Physical Interpretation of Quantum Field Theory in Noninertial Coordinate Systems

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In the upper and lower quadrants of flat spacetime ($\pm T > |X|$) it is possible to introduce a system of coordinates in which the metric is explicitly time dependent. The conventional interpretation of quantum field theory, based on a Fock basis, cannot be implemented in a natural fashion in such coordinates. This difficulty suggests that quantum theory is covariant only for a subset of the coordinate transformations which are allowed in the classical theory.

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According to general relativity, a given gravitational field is characterized by the curvature of the spacetime manifold and can be described by the Riemann-Christoffel tensor R_{iklm}. However, to study the interaction of gravity with other fields we use auxiliary quantities, such as the metric tensor g_{ik} and the connections $\Gamma_{i,kl}$. It is possible to describe the same manifold in terms of different coordinate systems, corresponding to different gauge choices for $\Gamma_{i,kl}$. In classical relativity, physically relevant quantities are expected to be generally covariant and hence the theory will be invariant under coordinate ("gauge") transformations.

Since we do not have a workable model for quantum gravity, it is not possible to ascertain whether the full quantum theory of gravity interacting with other fields is coordinate invariant. However, an intermediate level of the theory, describing a quantized matter field in a given background spacetime, has been investigated extensively in the literature. One of the main conclusions which emerged from this study is the following: The concept of a particle is not generally covariant and depends on the gauge chosen to describe the particular spacetime. This effect persists even in flat spacetime; the concept of a particle defined using the global, inertial coordinate system will not be the same as defined, for example, in a noninertial coordinate system.[1]

The mathematical reason for this effect is the following: To define the concept of particles, one needs to separate the solutions of the operator field equations into the "positive"-frequency part and "negative"-frequency part [which vary as $\exp(\mp i\omega t)$, respectively, with respect to some chosen time coordinate t]. This separation is Lorentz invariant but is *not* invariant under arbitrary coordinate transformations. It is possible to introduce several noninertial coordinate systems in flat spacetime, with different time coordinates, in such a way that the positive-frequency modes defined with respect to any one time coordinate is a mixture of positive- and negative-frequency modes defined with respect to another time coordinate. The concept of particles defined in each of these noninertial frames can be different.

Consider, for example, the flat spacetime expressed in the globally defined Minkowski coordinates (T,X,Y,Z). The lines $T = \pm X$ divide the T-X plane of the flat spacetime into four quadrants which we will call "upper" ($T > |X|$), "lower" ($-T > |X|$), "left" ($-X > |T|$), and "right" ($X > |T|$) regions. In the right and left quadrants, it is possible to introduce the Rindler coordinates by the transformation $gX = \pm e^{g\rho}\cosh(g\tau)$, $gT = \pm e^{g\rho}\sinh(g\tau)$. The quantum theory based on these coordinates has been a subject of extensive study in the past. It is known, for example, that the particle concept defined in a quantum theory based on Rindler coordinates is not the same as that defined in the quantum theory based on Minkowski coordinates.[1]

The situation becomes more complicated when one introduces coordinate systems in which the background metric depends on time explicitly. If we quantize a field in a background spacetime which is time dependent (say, in an expanding universe) then we may expect to find "genuine" particle creation from the vacuum due to the time-dependent external potential. *However, as we shall see below, there exist coordinate transformations in flat spacetime which exactly mimic this effect.* We shall see that there is no simple way in which one can distinguish genuine particle creation from spurious effects arising in a noninertial frame. It seems necessary to disallow a certain class of noninertial coordinate transformations, if the conventional interpretation of quantum field theory has to be implemented.

To study this situation, we will consider the upper and the lower quadrants of the X-T plane in flat spacetime which do not seem to have received a similar amount of attention in the literature. In these regions, it is possible to introduce a Milne-type coordinate system by the transformations $gX = \pm e^{\pm gt}\sinh(gx)$, $gT = \pm e^{\pm gt} \times \cosh(gx)$, where the top and bottom signs are used in upper and lower quadrants, respectively. Hereafter, we will concentrate on the upper quadrant $\mathcal{U}$ and will merely mention the results for the lower quadrant $\mathcal{L}$. In $\mathcal{U}$,

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the metric has the form

$$ds^2 = dT^2 - dX^2 - dY^2 - dZ^2 = e^{2gt}(dt^2 - dx^2) - dy^2 - dz^2, \quad (1)$$

with all the coordinate labels ranging in the interval $(-\infty, +\infty)$. The spacelike hypersurfaces with constant t provide a foliation of the upper quadrant; the data on some t =const surface will allow us to predict the future dynamics within $\mathcal{U}$. The metric in (t,x,y,z) is the Cartesian analog of the metric in the expanding spherical Milne coordinates.

