplification of deviations from spherical symmetry. Secondly, the existence of substructure will influence the evolution in a non-spherically symmetric way. Finally, in the real universe, there will be merging of such clusters [each of which could have been centres of spherical overdense regions in the begining] which will again destroy the spherical symmetry. This will be the nonlinear phase.

The description given above is sufficiently vague and sufficiently well known that one may suspect it can not lead to any insight into the problem. In particular, structures observed in the real universe are hardly spherical. I will show that it is, however, possible to model the above process in a manner which allows direct generalisation to the real universe.

To do this we will begin by studying the evolution of system starting from a gaussian initial fluctuations with an initial power spectrum, $P_{in}(k)$. The fourier transform of the power spectrum defines the correlation function $\xi(a,x)$ where $a \propto t^{2/3}$ is the expansion factor in a universe with $\Omega = 1$. It is more convenient for our purpose to work with the average correlation function inside a sphere of radius x, defined by

$$\bar{\xi}(a,x) \equiv \frac{3}{x^3}\int_0^x \xi(a,y)y^2 dy \tag{3.1}$$

In the linear regime we have $\bar{\xi}_L(a,x) \propto a^2\bar{\xi}_{in}(a_i,x)$. In the quasilinear and nonlinear regimes, we would like to have prescription which relates the exact $\bar{\xi}$ to the mean correlation function calculated from the linear theory. One might have naively imagined that $\bar{\xi}(a,x)$ should be related to $\bar{\xi}_L(a,x)$. But one can convince oneself that the relationship is likely to be nonlocal by the following analysis:

Recall that, the conservation of pairs of particles, gives an exact equation satisfied by the correlation function (Peebles, 1980):

$$\frac{\partial\xi}{\partial t} + \frac{1}{ax^2}\frac{\partial}{\partial x}[x^2(1+\xi)v] = 0 \tag{3.2}$$

where $v(t,x)$ denotes the mean relative velocity of pairs at separation x and epoch t. Using the mean correlation function $\bar{\xi}$ and a dimensionless pair velocity $h(a,x) \equiv -(v/\dot{a}x)$, equation (3.2) can be written as

$$\left(\frac{\partial}{\partial \ln a} - h\frac{\partial}{\partial \ln x}\right)(1+\bar{\xi}) = 3h(1+\bar{\xi}) \tag{3.3}$$

This equation can be simplified by first introducing the variables

$$A = \ln a, \qquad X = \ln x, \qquad D(X,A) = \ln(1+\bar{\xi}) \tag{3.4}$$

in terms of which we have (Nityananda and Padmanabhan, 1994)

$$\frac{\partial D}{\partial A} - h(A,X)\frac{\partial D}{\partial X} = 3h(A,X) \tag{3.5}$$

Introducing further a variable $F = D + 3X$, (3.5) can be written in a remarkably simple form as

$$\frac{\partial F}{\partial A} - h(A,X)\frac{\partial F}{\partial X} = 0 \tag{3.6}$$

The charecteristic curves to this equation - on which F is a constant - are determined by $(dX/dA) = -h(X,A)$ which can be integrated if h is known. But note that the charecteristics satisfy the condition

$$F = 3X + D = \ln[x^3(1+\bar{\xi})] = \text{constant} \tag{3.7}$$

or, equivalently,

$$x^3(1+\bar{\xi}) = l^3 \tag{3.8}$$

where l is another length scale. When the evolution is linear at all the relevant scales, $\bar{\xi} \ll 1$ and $l \approx x$. As clustering develops, $\bar{\xi}$ increases and x becomes considerable smaller than l. It is clear that the behaviour of clustering at some scale x is determined by the original *linear* power spectrum at the scale l through the "flow of information" along the charesteristics. This suggests that *we should actually try to express the true correlation function $\bar{\xi}(a,x)$ in terms of the linear correlation function $\bar{\xi}_L(a,l)$ evaluated at a different point.*

