

NONLINEAR EVOLUTION OF DENSITY PERTURBATIONS USING APPROXIMATE CONSTANCY OF GRAVITATIONAL POTENTIAL

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ABSTRACT

During the evolution of density inhomogeneities in an $\Omega = 1$, matter dominated universe, the typical density contrast changes from $\delta \simeq 10^{-4}$ to $\delta \simeq 10^2$. However, during the same time, the typical value of the gravitational potential generated by the perturbations changes only by a factor of order unity. This significant fact can be exploited to provide a new, powerful, approximation scheme for studying the formation of nonlinear structures in the universe. This scheme, discussed in this paper, evolves the initial perturbation using a Newtonian gravitational potential frozen in time. We carry out this procedure for different initial spectra and compare the results with the Zeldovich approximation and the frozen flow approximation (proposed by Mattarrese et al. recently). Our results are in far better agreement with the N-body simulations than the Zeldovich approximation. It also provides a dynamical explanation for the fact that pancakes remain thin during the evolution. While there is some superficial similarity between the frozen flow results and ours, they differ considerably in the velocity information. Actual shell crossing does occur in our approximation; also there is motion of particles along the pancakes leading to further clumping. These features are quite different from those in frozen flow model. We also discuss the evolution of the two-point correlation function in various approximations.

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It is generally believed that structures like galaxies etc. formed through the growth of density perturbations via gravitational instability. To understand the formation of these structures, it is necessary to evolve the initial perturbations to highly non-linear regimes, which turns out to be a technically formidable task. Usually one applies linear perturbation theory to evolve the inhomogeneities when they are small, but relies on extensive N-body simulations to model the non-linear regime. As a result, we have only a limited knowledge of the physics of the non-linear regime.

It would, therefore, be worthwhile to develop semianalytic approximations which could help us to understand the non-linear evolution of the perturbations. In this paper, we propose one such approximation scheme. We describe qualitatively its main features and compare its results with some other approximation schemes.

To motivate the approximation scheme suggested in this paper, it is best to begin by recalling some well known theoretical facts. Consider the evolution of a density perturbation δ in the matter dominated era of an $\Omega = 1$ Friedmann model, described by an expansion factor $a(t)$. It is well known that $\delta \propto a$ for the growing mode, implying that the perturbed Newtonian gravitational potential ϕ (generated by this perturbation) remains constant in time. As evolution proceeds, δ will approach unity invalidating the application of linear theory. Even though non-linear evolution is extremely complex, it is reasonable to assume that the constancy of gravitational potential is approximately maintained even in the non-linear regime. For example, if one uses the spherical model (Peebles, 1980) to study the non-linear evolution then the gravitational potential changes only by a factor 2 or so as the structures turn around, collapse and virialise. After the bound structures are formed, virial theorem ensures that no significant evolution occurs to the potential. *In other words, the gravitational potential due to perturbations varies only by a factor of the order unity when density contrast changes from, say, 10^{-4} to 10^2 .* We believe this is an extremely important fact which can be profitably exploited to study non-linear evolution.

In order to implement this feature effectively, one can take a cue from another popular approximation scheme, usually called the Zeldovich approximation (Zeldovich, 1970; Shandarin and Zeldovich, 1989). In the Zeldovich approximation, one moves the particles using a fixed velocity field which is taken to be the initial velocity field. Mathematically, this scheme is implemented by assuming that

$$\mathbf{x}(t) = \mathbf{q} + a\mathbf{v}(\mathbf{q}) \quad (1)$$

where \mathbf{q} is the Lagrangian coordinate of a particle, \mathbf{x} is the co-moving Eulerian coordinate and $\mathbf{v}(\mathbf{q}) = d\mathbf{q}/da$ is the initial velocity field. This field $\mathbf{v}(\mathbf{q})$ is related to the gravitational

potential ϕ by the relation

$$\mathbf{v} \equiv \nabla\Phi \equiv -\frac{2}{3a\dot{a}^2}\nabla\phi \quad (2)$$

Since Zeldovich approximation perturbs the trajectories rather than the density, it can be used to study a larger range of density contrast than the linear perturbation theory. Unfortunately this approximation suffers from two significant shortcomings. The density contrast at any time, calculated from mass conservation, can be expressed as

$$\delta = \left[\prod_{i=1}^3 (1 - a\lambda_i) \right]^{-1} - 1 \quad (3)$$

where λ_i are the eigen values of the matrix $(-\partial v_i/\partial q_j)$ with $\lambda_1 \geq \lambda_2 \geq \lambda_3$. Hence, strictly speaking, the approximation breaks down when $a\lambda_1 = 1$ which occurs at fairly low density contrasts (say, around $\delta \simeq 3$) in realistic models.

A more serious shortcoming is the following: Zeldovich evolution gives a distorted view of the density distribution once pancakes form and particles move *through* the pancakes. Since the particles only 'remember' the *initial* velocities, they move past the pancakes unhindered, thereby leading to fair amount of thickening of pancakes. Numerical simulations, on the other hand, clearly show that pancakes remain thin for long periods of time. We wish to emphasise that the question, "why pancakes do not thicken during the evolution of density perturbations?" is an important dynamical issue which needs to be understood from first principles. It is, of course, possible to introduce artificial models, like the adhesion model containing damping mechanisms, which correctly reproduce the N-body results. The key question, in our opinion, is to understand why models like the adhesion model work as well as they do. In other words, there is something intrinsic in the gravitational dynamics of particles in an expanding universe which dampens the kinetic energy in the component perpendicular to the pancake more than the kinetic energy parallel to the pancake. This effect can be understood qualitatively as follows :

