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BLUESHIFT AND SPECTRAL FEATURES OF THE GRAVITATIONAL SEARCHLIGHT

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ABSTRACT

It is shown that a photon emitted in the forward direction by a charged particle moving in an equatorial circular orbit centred on a highly collapsed mass M, the radius being slightly in excess of one and a half times the Schwarzschild radius, is strongly blueshifted when it arrives at a distant receiver. A ring shaped emitting region composed of such orbiting particles has a power law spectrum of the form $d\nu/\nu$ *as seen by a distant stationary observer.*

§(1): *INTRODUCTION*

Circular orbits in Schwarzschild's geometry have attracted considerable attention in recent years [*1,2,3*]. Interesting effects are observed if a source of radiation moves in a circle of radius slightly in excess of $3GM/c^2$ in an equatorial plane of a highly collapsed spherical object or a black hole of mass M. For example, an electric charge moving in this way can emit a synchrotron type of radiation. Similarly, gravitational radiation can also be emitted by a particle in such a circular orbit. Most such effects are, however, of only theoretical significance, since they are too small to be measurable. Here we investigate an effect which may have possible astrophysical applications. This is the blueshift of radiation emitted in the forward direction by a source of radiation, while moving in a circular orbit of the type described above. This radiation is blueshifted by the Doppler effect to such an extent that it exceeds the strong gravitational redshift in the neighbourhood of the mass M.

To begin with, we calculate the extent of the blueshift and its effect on the spectrum of continuum radiation emitted by a ring of sources moving around the mass M. We also investigate the geometrical properties of the null rays emerging from such moving sources.

§(2): *BLUESHIFT OF THE FORWARD SEARCHLIGHT*

We assume the mass M to have the standard Schwarzschild line element in its exterior region,

$$ds^2 = \left(1 - \frac{r_0}{r}\right)c^2dt^2 - \left(1 - \frac{r_0}{r}\right)^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2), \qquad (1)$$

where

$$r_0 = \frac{2GM}{c^2}, \qquad (2)$$

G being the gravitational constant. The surface of the object is assumed to be given by

$$r = R_S \leqslant \frac{3}{2}r_0. \qquad (3)$$

A circular equatorial orbit is specified by the following relations:

$$r = a = \frac{3}{2}r_0(1 + \varepsilon), \qquad \theta = \frac{\pi}{2}, \qquad \phi_0 = nt_0 = n\frac{\gamma s}{c}, \qquad (4)$$

where $(t_0, a, \frac{1}{2}\pi, \phi_0)$ are the coordinates of a particle in the orbit at the proper time parameter s and

$$n = \left(\frac{r_0}{2a^3}\right)^{\frac{1}{2}}c, \qquad \gamma = \left(1 - \frac{3r_0}{2a}\right)^{-\frac{1}{2}}. \qquad (5)$$

for $\varepsilon \ll 1$, $\gamma \simeq \varepsilon^{-\frac{1}{2}}$ and $n \simeq 2c/3\sqrt{3}r_0$.

We now consider an observer O in the equatorial plane $\theta = \frac{1}{2}\pi$ with space coordinates $r = R \gg r_0$, $\phi = 0$. The time coordinate of O also measures its proper time in this approximation. The typical position of the particle moving in the circular orbit will be specified by the world point Q with coordinates given by (4).

Suppose a ray of light is sent from Q and received by O at time $t = T$. Then the equations of this null geodesic from Q are given by

$$\theta = \tfrac{1}{2}\pi, \qquad \frac{dr}{dt} = cf(r,q), \qquad \frac{dr}{d\phi} = r^2 f(r,q)q^{-1}\left(1 - \frac{r_0}{r}\right)^{-1}, \qquad (6)$$

where

$$f(r,q) = \left(1 - \frac{r_0}{r}\right)\left[1 - \frac{q^2}{r^2}\left(1 - \frac{r_0}{r}\right)\right]^{\frac{1}{2}}, \tag{7}$$

and q is a parameter specifying the null geodesic. The value of q is to be chosen by the requirements that (i) the ray is emitted in the instantaneous direction of motion of the source, and (ii) that it arrives at O. These requirements in fact overdetermine the problem, indicating that not from all positions on the circular orbit will the ray emitted in the forward direction arrive at O. We will show that there exists at least one such emitting point Q from whence the ray arrives at O.