Let us see how we can do this starting from the quasilinear regime. Consider a region surrounding a density peak in the linear stage, around which we expect the clustering to take place. It is well known that density profile around this peak can be described by

$$\rho(x) \approx \rho_{bg}[1+\xi(x)] \tag{3.9}$$

Hence the initial mean density contrast scales with the initial shell radius l as $\bar{\delta}_i(l) \propto \bar{\xi}_L(l)$ in the initial epoch, when linear theory is valid. This shell will expand to a maximum radius of $x_{max} \propto l/\bar{\delta}_i \propto l/\bar{\xi}_L(l)$. In scale-invariant, radial collapse, models each shell may be approximated as contributing with a effective radius which is propotional to x_{max}. Taking the final effective radius x as proportional to x_{max}, the final mean correlation function will be

$$\bar{\xi}_{QL}(x) \propto \rho \propto \frac{M}{x^3} \propto \frac{l^3}{(l^3/\bar{\xi}_L(l))^3} \propto \bar{\xi}_L(l)^3 \tag{3.10}$$

That is, the final correlation function $\bar{\xi}_{QL}$ at x is the cube of initial correlation function at l where $l^3 \propto x^3\bar{\xi}_L^3 \propto x^3\bar{\xi}_{QL}(x)$. This is in the form demanded by (3.8) if $\bar{\xi} \gg 1$. *Note that we did not assume that the initial power spectrum is a power law to get this result.*

In case the initial power spectrum is a power law, with $\bar{\xi}_L \propto x^{-(n+3)}$, then we immediately find that

$$\bar{\xi}_{QL} \propto x^{-3(n+3)/(n+4)} \tag{3.11}$$

[If the correlation function in linear theory has the powerlaw form $\bar{\xi}_L \propto x^{-\alpha}$ then the process desribed above changes the index from α to $3\alpha/(1+\alpha)$. We shall comment more about this aspect later]. For the power law case, the same result can be obtained by more explicit means. For example, in power law models the energy of spherical shell will scale with its radius as some power which we write as $E \propto x_i^{2-b}$. Since $M \propto x_i^3$, it follows that the maximum radius reached by the shell scales as $x_{max} \propto (M/E) \propto x_i^{1+b}$. Taking the effective radius as $x = x_{eff} \propto x_i^{1+b}$, the final density scales as

$$\rho \propto \frac{M}{x^3} \propto \frac{x_i^3}{x_i^{3(1+b)}} \propto x_i^{-3b} \propto x^{-3b/(1+b)} \tag{3.12}$$

In this quasilinear regime, $\bar{\xi}$ will scale like the density and we get $\bar{\xi}_{QL} \propto x^{-3b/(1+b)}$. The index b can be related to n by assuming the the evolution starts at a moment

when linear theory is valid. The gravitational potential energy [or the kinetic energy] scales as $E \propto x_i^{-(n+1)}$ in the linear theory. This may be seen as follows: The power spectrum for velocity field, $P_v(k)$ in the linear regime is related to that of density by $P_v \propto P(k)/k^2 \propto k^{n-2}$. Hence the contribution to v^2 in each logarithmic scale in k-space is $k^3 P_v/2\pi^2 \propto k^{n+1} \propto x^{-(n+1)}$. Similarly, the gravitational potential energy due to *fluctuations* is

$$\phi \propto \int_0^x 4\pi y^2 dy \frac{\xi(y)}{y} \propto x^2 \xi(x) \propto x^{-(n+1)} \tag{3.13}$$

So the total energy in the initial configuration scales as $x_i^{-(n+1)}$ allowing us to determine $b = n + 3$. This shows that the correlation function in the quasilinear regime to be the one given by (3.11) .

The case with power law initial spectrum has no intrisic scale, if $\Omega = 1$. It follows that the evolution has to be self similar and $\bar{\xi}$ can only depend on $q = xa^{-2/(n+3)}$. This allows to determine the a dependence of $\bar{\xi}_{QL}$ by substituting q for x in (3.11). We find

$$\bar{\xi}_{QL}(a, x) \propto a^{6/(n+4)} x^{-3(n+3)/(n+4)} \tag{3.14}$$

Direct algebra shows that

$$\bar{\xi}_{QL}(a, x) \propto [\bar{\xi}_L(a, l)]^3 \tag{3.15}$$

reconfirming the local dependence in a and nonlocal dependence in spatial coordinate. This result has no trace of original assumptions [spherical evolution, scale-invariant spectrum ....] left in it and hence once would strongly suspect that it will have far general validity.