Consider a set of particles interacting via Newtonian gravity in an expanding universe. If we confine our attention to regions with dimensions much smaller than the Hubble radius, then the equation of motion for the i^{th} particle is well approximated by:

$$\ddot{\mathbf{r}}_i = - \sum_{j \neq i} \frac{Gm}{|\mathbf{r}_{ij}|^3}; \quad \mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j \quad (4)$$

Here \mathbf{r}_i stands for the proper coordinate related to the comoving coordinate \mathbf{x}_i by $\mathbf{r}_i = a(t)\mathbf{x}_i$. In terms of \mathbf{x}_i , we have

$$\ddot{\mathbf{x}}_i + \frac{2\dot{a}}{a}\dot{\mathbf{x}}_i = -\nabla\phi_i \quad (5)$$

where

$$\nabla^2 \phi = \frac{4\pi G}{a^3} \left[\sum_j m \delta(\mathbf{x} - \mathbf{x}_j) - \rho_0 a_0^3 \right] = 4\pi G \rho_b \delta \quad (6)$$

in the matter dominated phase of the Friedmann model. In the above form the equations depend explicitly on t due to the presence of the terms (\dot{a}/a) and a^3 . This can be avoided by introducing new dimensionless time and space coordinates τ and \mathbf{y} ; via

$$\tau \equiv \ln(t/T); \quad \mathbf{x}_i \equiv L\mathbf{y}_i; \quad L^3 = Gmt_0^2/a_0^3 \quad (7)$$

with an arbitrary constant T . Transforming the equations, we easily find that:

$$\frac{d^2 \mathbf{y}_i}{d\tau^2} + \frac{1}{3} \frac{d\mathbf{y}_i}{d\tau} = -\nabla_{\mathbf{y}} U \quad (8)$$

$$\nabla_{\mathbf{y}}^2 U = 4\pi \sum_i \delta(\mathbf{y} - \mathbf{y}_i) - \frac{2}{3}. \quad (9)$$

Equation (8) can be used to understand the motion of the particles near a pancake. Locally, the gravitational acceleration, $\mathbf{g} = -\nabla U$ is towards the pancake and has no appreciable component parallel to the pancake. Hence the velocity parallel to pancake (v_{\parallel}) decays only due to the cosmic expansion (described by the $(1/3)\dot{\mathbf{y}}$ term) while the velocity normal to the pancake (v_{\perp}) is affected by the potential as well. It turns out that this effect restricts the motion in direction perpendicular to the pancake (see Fig.4 and our discussion later). Quite clearly, the local gravitational force produced by the pancake plays an important role in this process. Since Zeldovich approximation misses out the effects of acceleration and evolves particles using the initial velocity field, it fails to reproduce the thin nature of the pancakes.

The above discussion suggests a possible new way of approximating non-linear evolution. The success of Zeldovich approximation clearly demonstrates that one should work in a Lagrangian description and consider perturbation of trajectories. However, it is also necessary that the effect of acceleration is incorporated into this scheme. On a rigorous N-body simulation, this is done by using the *exact* gravitational force acting on the particle at every instant of time. However, if our hypothesis of approximate constancy of gravitational potential is correct, then one should be able to make considerable progress by evolving the trajectories using a prespecified gravitational potential which remains frozen in time. This is the idea which we pursue here.

The Euler equation governing the motion of fluid particles can be written in the comoving coordinates as

$$\frac{d\mathbf{v}}{da} = -\frac{3}{2a} \left[\mu \nabla \phi + \mathbf{v} \right] \quad (10)$$

where the velocity is defined as $\mathbf{v} = dx/da$ and μ is a constant that equals $(2/3a\dot{a}^2)$. The left hand side of (10) is, of course, the convective derivative evaluated along the path of the particle. In the proposed scheme we use the Euler's equation with the potential as specified at the initial time. The initial velocity of a particle is assumed to be $-\mu\nabla\phi$, just as in the Zeldovich approximation. At subsequent instants of time, the acceleration is computed by using the *instantaneous* velocity of the particle and the *initially specified* potential at its *current* position. In reality, the potential will also change with time as the density distribution evolves. It is this fact which we ignore by invoking the hypothesis of constancy of the potential.

We find that this approximation works well and that the pancakes do not thicken. The particles tend to move along the pancakes towards regions of lower potential to end up in a few clumps. The acceleration we use in this approximation is largest in regions where the instantaneous velocity vector points along with the gradient of potential, as happens for particles after they cross the pancake.

This may be thought of as a "frozen potential" formulation for the evolution of density perturbations. Recently, Mattarrese et. al.(1992) have proposed a frozen flow approximation which essentially freezes the velocity field rather than the potential. In this approach the particles move along streamlines computed from the initial potential, using the same relation as in Zeldovich approximation (2). However, here the inertia of particles is ignored, whereas in Zeldovich approximation inertia is assumed to dominate over change in the force field. We take into consideration both factors but assume a constant potential. As we shall see, our approximation works better than the frozen flow approximation.

The idea was explored numerically in the following manner. We start with an initial potential ϕ (in 2D with a 64×64 grid) which is a realisation of a gaussian random field with the power spectrum $P_\phi \propto k^{(n-2)}$ for various values of n . (Note that n in 2D is "equivalent" to $n-1$ in 3D). The particles are now moved by the ansatz described above. We normalise our expansion factor such that $a = \langle \delta_L \rangle$, the average density contrast calculated by the linear theory. Thus we would expect nonlinear structures to form around $a \gtrsim 1$.