To calculate the blueshift we will proceed in the following way. We will start with condition (ii) first and work out the blueshift from a general position. Next we will calculate the explicit value of the blueshift for the specific case satisfying condition (i).

Using (ii) we get the following relations:

$$T - t_0 = \frac{1}{c}\int_a^R \frac{\mathrm{d}r}{f(r,q)}\ , \tag{8}$$

and

$$-\ \phi_0 = \int_a^R \frac{q\left(1 - \frac{r_0}{r}\right)}{r^2 f(r,q)}\ \mathrm{d}r. \tag{9}$$

Thus (9) determines q and then (8) determines the time of reception of the light signal sent out at Q. The integrals (8,9) are of the elliptic kind, but fortunately it is not necessary to evaluate these explicitly for the calculation of the blueshift. We shall need the values of (8,9) for another purpose later in this work.

Consider now another light ray emitted from the circular orbit at a world point Q' with the time coordinate $t_0 + \Delta t_0$. This ray will have a parameter $q + \Delta q$ and will arrive at O at the time $T + \Delta T$. Then we can write equations similar to (8,9) for this null ray. Taking the difference between these equations and the corresponding equations (8,9), we get

$$\Delta T - \Delta t_0 = c^{-1}\Delta q\int_a^R \frac{\partial}{\partial q}\left\{\frac{1}{f(r,q)}\right\}\ \mathrm{d}r = I(q)\Delta q, \qquad \text{(say)}, \tag{10}$$

$$-\ \Delta\phi_0 = -\ n\Delta t_0 = \Delta q\int_a^R \frac{1}{r^2}\left(1 - \frac{r_0}{r}\right)\frac{\partial}{\partial q}\left\{\frac{q}{f(r,q)}\right\}\ \mathrm{d}r$$

$$= J(q)\Delta q, \qquad \text{(say)}, \tag{11}$$

If ν_0 is the frequency of the wave emitted between QQ' and ν the frequency of the wave received by O, the blueshift is given by

$$1 + b = \frac{\nu}{\nu_0} = \frac{\Delta s}{c\Delta T} . \tag{12}$$

where Δs is the proper time of the source corresponding to the time interval Δt_0. From (4,10,11,12) we get

$$1 + b = \frac{J(q)}{\gamma[J(q) - nI(q)]} . \tag{13}$$

In general the evaluation of $I(q)$ and $J(q)$ would require numerical integration. However, we can compute (13) analytically. This follows from a straightforward calculation that

$$cI(q) = qJ(q) \tag{14}$$

and then equation (13) reduces to

$$1 + b = \gamma^{-1}\left(1 + \frac{nq}{c}\right)^{-1} . \tag{15}$$

To evaluate q for the special position satisfying condition (i) we may use the condition that the null geodesic should be tangential to the circular orbit at this point. It is easily verified that this condition requires

$$q^2 = a^2\left(1 - \frac{r_0}{a}\right)^{-1} . \tag{16}$$

The two solutions of (16) correspond to the forward and backward emission. The forward emission is given by

$$q = - a\left|\left(1 - \frac{r_0}{a}\right)^{-\frac{1}{2}}\right| . \tag{17}$$

Substituting (5,17) into (15) we finally get

$$1 + b = \frac{\left(1 - \frac{3}{2}\xi\right)^{\frac{1}{2}}[2(1 - \xi)]^{\frac{1}{2}}}{[2(1 - \xi)]^{\frac{1}{2}} - \xi^{\frac{1}{2}}} , \tag{18}$$