Let us now proceed to the third and nonlinear regime. If we ignore the effect of mergers, then it seems reasonable that virialised systems should maintain their densities and sizes in proper coordinates, i.e. the clustering should be "stable". This would require the correlation function to have the form $\bar{\xi}_{NL}(a, x) = a^3 F(ax)$. [The factor a^3 arising from the decrease in background density]. From our previous analysis we expect this to be a function of $\bar{\xi}_L(a, l)$ where $l^3 \approx x^3 \bar{\xi}_{NL}(a, x)$. Let us write this relation as

$$\bar{\xi}_{NL}(a, x) = a^3 F(ax) = U[\bar{\xi}_L(a, l)] \tag{3.16}$$

where $U[z]$ is an unknown function of its argument which needs to be determined. Since linear correlation function evolves as a^2 we know that we can write $\bar{\xi}_L(a, l) = a^2 Q[l^3]$ where Q is some known function of its argument. [We are using l^3 rather than l in defining this function just for future convenience of notation]. In our case $l^3 = x^3 \bar{\xi}_{NL}(a, x) = (ax)^3 F(ax) = r^3 F(r)$ where we have changed variables from (a, x) to (a, r) with $r = ax$. Equation (3.16) now reads

$$a^3 F(r) = U[\bar{\xi}_L(a, l)] = U[a^2 Q[l^3]] = U[a^2 Q[r^3 F(r)]] \tag{3.17}$$

Consider this relation as a function of a at constant r. Clearly we need to satisfy $U[c_1 a^2] = c_2 a^3$ where c_1 and c_2 are constants. Hence we must have

$$U[z] \propto z^{3/2}. \tag{3.18}$$

Thus in the extreme nonlinear end we should have

$$\bar{\xi}_{NL}(a, x) \propto [\bar{\xi}_L(a, l)]^{3/2} \tag{3.19}$$

[Another way deriving this result is to note that if $\bar{\xi} = a^3 F(ax)$, then $h = 1$. Integrating (3.5) with appropriate boundary condition leads to (3.19) .] Once again we did not need to invoke the assumption that the spectrum is a power law. If it is a power law, then we

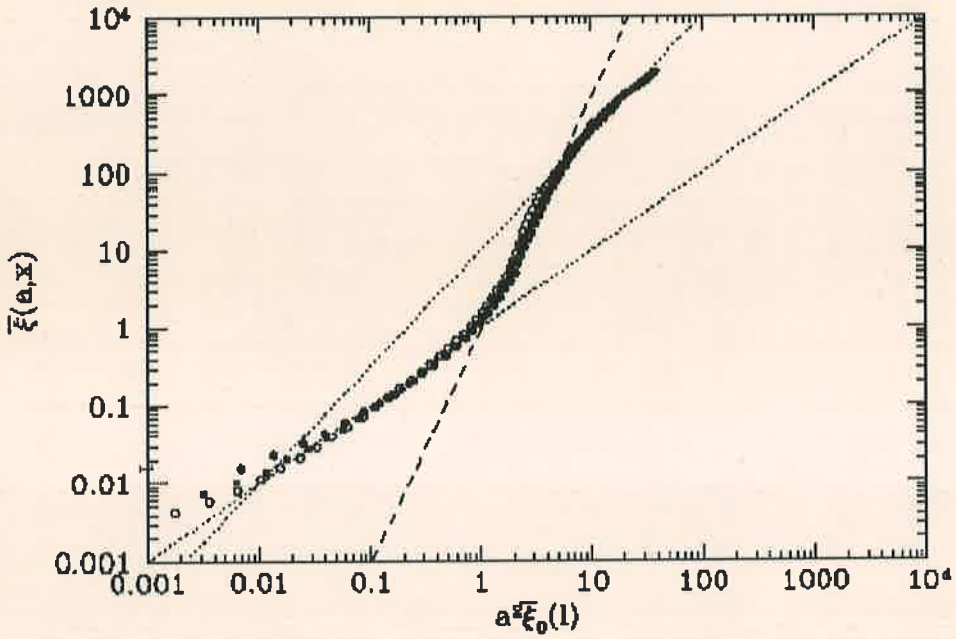

FIGURE 1. Plot of $\bar{\xi}(a,x)$ against $\bar{\xi}_L(a,l)$ for CDM model. The slopes in the three different regimes are indicated. The data points are for three different redshifts [0.1,0.5 and 1.0] and are based on the simulations described in Padmanabhan et al (1995).

get,

$$\bar{\xi}_{NL}(a,x) \propto a^{(3-\gamma)}x^{-\gamma}; \qquad \gamma = \frac{3(n+3)}{(n+5)} \tag{3.20}$$

This result is based on the assumption of "stable clustering" and was originally derived by Peebles (Peebles, 1965). It can be directly verified that the right hand side of this equation can be expressed in terms of q alone, as we would have expected.