In figures 1 and 2 we have shown the evolution for $n = 1$ and $n = -1$ (corresponding to $n_{3D} = 0, -2$). The top three frames are based on our approximation, the middle three are based on frozen flow and the bottom ones are from using the Zeldovich approximation. The time evolution proceeds from left to right in all the cases and the frames correspond to $a = 1, 2$ and 3.

It is quite obvious that the Zeldovich approximation is the worst of the three for $a \gtrsim 1$. The pancakes thicken enormously and - of course - this is to be expected. The top and middle frames show that there is some superficial similarity in the spatial distribution

calculated by frozen flow and frozen potential; however, there is significant difference in the details. The velocities of particles in the nonlinear regimes (and the nature of their motion near pancakes) are quite different in the two cases. In frozen flow, no shell crossing occurs and the particles approach the pancakes more and more slowly. In the frozen potential approximation, there is shell crossing and significant motion near the pancakes. To see this difference clearly, we have plotted the trajectories of a few particles near the pancakes in fig. 4. In the frozen flow, trajectories asymptotically approach one another defining the "pancakes" (thin lines in fig.4). But in the frozen potential approximation (thick lines), the particles cross the pancake and oscillate around the plane because of the local acceleration. It is this feature which limits the thickness of the pancake. The pancake in figure 4 is shown at $\langle \delta_L \rangle = 2$ while the trajectories are evolved for a much longer time to show that particles remain confined in the pancake. One can also see that the results of our approximation are similar to that of N-body simulations and that in evolving from $a = 2$ to $a = 3$, the particles form more clumpy regions in comparison with frozen flow approximation.

Figure 3 shows the frozen potential (top) and frozen flow (bottom) results for $n = 0$ (left) and $n = 2$ (right). All frames are evaluated at $a = 3$. The frozen flow does not bring out the clumpiness expected in the $n = 2$ case as well as the frozen potential does. Comparison with figures 1,2 show that the approximation works quite well for a wide variety of n which are of interest.

The nature of these approximations - Zeldovich, Frozen flow and Frozen potential - can be compared qualitatively in the following simple manner. Let $\mathbf{v}(\mathbf{q})$ be the initial velocity field which we start with as a realization of some given power spectrum. Suppose that, by random occurrence, all the velocity vectors in some given, finite subregion of space point roughly towards a plane (on both sides). Quite clearly particles in that region will move towards this plane and, in Zeldovich approximation, we will obtain a pancake coinciding with the plane. When we evolve the system further (using Zeldovich approximation) the particles will continue to move with the *original* velocity they have. This will result in the particles crossing the plane and moving away from the plane and a consequent thickening of the pancake.

In the frozen flow approximation the *local* velocity field is used to move the particles (rather than the *initial* velocity field at the original Lagrangian coordinate). For the velocity field which is generally pointing towards a plane, continuity demands the velocities would get smaller and smaller near the plane. This would mean that the velocity of particles normal to the plane will become smaller and smaller as they approach the would-be pancake. Quite obviously no thickening of pancake will take place, since no pancake

has really formed.

In the frozen potential formulation, the evolution is quite different. Along with the initial velocity field, we also have an initial acceleration field g , defined in that region. Since the density contrast was small at the initial epoch, allowing the use of linear theory, we know that $g = -\mu\nabla\phi$ and v point in the same direction. In other words, the acceleration vector field also points towards the plane. If we evolve the particles using the acceleration field, then the particles which cross over the pancake get pushed back towards the pancakes. This force prevents the particles from moving too far away from the pancake; consequently the pancakes remain thin.

We stress that in the frozen flow approximation no actual shell crossing can occur. Thus even though frozen flow approximation prevents thickening of pancakes, it does it in a somewhat crude manner. In the case of frozen potential approximation, shell crossing does occur but the particles are dynamically influenced by the gravitational potential which reverses their direction of motion when they cross the pancake.

In Fig.5 and 6 we have plotted the two point correlation function, as it evolves with time, for the frozen potential and frozen flow approximation. The distance scale used has been left arbitrary as we are working in two dimensions and comparison with the real universe is not very meaningful. A somewhat qualitative comparison can be made using the fact that the correlation function is about unity at a grid size of 1.5 at $a = 1$ for frozen potential approximation. It is clearly seen that for frozen flow approximation, the two point correlation function is much more strongly peaked at small scales than the frozen potential approximation at $a = 3$.

It would be interesting to compare our results with actual N-body simulations in detail. We have to be content in this paper with visual comparisons with published results since we do not possess an N-body code now. We hope to do this in a future collaboration.

There are several further directions and possibilities which we plan to explore using this approximation. Since the velocity information in our approximation is quite different from that of frozen flow, it is worthwhile to calculate the scaled average pair velocity, $h(a, x) = [-v_{pair}(a, x)/\dot{a}x]$ and compare it with the mean square fluctuation in mass, $\sigma^2(a, x)$, at the same epoch and scale. Theoretical reasoning (see Nityananda and Padmanabhan, 1993; Padmanabhan, 1992) as well as results of numerical simulations (Hamilton et al., 1991) suggest that: (i) The function $h(a, x)$ depends on a and x only through σ^2 ; i.e., $h(a, x) = h[\sigma^2(a, x)]$ and (ii) $h \propto \sigma^2$ for $\sigma^2 \ll 1$, rises to a value greater than unity and falls to unity for $\sigma^2 \gg 1$. This overshooting of hubble velocity in the formation of bound structures is unlikely to occur in the frozen flow models but will happen in the present approach.