where

$$\xi = \frac{r_0}{a} . \tag{19}$$

In the limiting case where $\xi \to 2/3$, i.e. $\varepsilon \to 0$, we get very large blueshifts. Using the approximation $\varepsilon << 1$, we get

$$b \simeq \frac{2}{3}\,\varepsilon^{-\frac{1}{2}}. \tag{20}$$

It is interesting to note that the formula (15) can be obtained in another way by using a general formula given by Schrödinger [4]. This formula gives the frequency shift in the following form

$$\frac{\nu}{\nu_0} = \frac{(u\cdot k)_R}{(u\cdot k)_S}\,, \tag{21}$$

where $(u\cdot k)_S$ represents the scalar product of the tangent vectors of the world line of motion of the source particle and of the light ray. $(u\cdot k)_R$ represents the corresponding product at the receiving end.

Using (4,6), u and k at S can be easily determined while at R, $(u\cdot k)$ is simple because u has only the time component u^0 which, in our approximation, is equal to c^{-1}. The result (15) then follows without difficulty.

§(3): *PROPAGATION OF THE LIGHT RAY*

It is interesting to study how the propagation of the light ray depends on the parameter ε. For this we evaluate the integrals (8,9) for q given by (17) and for a given (4). For numerical integration it is convenient to express (9) in the form

$$\phi_0 = \int_0^1 \frac{dv}{[(1-\xi) - v^2(1-\xi v)]^{\frac{1}{2}}}\,, \tag{22}$$

where $v = a/r$, and we have taken the limiting case of $R \to \infty$. This is the result of ϕ_0 changing only slightly with R as $R \to \infty$. The main contribution to ϕ_0 comes from the vicinity of $r = a$.

For $\xi = 0$, $\phi_0 = \frac{1}{2}\pi$. This is the simple case of a light ray in Minkowski space. As $\xi \to 2/3$, $\phi \to \infty$ according to the formula:

$$\phi_0 \simeq \ln \frac{12(2-\sqrt{3})}{\varepsilon}\,. \tag{23}$$

This result is proved in the appendix. In table I we give the values of ϕ_0 for various values of ξ close to 2/3. We see that as indicated by (22), the light ray makes several circuits around the object arriving at the distant receiver.

Consider now what happens at a finite but large value of R. For $r \to R$, the equation (6) is approximated by

TABLE I

n	$\xi = \frac{2}{3}(1 - 10^{-n})$	ϕ_0 (radians)	Number of revolutions around M ($\phi_0/2\pi$)
2	0.66	5.776	0.92
3	0.666	8.076	1.28
4	0.6666	10.378	1.65
5	0.66666	12.681	2.02
6	0.666666	14.934	2.38
7	0.6666666	17.230	2.74
8	0.66666666	19.542	3.11
9	0.666666666	21.867	3.48

$$\frac{d\phi}{dr} \simeq \frac{q}{r^2} .$$

With O as the origin we choose rectangular Cartesian coordinates (x,y) in the plane $\theta = \frac{1}{2}\pi$. This is possible since we are in the region of space-time which is nearly Minkowskian. In the region close to the new origin

$$dy \simeq R d\phi, \qquad dx \simeq dr,$$

so that

$$\frac{dy}{dx} \simeq \frac{q}{R} . \tag{25}$$

In other words, the angle made by the light ray reaching O with the line of sight to the centre of the object is given by

$$\alpha = \left|\frac{q}{R}\right| \simeq \frac{3\sqrt{3}r_0}{2R} . \tag{26}$$

This determines the angular radius of the orbit as seen by the observer at O.

Finally, in table II below we give the numerical results of the T-integral for $R = 100a$ and for small values of ε. This choice of the value of R is somewhat arbitrary, the only criterion being that

TABLE II

ξ	$T - t_0$
0.66	33.03
0.666	34.49
0.6666	35.69
0.66666	36.87

R should be large enough for the space to be almost flat for $r > R$. It will be noticed from table II that as ε becomes smaller and smaller, the time taken increases logarithmically with ε^{-1}. (To convert these numbers into cgs units it is necessary to multiply them by GM/c^3).