Putting all our results together, we find that the nonlinear mean correlation function can be expressed in terms of the linear mean correlation function by the relation:

$$\bar{\xi}(a,x) = \begin{cases} \bar{\xi}_L(a,l) & (\text{for } \bar{\xi}_L < 1, \bar{\xi} < 1) \\ \bar{\xi}_L(a,l)^3 & (\text{for } 1 < \bar{\xi}_L < 5.85, 1 < \bar{\xi} < 200) \\ 14.14\bar{\xi}_L(a,l)^{3/2} & (\text{for } 5.85 < \bar{\xi}_L, 200 < \bar{\xi}) \end{cases} \tag{3.21}$$

The numerical coefficients have been determined by continuity arguments. We have assumed the linear result to be valid upto $\bar{\xi} = 1$ and the virialisation to occur at $\bar{\xi} \approx 200$ which is result arising from the spherical model. The exact values of the numerical coefficients can be obtained only from simulations.

The true test of such a model, of course, is N-body simulations and remarkably enough, simulations are very well represented by relations of the above form. Figure 1 shows the results of a CDM simulation based on the investigations carried out in Padmanabhan et

al. (1995). This data can be fitted by the relations (Bagla & Padmanabhan, 1993):

$$\bar{\xi}(a,x) = \begin{cases} \bar{\xi}_L(a,l) & (\text{for } \bar{\xi}_L < 1.2, \bar{\xi} < 1.2) \\ \bar{\xi}_L(a,l)^3 & (\text{for } 1 < \bar{\xi}_L < 5, 1 < \bar{\xi} < 125) \\ 11.7\bar{\xi}_L(a,l)^{3/2} & (\text{for } 5 < \bar{\xi}_L, 125 < \bar{\xi}) \end{cases} \tag{3.22}$$

[The fact that numerical simulations show a correlation between $\bar{\xi}(a,x)$ and $\bar{\xi}_L(a,l)$ was originally pointed out by Hamilton et al. (1991) who, however, tried to give a multiparameter fit to the data. This fit has somewhat obscured the simple physical interpretation of the result though has the virtue of being very accurate for numerical work.]

A comparison of (3.21) and (3.22) shows that the physical processes which operate at different scales are well represented by our model. In other words, the processes descibed in the quasilinear and nonlinear regimes for an *individual* lump still models the *average* behaviour of the universe in a statistical sense. It must be emphasised that the key point is the "flow of information" from l to x which is an exact result. Only when the results of the specific model are recast in terms of suitably chosen variables, we get a relation which is of general validity. It would have been, for example, incorrect to use spherical model to obtain relation between linear and nonlinear densities at the same location or to model the function h. With hindsight, it is clear why such attempts have not succeeded in the past.

It may be noted that to obtain the result in the nonlinear regime, we needed to invoke the assumption of stable clustering which has not been deduced from any fundamental considerations. In case mergers of structures are important, one would consider this assumption to be suspect (see Padmanabhan et al., 1995). We can, however, generalise the above argument in the following manner: If the virialised systems have reached stationarity in the statistical sense, the function h - which is the ratio between two velocities - should reach some constant value. In that case, one can integrate (3.6) and obatin the result $\bar{\xi}_{NL} = a^{3h}F(a^h x)$. A similar argument will now show that

$$\bar{\xi}_{NL}(a,x) \propto [\bar{\xi}_L(a,l)]^{3h/2} \tag{3.23}$$

in the general case. For the power law spectra, one would get

$$\bar{\xi}(a,x) \propto a^{(3-\gamma)h} x^{-\gamma}; \qquad \gamma = \frac{3h(n+3)}{2+h(n+3)} \tag{3.24}$$

Simulations are not accurate enough to fix the value of h; in particular, the asymptotic value of h could depend on n within the accuracy of the simulations. It may be possible to determine this dependence by modelling mergers in some simplified form.

If $h = 1$ asymptotically, the correlation function in the extreme nonlinear end depends on the linear index n. One may feel that physics at highly nonlinear end should be independent of the linear spectral index n. This will be the case if the asymptotic value of h satisfies the scaling

$$h = \frac{3c}{n+3} \tag{3.25}$$

in the nonlinear end with some constant c. Only high resolution numerical simulations can test this conjecture that $h(n+3) =$ constant.