The approximation suggested here could also replace the adhesion model in some contexts. It may be recalled that the adhesion models (with an artificial viscosity term) were originally introduced to keep the pancakes thin and to provide information regarding the locations of the pancakes. The approach described here keeps the pancakes thin because of natural, dynamical reasons eliminating the need for and introduction of a viscosity term. The approach can also provide the location of pancakes. What is more, the code for the present approximation is both conceptually and numerically simple to implement as compared to the adhesion code. The formation and properties of the voids, for example, can be studied quite easily by this approach.

The approximation outlined in this paper is ideally suited to study another important problem: the dynamics of baryons in the dark matter potential wells. Right now, attacking this problem using N-body simulations is extremely hard and time consuming. The approximation schemes like adhesion model will not be useful in this context; the frozen-flow, on the other hand, will provide a distorted picture of the motion near pancakes. The frozen potential approximation, with a better velocity information, may give us valuable insights into the physical processes at work while economising greatly on the computation involved.

Lastly, one would like to study the limitation on the validity of the approximation by trying to understand accuracy to which the gravitational potential remains constant in the course of real evolution. In an $\Omega = 1$ matter dominated model the potential does not change during the linear regime. When the nonlinear phase begins, if we approximate overdense regions by suitably averaged regions with spherical symmetry then the potential changes in the nonlinear phase only by a factor of order unity. The situation becomes more complex when we take into account the deviations from the spherical symmetry. Even though the gravitational potential at any one point is contributed by matter distribution all over, there is some amount of cancellation between the contributions from overdense and underdense regions in an expanding universe. This suggests that even in a realistic case, the change in the potential is only of order unity in the nonlinear phase. We plan to address these and related questions in a future publication (Bagla and Padmanabhan, 1993).

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Figure Captions

Fig.1 Evolution of density perturbations for various approximations: time increases from left to right. These frames correspond to $\langle \delta_L \rangle = 1, 2$ and 3 respectively. The top row of frames is for frozen potential approximation, the middle shows evolution according to the frozen flow approximation and the bottom row is for Zeldovich approximation. Power spectrum used here has the index $n = 1$.

Fig.2 Same as figure 1 but for $n = -1$.

Fig.3 Evolved perturbations at $\langle \delta_L \rangle = 3$ for frozen potential (top row) and frozen flow (bottom row) approximations. These are for $n = 0$ (left column) and $n = 2$ (right column) power spectra.

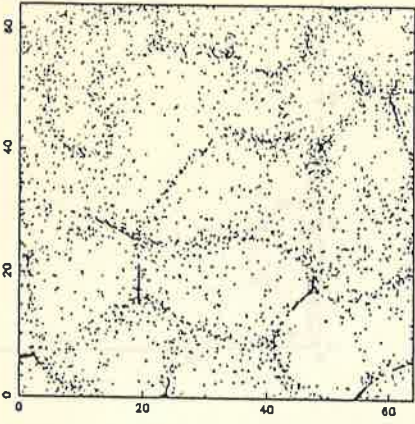
Fig.4 Trajectories of a few particles near a pancake. The pancake is shown at $\langle \delta_L \rangle = 2$, the trajectories are evolved for a much longer time to show that particles remain confined in the pancake. Thin lines correspond to trajectories in frozen flow approximation and thick lines for frozen potential approximation.

Fig.5 Evolution of the correlation function in frozen potential approximation. Note that $a = \langle \delta_L \rangle$.

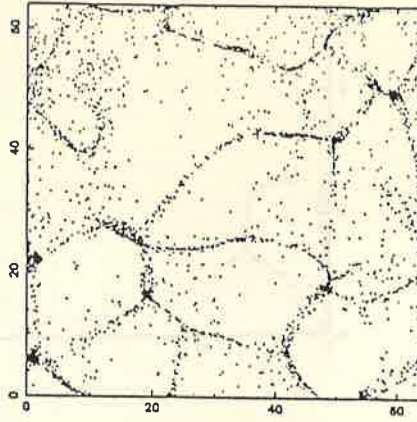
Fig.6 Same as figure 5 but for frozen flow approximation.

Figure 1

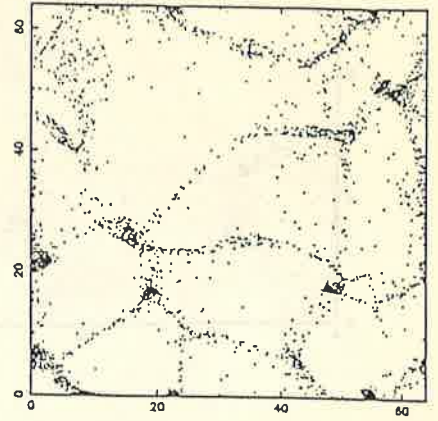
$\alpha=1.0$: Frozen Potential



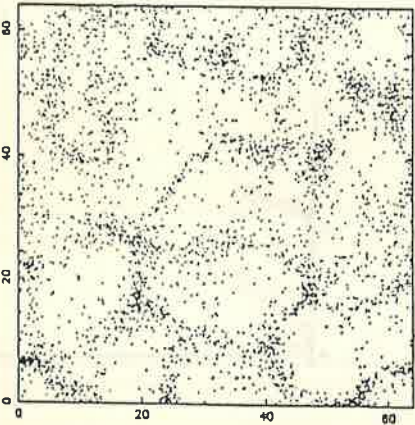
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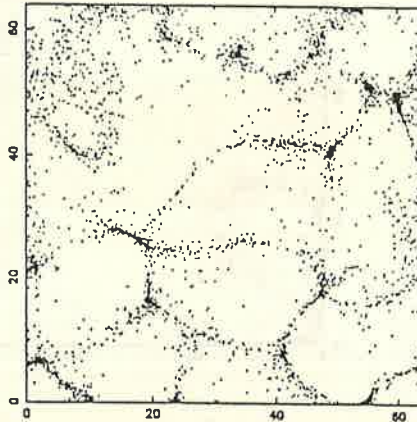
$\alpha=3.0$: Frozen Potential