Owing to this slow dependence of T on ε (see appendix) even values of ε as small as 10^{-10} will not significantly alter the time scale for the light ray to emerge from the vicinity of the object. We make this point here because in the following section we will be concerned with radiation arriving from orbits very close to the unstable circular orbit at $a = 3r_0/2$.

§(4): *EMISSION SPECTRUM*

We now imagine the mass to be surrounded by a ring of particles moving in circular orbits with different $\varepsilon \ll 1$. The volume of the ring will be determined by the overall range of ε. The volume element between ε and $\varepsilon + d\varepsilon$ is

$$dV = 2\pi r^2 \left(1 - \frac{r_0}{a}\right)^{-\frac{1}{2}} dr d\theta$$

$$\simeq \frac{9\sqrt{3}}{2} \pi r_0^{\,2} H d\varepsilon, \tag{27}$$

where $r d\theta = H$ is the height of the ring. In approximating dV to the final expression we have used the fact that $\varepsilon \ll 1$, i.e., $a \simeq 3r_0/2$.

Suppose the ring emits N photons of frequency ν_0 per second in the rest frame per unit volume. The number emitted in time $c^{-1}ds$ is $c^{-1}N ds dV$. At large distances with $r = R$, they will be distributed over a sphere of surface area $4\pi R^2$. Although most of them will be in the equatorial plane, the average over the whole sphere gives a flux (per unit area per unit time) as

$$\frac{N\mathrm{d}s\mathrm{d}V}{4\pi R^2 c\mathrm{d}T} = \frac{N\nu}{4\pi R^2 \nu_0}\, dV. \tag{28}$$

Here we have used the result (12). The average flux of energy is therefore given by

$$\frac{Nh\nu_0}{4\pi R^2}(1 + b)^2 \mathrm{d}V = \frac{E(\nu_0)\mathrm{d}V}{4\pi R^2}(1 + b)^2 = \frac{E(\nu_0)\nu^2}{4\pi R^2 \nu_0{}^2}\,\mathrm{d}V, \tag{29}$$

where $E(\nu_0)$ is the volume emissivity at frequency ν_0. Using (27, 20) we get

$$\mathrm{d}V = \frac{9\sqrt{3}}{2}\pi r_0{}^2 H \left|\frac{\mathrm{d}\varepsilon}{\mathrm{d}\nu}\right| \mathrm{d}\nu$$

$$= 4\sqrt{3}\pi r_0{}^2 H \nu_0{}^2 \frac{\mathrm{d}\nu}{\nu^3}\,. \tag{30}$$

Combining (29,20) we get the flux in the range ν, $\nu + \mathrm{d}\nu$ as $S(\nu)\mathrm{d}\nu$ where

$$S(\nu) = \frac{\sqrt{3} r_0{}^2 H E(\nu_0)}{\nu} = \frac{A(\nu_0)}{\nu}\,, \tag{31}$$

and

$$A(\nu_0) = 4\sqrt{3}\,\frac{G^2 H M^2}{c^4}\, E(\nu_0). \tag{32}$$

§(5): *CONCLUSION*

The properties of the gravitational searchlight discussed above are of interest because they arise from the non-Euclidian nature of the geometry of the space-time in the neighbourhood of a highly collapsed object. The ν^{-1} form of the spectrum is quite common in astrophysics and it is worth investigating the circumstances in which the formula (31) could be applicable. The astrophysical implications will be discussed in a later paper. Here we only emphasize the fact that the shape of the spectrum obtained here is of purely geometrical origin and may therefore apply in a variety of cases.