It is possible to obtain similar relations between $\xi(a,x)$ and $\xi_L(a,l)$ - in two dimensions as well. In this case the pair conservation equation can be transformed to

$$\frac{\partial D}{\partial A} - h(A,x)\frac{\partial D}{\partial x} = 2h(A,X), \tag{3.26}$$

and the characteristics are determined by the relation

$$x^2(1+\bar{\xi}) = l^2 \tag{3.27}$$

The self similar models due to Filmore and Goldreich (1984) show that even in two dimensions $x_{\max} \propto l/\bar{\delta}_i \propto l/\bar{\xi}_L(l)$. Repeating the previous analysis and noting that in two dimensions $M \propto x^2$, we find $\bar{\xi}_{QL}(x) \propto [\bar{\xi}_L(l)]^2$. In the nonlinear limit if we invoke stable clustering, then $\bar{\xi}_{NL}(a,x) = a^2 F(ax)$. An analysis similar to the one performed before will now show that $\bar{\xi}_{NL}(a,x) \propto \bar{\xi}_L(a,l)$. Thus in 2-D the scaling relations are

$$\bar{\xi}(a,x) \propto \begin{cases} \bar{\xi}_L(a,l) & \text{(Linear)} \\ \bar{\xi}_L(a,l)^2 & \text{(Quasi-linear)} \\ \bar{\xi}_L(a,l) & \text{(Nonlinear)} \end{cases} \tag{3.28}$$

For power law spectrum the nonlinear correction function will $\bar{\xi}_{NL}(a,x) = a^{2-\gamma}x^{-\gamma}$ with $\gamma = 2(n+2)/(n+4)$.

If we generalize the concept of stable clustering to mean constancy of h in the nonlinear epoch, then the correlation function will behave as $\bar{\xi}_{NL}(a,x) = a^{2h}F(a^h x)$. In this case, if the spectrum is a power law then the nonlinear and linear indices are related to

$$\gamma = \frac{2h(n+2)}{2+h(n+2)} \tag{3.29}$$

All the features discussed in the case of 3 dimensions are present here as well. For example, if the asymptotic value of h scales with n such that $h(n+2) =$ constant then the nonlinear index will be independent of the linear index. (Numerically it would be lot easier to test this result in 2-D rather than in 3-D; work is in progress to test these results).

4. Conclusions and Speculations

The relations obtained above will, of course, have certain limitations on their validity. To begin with, we do expect a weak n-dependence in these relations due to averaging over peaks of different heights. This has been discussed using a simple analytic model, as well as numerically, in Padmanabhan et al. (1995). [Also see Mo et al., (1995) for a similar discussion]. Broadly speaking, we expect h to be lower at a given $\bar{\xi}$ (in the nonlinear regime with $\bar{\xi} > 10$ or so) as we add more small scale power. If stable clustering is invoked, then it is possible to motivate this conclusion along the following lines: Consider a spherical region of initial radius r_i and an overdensity of $\nu\sigma$, where $\sigma = \sigma_0 r_i^{-(n+3)/2}$ is the variance of the gaussian density fluid. In a spherical model, this region will expand to a maximum radius of about $(r_i/\nu\sigma)$ and virialise to a final radius $r \equiv \lambda(r_i/\nu\sigma) = (\lambda/\nu\sigma_0) r_i^{\frac{n+5}{2}}$ where $\lambda \approx 0.5$. We shall assume that the coorelation function at the nonlinear end $\xi_{NL}(r)$ is contributed by such virialised objects and can be computed as

$$1 + \xi(r) \approx \xi(r) = \left\langle \left(\frac{r_i}{r}\right)^3 \right\rangle. \tag{4.30}$$

[This assumption is equivalent to the stable clustering hypothesis.] From the relation $r = (\lambda/\nu\sigma_0) r_i^{(n+5)/2}$, it follows that

$$\xi(r) = \left(\frac{\sigma_0}{\lambda}\right)^{6/(n+5)} r^{-\frac{3(n+3)}{(n+5)}} \left\langle \nu^{6/(n+5)} \right\rangle. \tag{4.31}$$

Assuming that ν is a gaussian variable, we get

$$\langle \nu^{6/(n+5)} \rangle = \frac{1}{\sqrt{2\pi}} 2^{\frac{1-n}{2n+10}} \Gamma\left(\frac{n+11}{2n+10}\right). \tag{4.32}$$

Equations (4.31 and (4.32) show that $\xi(r) \propto \xi_L^{3/2}[l]$ with $l \approx r\xi^{1/3}$; however, the proportionality constant has a weak n-dependence.