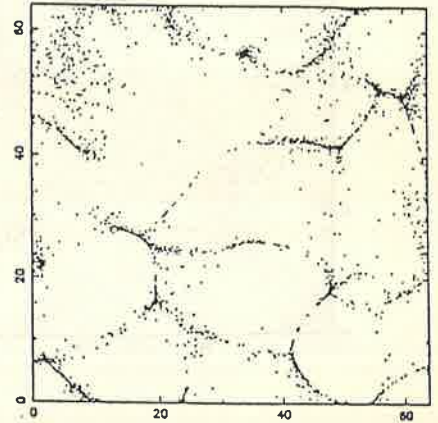
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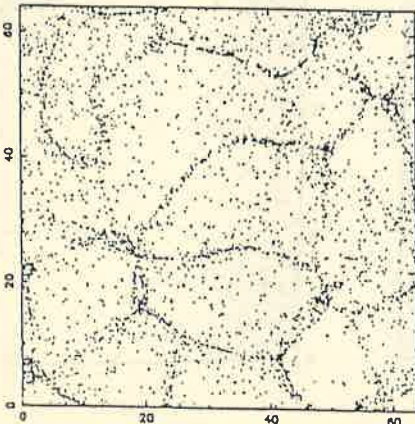
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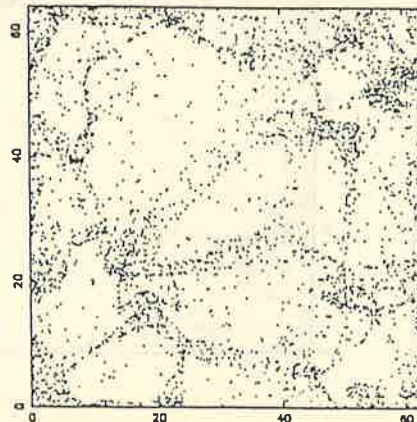
$\alpha=3.0$: Frozen Flow



$\alpha=1.0$: Zeldovich Approximation



$\alpha=2.0$: Zeldovich Approximation



$\alpha=3.0$: Zeldovich Approximation

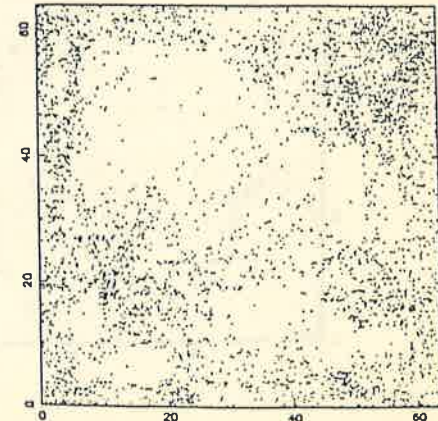
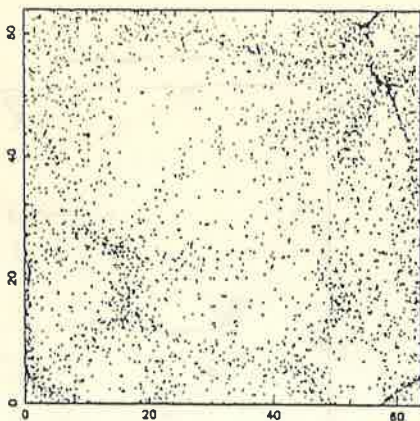
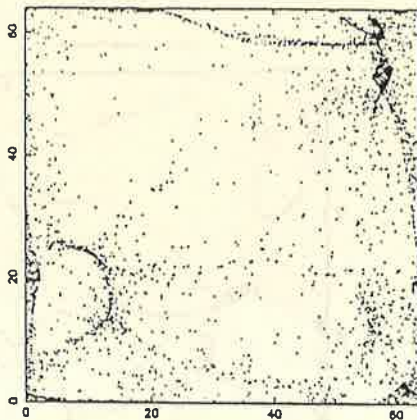