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APPENDIX

We wish to study the behaviour of ϕ_0 given by (22) as $\xi \to 2/3$, i.e., as $\varepsilon \to 0$. First we note that at $\xi = 2/3$, the integrand has a factor $(1 - v)^{-1}$ and the integral diverges logarithmically. So ϕ_0 is expected to diverge as $\xi \to 2/3$ largely because of contributions to the integral at $v \approx 1$.

It is convenient to write

$$\xi = \frac{2}{3} - \Delta, \qquad \Delta \approx \frac{2}{3}\varepsilon \quad \text{as} \quad \varepsilon \to 0. \tag{A.1}$$

Then for small ε and Δ we can rewrite ϕ in the form

$$\phi_0 = \left(\frac{2}{3} - \Delta\right)^{-\frac{1}{2}} \int_0^1 \frac{dv}{[(1 - v)(v + v_2)(v_1 - v)]^{\frac{1}{2}}}, \tag{A.2}$$

where

$$v_1 \simeq 1 + 3\Delta, \qquad v_2 \simeq \frac{1}{2} + \frac{3}{4}\Delta. \tag{A.3}$$

In the approximation implied by '$\simeq$' above, powers of Δ higher than unity are neglected in comparison with unity.

Using the transformation

$$v = v_1 \sin^2\theta - v_2 \cos^2\theta,$$
$$\theta_1 = \tan^{-1}\left(\frac{v_2}{v_1}\right)^{\frac{1}{2}}, \qquad \theta_2 = \sin^{-1}\left(\frac{1 + v_2}{v_1 + v_2}\right)^{\frac{1}{2}}, \tag{A.4}$$

we get

$$\phi_0 \simeq 2\int_{\theta_1}^{\theta_2} \frac{d\theta}{[\sin^2\theta_2 - \sin^2\theta]^{\frac{1}{2}}} . \tag{A.5}$$

This can be brought into the standard elliptic integral form by the transformation $\theta \to \psi$ given by

$$\sin\theta = \sin\theta_2 \sin\psi, \qquad \theta = \theta_1 \Rightarrow \psi = \psi_1. \tag{A.6}$$

We get

$$\phi_0 \simeq 2\int_{\psi_1}^{\frac{1}{2}\pi} \frac{d\psi}{[1 - (1 - 2\Delta)\sin^2\psi]^{\frac{1}{2}}}$$

$$\simeq 2K((1 - 2\Delta)^{\frac{1}{2}}) - 2\ln(\sec\psi_1 + \tan\psi_1). \tag{A.7}$$

But as $\Delta \to 0$,

$$\sec\psi_1 + \tan\psi_1 \simeq \frac{(\sqrt{3} + 1)}{\sqrt{2}} .$$

Also

$$K((1 - 2\Delta)^{\frac{1}{2}}) \simeq \ln \frac{4}{\sqrt{2\Delta}} . \tag{A.8}$$

Therefore,

$$\phi_0 \simeq \ln \frac{8(2 - \sqrt{3})}{\Delta} \simeq \ln \frac{12(2 - \sqrt{3})}{\varepsilon} . \tag{A.9}$$

This is the logarithmic divergence of ϕ_0 with ε. The values in table I correspond to $\varepsilon = 10^{-n}$, $2 \leqslant n \leqslant 9$. For such cases the change in the values of ϕ introduced by decreasing Δ by a factor 10 is given by

$$\delta\phi_0 \simeq \ln 10 \simeq 2.30 . \tag{A.10}$$

This is borne out by numerical integration.

The integral for $T - t_0$ will diverge if we let $R \to \infty$. However, for any finite, but fixed, R we can study its dependence on ε as $\varepsilon \to 0$. Here again the integral diverges logarithmically. A simple calculation shows that if ε, $\varepsilon + d\varepsilon$ are two neighbouring values very small compared with unity, the difference between the corresponding values of $T - t_0$ is given in terms of the difference in the ϕ_0 values by

$$\delta(T - t_0) \simeq \frac{3\sqrt{3}r_0}{2c}\,\delta\phi_0. \tag{A.11}$$