The averaging in (4.30) can be made more sophisticated by using a weightage proportional to r_i^m. In that case, we still obtain the same r dependence but the proportionality constant becomes

$$c = \frac{\Gamma\left(\frac{n+11+2m}{2n+10}\right)}{\Gamma\left(\frac{n+5+2m}{2n+10}\right)}. \tag{4.33}$$

This reduces to the above result when $m = 0$; recently, Mo et al. (1995) have suggested a model based on $m = 3$, which – of course – leads to similar conclusions.

Secondly, the asymptotic behaviour will be sensitive to the value of Ω. When $\Omega < 1$, structures "freeze out" during the late stages of evolution and "stable clustering" is likely to be a reasonable assumption.

The description presented above is most suitable for models with hierarchical clustering. As is well known, models like HDM which have very little small scale power evolve quite differently. As an extreme example of this case one may study the evolution of a model which has power peaked at some scale L with a band of width ΔL. During the linear epoch the power spectrum retains its shape and evolves in a self similar manner. [This statement is not precisely true. If the power spectrum falls faster than λ^{-4} for large λ, then the discreteness of particles will generate a λ^{-4} tail fairly rapidly. This effect, however, is important only at large scales and we will be concerned with small scales.] At some epoch, the scale L will go nonlinear and we will form structures which look typically like shells of radius L and width ΔL. As evolution proceeds, small scale power is rapidly generated due to the instability of the matter in the shell. At late epochs, there is more power at small scales than at large scales due to this effect. [See figure 2.]

Since the small scale power in the linear theory is exponentially small, the nonlinear correlation function should be a rapidly growing function of ξ_L in order to reproduce the results of simulation. This shows that the approach developed in the previous sections is not very useful for HDM like models. It is however possible to reproduce the simulation results of such models along the following lines. Consider an individual unit with a density profile $f(\mathbf{x}) = f(|\mathbf{x}|)$. If we now build a universe by superposing several such clumps, then the density at any location will be given by

$$\rho(\mathbf{x}) = \sum_i f(\mathbf{x} - \mathbf{x}_i) \tag{4.34}$$

It is easy to see that the power spectrum due to such a distribution is given by

$$P(\mathbf{k}) = P_{BG}(\mathbf{k})|f_{\mathbf{k}}|^2 \tag{4.35}$$

where $P_{BG}(\mathbf{k})$ is the power spectrum corresponding to the distribution of centres of the clumps. That is, $P_{BG}(\mathbf{k})$ is the fourier transform of the two-point correlation function of the individual clumps. At high redshifts, virialized structures have not formed and one may think of the original set of particles as distributed according to some power spectrum. As time goes on, individual clumps with some density profile originate leading to significant amount of $|f_{\mathbf{k}}|^2$. We may now think of $P_{BG}(\mathbf{k})$ as the power spectrum corresponding to the distribution of power centres (say, the minimum of gravitational potentials) and $|f_{\mathbf{k}}|^2$ as a modulation of this power spectrum. As time goes on, transfer