Figure 2

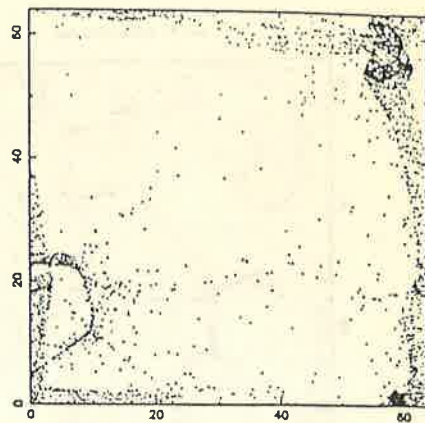
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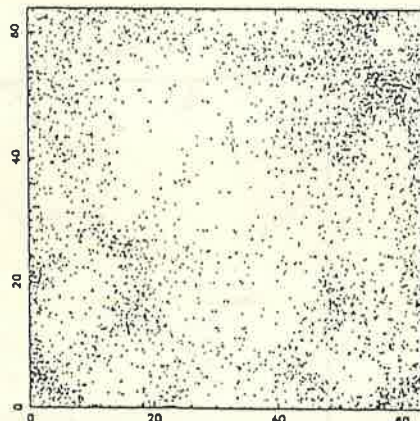
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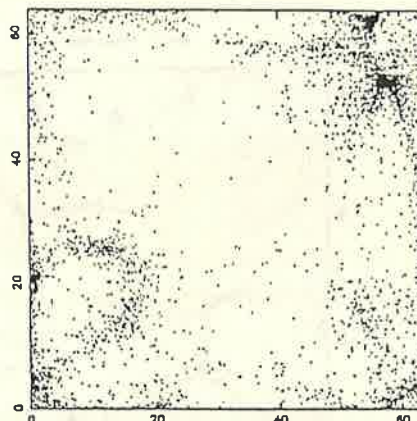
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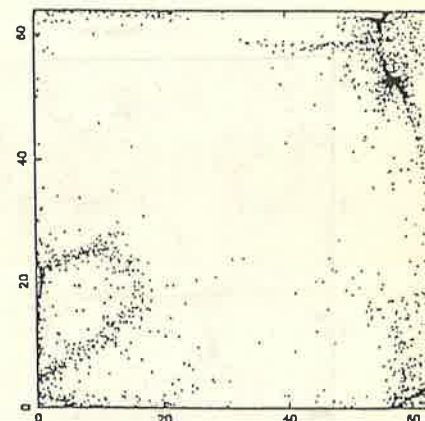
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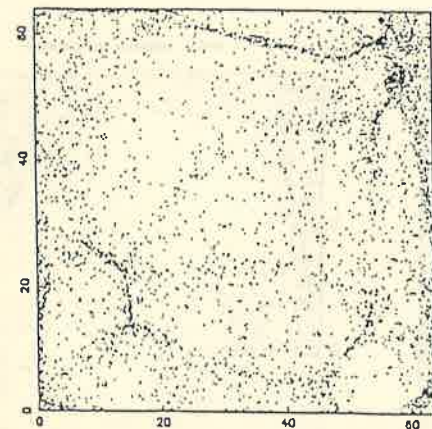
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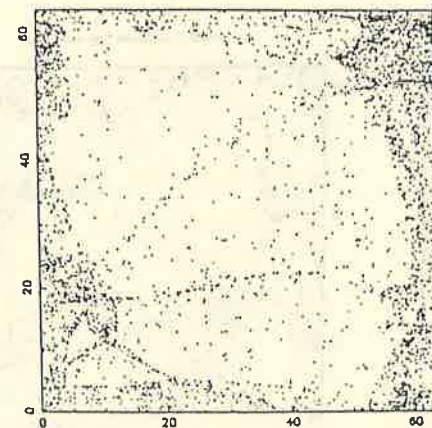
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$\alpha=1.0$: Zeldovich Approximation



$\alpha=2.0$: Zeldovich Approximation



$\alpha=3.0$: Zeldovich Approximation

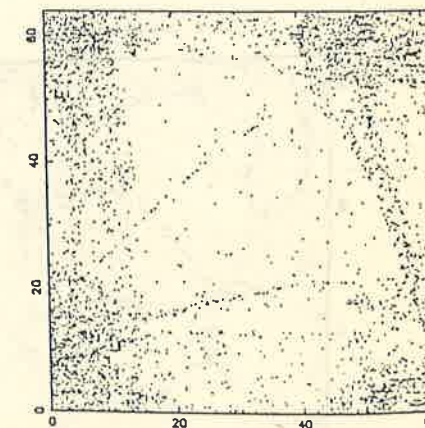
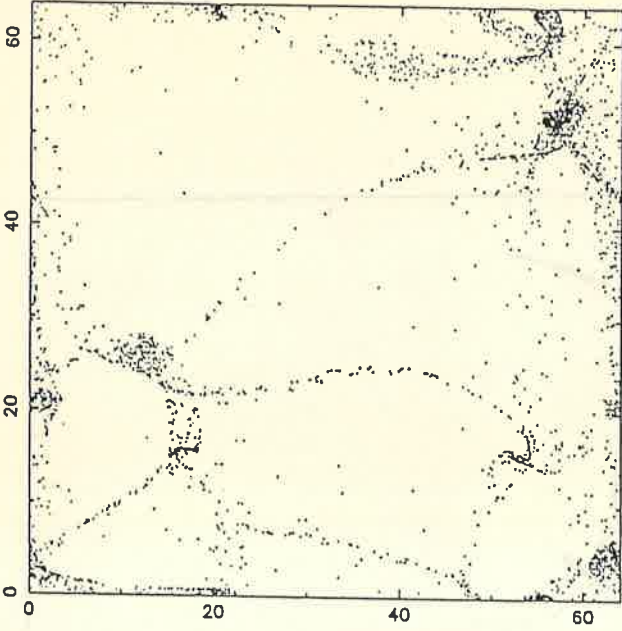
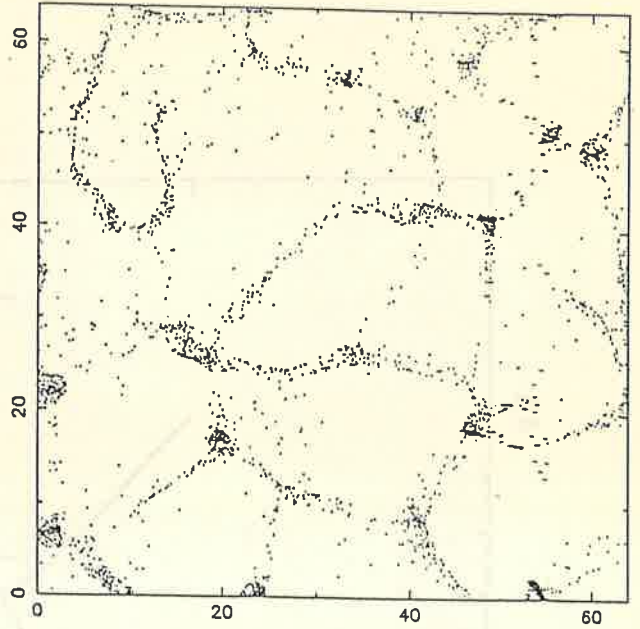