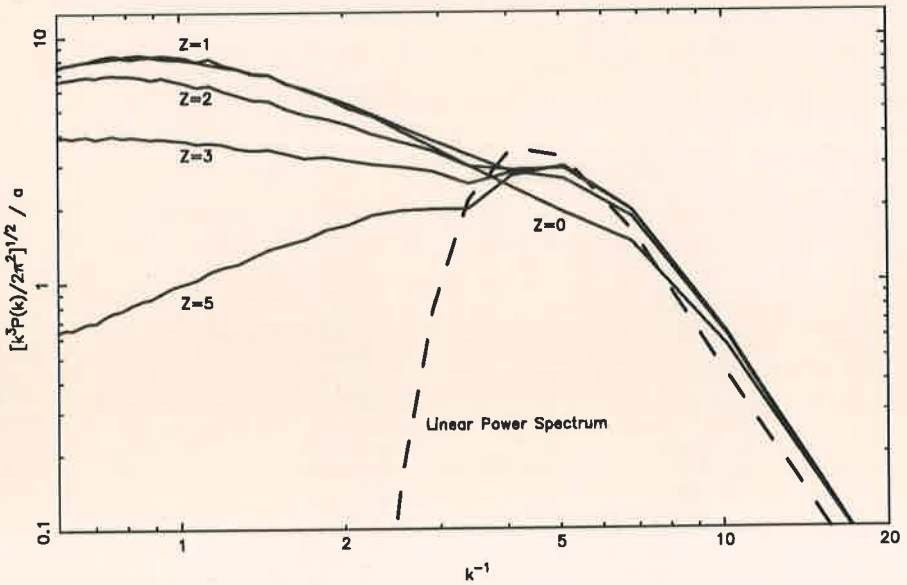

FIGURE 2. Evolution of power spectrum for a model with power concentrated in a small range of scales. The dashed curve shows the linear power spectrum, other curves show nonlinear power spectrum at five redshifts, scaled to the final epoch. Sudden increase in power at small scales and later saturation at a large value is apparent from this graph.

of power takes place from scale to scale changing the definitions of P_{BG} and $f_{\mathbf{k}}$. At late stages if one can identify some approximate nonlinear units then it will be possible to build a specific model. For HDM-like spectra, one models the universe as made of a bunch of spherical shells with different radii and shell thickness. The individual evolution of each such unit can be studied using spherical model and leads to results which are in qualitative agreement with the simulations (Bagla and Padmanabhan, 1995a).

These results suggest the following paradigm for nonlinear clustering: One is interested in expanding the density field as

$$\rho(a,\mathbf{x}) = \int \prod_{i=1}^{N} dc_i A(c_i,a) f(c_i,\mathbf{x}) \tag{4.36}$$

where $f(c_i,\mathbf{x})$ is a function which depend on a set of N parameters c_i and $A(c_i,a)$ determines the weightage to different scales. [In linear theory, $f = \exp i\mathbf{k}.\mathbf{x}$ and the c_i's are just $\mathbf{k}$] As evolution proceeds, one can ask how A evolves and use this information to quantify the notion of power transfer. Of course, the important thing is to decide on a set of functions f which are intutively simple, well matched to the actual evolution and easy to work with. [In the case of fluid turbulence, one can use the concept of eddies to model the dynamics to some extent; in a way, we are looking for a concept analogous to that of eddies in the case of gravity]. It is fairly straight forward to perform such analysis with functions f which have support in a small band of wavelengths [like the fashionable "wavelets"] and obtain the evolution of A's either from simulations or in some analytic approximation. But it is doubtful whether such an exercise is of any use unless one can show that the basis functions f are better tuned to the actual dynamics than the plane waves. This requires a careful analysis of the fully nonlinear equations nonperturbatively using different basis functions (Padmanabhan, 1995). The distinction between hierarchical clustering and HDM models will then only be in the choice of specific

nonlinear units chosen to describe the evolution. Hopefully such an approach like this will provide greater insight into the problem of nonlinear clustering.

Finally note that the radial, scale invariant infall described in the quasilinear regime has the effect of changing the linear correlation function $\bar{\xi}_L = x^{-(n+3)} = x^{-b}$ to the quasilinear correlation function $\bar{\xi}_{QL} = x^{-3b/(1+b)}$. It is amusing to ask what will be the effect of iterating this process N-times. It is easy to see that the index after N iterations can be expressed in the form:

$$\gamma_N = \frac{A_N b}{1 + B_N b}; \quad A_N = 3^N; B_N = \frac{3^N - 1}{2} \tag{4.37}$$

The fixed point, of course, is $\gamma_\infty = 2$ which is the only nontrivial fixed point for such an evolution [with the other, trivial, fixed point being zero]. If one could model the evolution as repeated application of this process, one would expect a continuum of scaling relations with the evolution being driven to a singular isothermal sphere. The quasilinear evolution does not change the x^{-2} profile, a result which was noted earlier in Bagla and Padmanabhan (1995). It is not clear whether the clustering can indeed be modelled using (4.37) .

5. Acknowledgements

I thank J.S. Bagla, D. Lynden-Bell, R. Nityananda, J.P. Ostriker and P.J.E. Peebles for several useful discussions. I thank my coauthors R. Cen, J.P. Ostriker and F. Summers for permission to adapt a figure from our joint work.

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