Figure 3

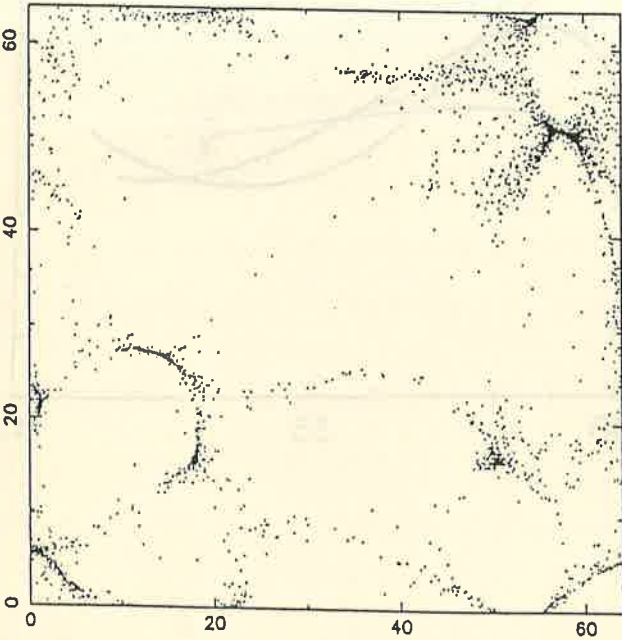
n=0 : Frozen Potential



n=2 : Frozen Potential



n=0 : Frozen Flow



n=2 : Frozen Flow

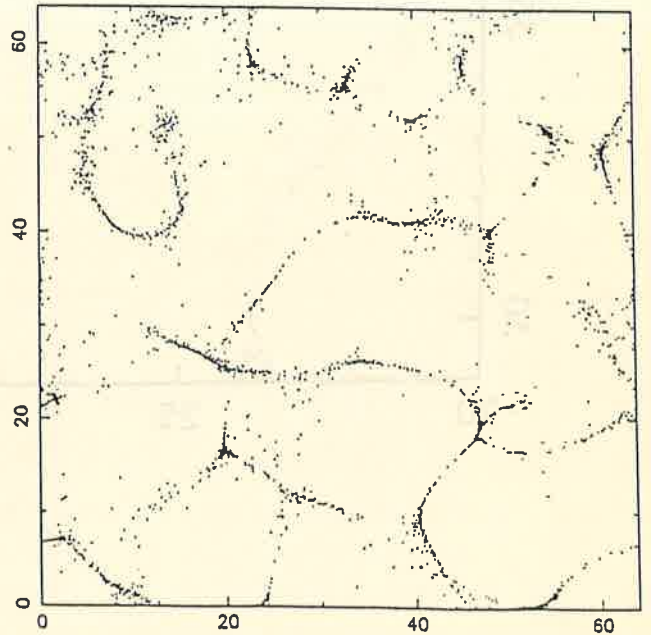


Figure 4

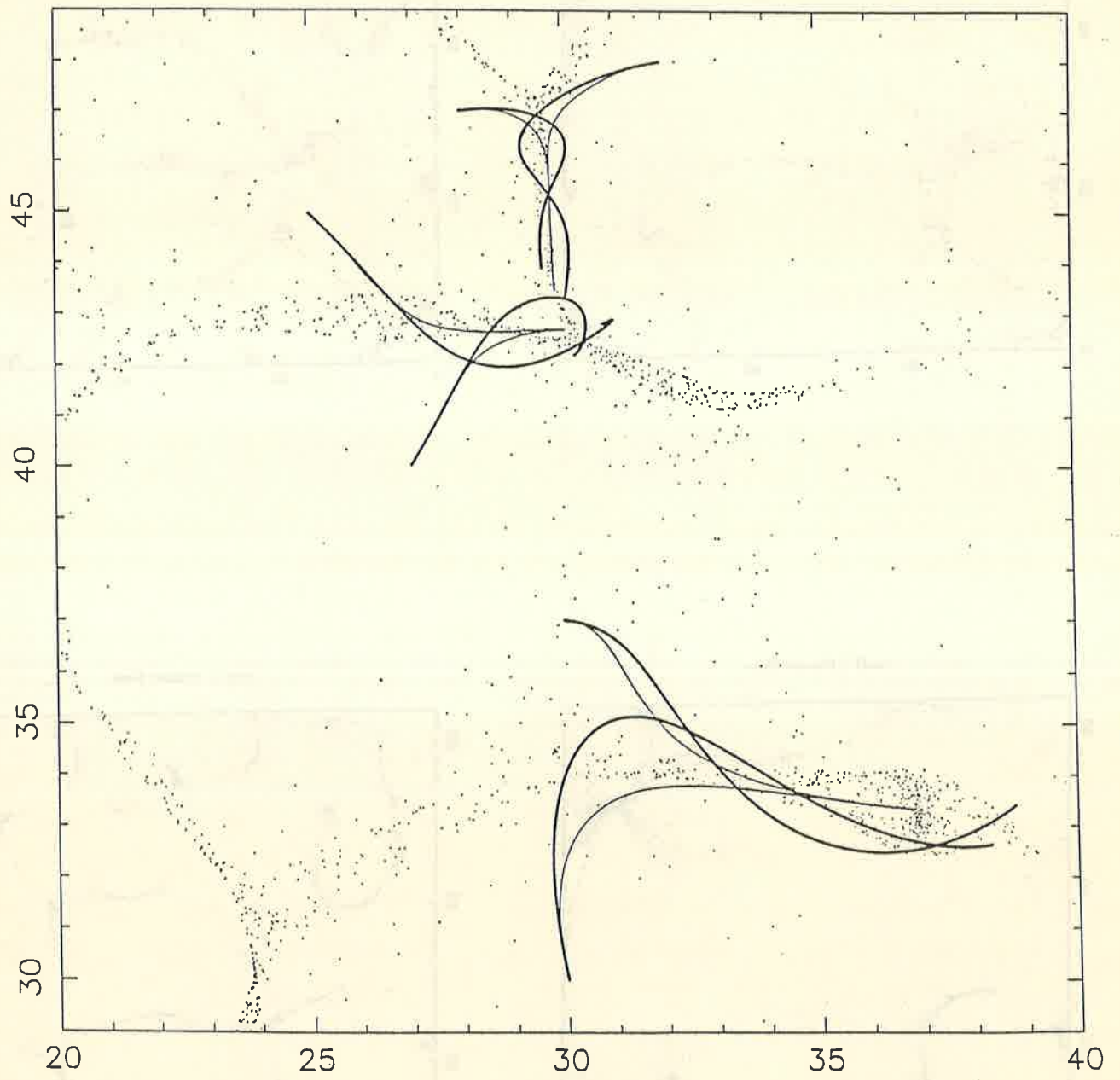


Figure 5

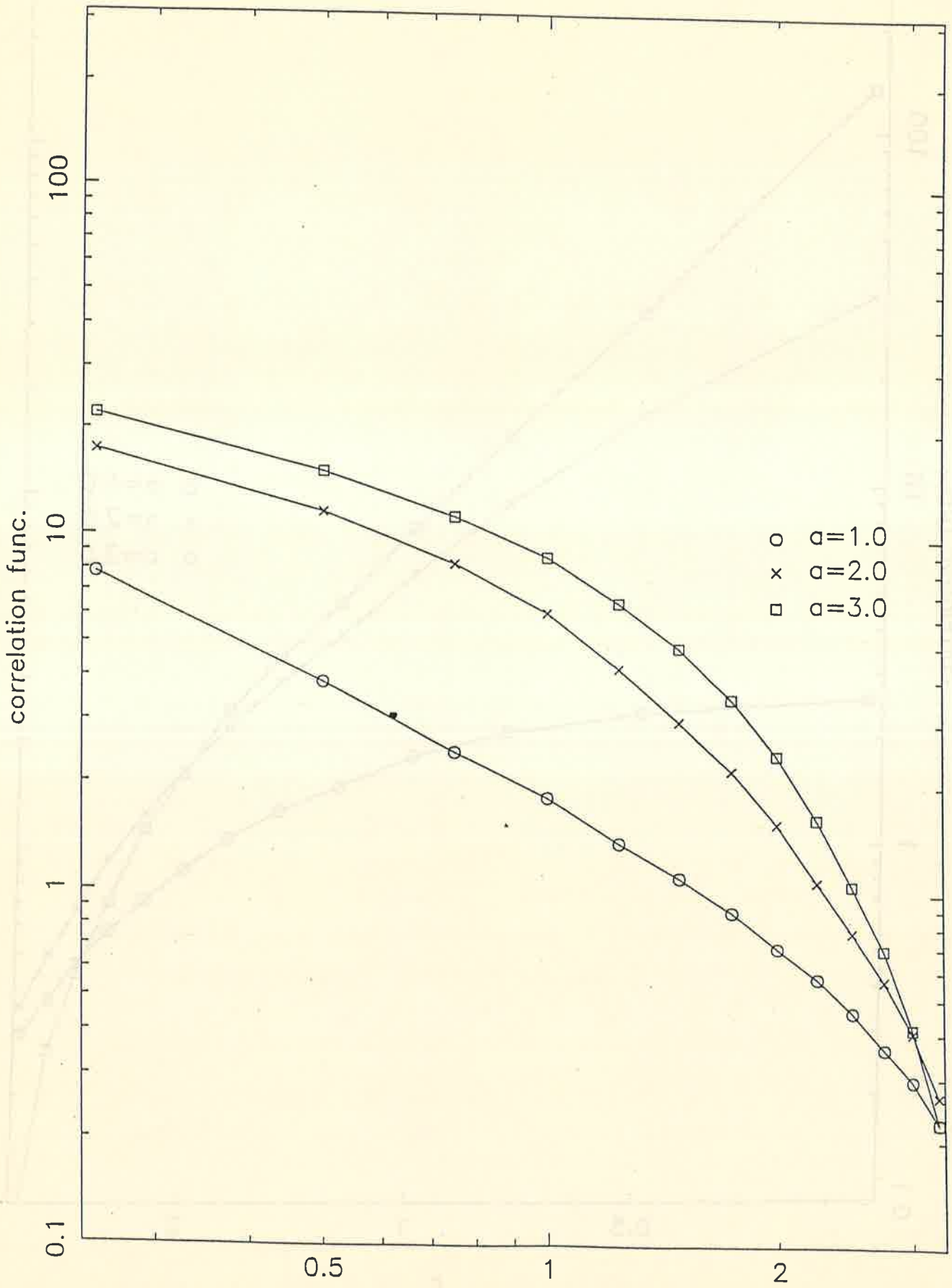


Figure 6