We are interested in the quantization of a scalar field ϕ in these coordinates. Because the metric is independent of the spatial coordinates, the solutions to the Klein-Gordon equation

$$\frac{1}{\sqrt{-g}}\partial_i(\sqrt{-g}g^{ik}\partial_k\phi) + m^2\phi = 0 \quad (2)$$

can be expressed in the form $f_{\mathbf{k}}(t)\exp(i\mathbf{k}\cdot\mathbf{x})$, where $f_{\mathbf{k}}(t)$ satisfies the equation

$$\frac{d^2 f_{\mathbf{k}}}{dt^2} + [k_x^2 + (k_y^2 + k_z^2 + m^2)e^{2gt}]f_{\mathbf{k}} = 0. \quad (3)$$

Since the metric has an explicit time dependence, the situation is similar to the one encountered in developing quantum field theory in the Friedmann universes. The usual strategy adopted in such cases is the following:[2] (1) Obtain a complete set of orthonormal solutions which can be identified as positive- and negative-frequency modes in the asymptotic past ($t \to -\infty$); (2) obtain, in a similar manner, the positive- and negative-frequency modes in the asymptotic future; (3) because of the time dependence of $g_{ik}(t)$, a positive-frequency mode in the infinite past will evolve into a mixture of positive- and negative-frequency modes in the infinite future, a phenomenon usually attributed to particle creation. The mean number of particles produced in a given mode can be found by evaluating the conserved, covariant Hilbert-space scalar product

$$(f,g) = i\int d\mathbf{x}\sqrt{-g}g^{00}\left(f^*\frac{\partial g}{\partial t} - g\frac{\partial f^*}{\partial t}\right) \quad (4)$$

between the positive-frequency mode of the past and the negative-frequency mode of the future. This is the procedure we will follow.[3]

The general solution to (3) can be expressed in terms of Bessel functions as

$$f_{\mathbf{k}} = c_1 H_{ik}^{(1)}(\tau) + c_2 H_{ik}^{(2)}(\tau) \equiv a_1 J_{ik}(\tau) + a_2 J_{-ik}(\tau), \quad (5)$$

where $\tau \equiv g^{-1}(k_z^2 + k_y^2 + m^2)^{1/2}e^{gt}$ and $k \equiv |k_x|/g$. Since the Bessel and Hankel functions are linear combinations of each other, the constants a_1 and a_2 can be expressed as linear combinations of c_1 and c_2. These constants have to be determined by identifying the positive- and negative-frequency modes asymptotically.

It is clear from (3) that, as $t \to -\infty$, the solutions behave as $\exp(\pm i|k_x|t)$. Therefore, the mode which behaves as $\exp(-i|k_x|t)$ in the infinite past may be taken to be the positive-frequency mode in the asymptotic past. This is ensured by the choice $a_1 = 0$. We fix the value of a_2 by normalizing the mode functions using the norm in (4). Straightforward calculation now gives the mode, which behaves as of pure positive frequency in the infinite past,

$$f(t) = \left(\frac{1}{2\pi}\right)^{3/2}\left(\frac{\pi}{2g\sinh(k\pi)}\right)^{1/2} J_{-ik}(\tau). \quad (6)$$

The mode which is pure positive frequency in the infinite future can be determined by a similar analysis. [It is clear from (3) that, as $t \to +\infty$, the solutions behave as $\exp(\pm i\tau)$; the positive-frequency mode is taken to be the one which behaves as $\exp(-i\tau)$.] We get

$$g(t) = \left(\frac{1}{2\pi}\right)^{3/2}\left(\frac{\pi}{4g}\right)^{1/2} e^{-(1/2)k\pi}H_{ik}^{(2)}(\tau). \quad (7)$$

It is now clear that the positive-frequency mode in the infinite past $f(t)$ evolves into a superposition of positive- and negative-frequency modes [$g(t)$ and $g^*(t)$] of the infinite future. In fact, using the results

$$2J_{-ik}(\tau) = e^{-k\pi}H_{ik}^{(1)}(\tau) + e^{k\pi}H_{ik}^{(2)}(\tau), \qquad [H_{ik}^{(2)}(\tau)]^* = e^{-\pi k}H_{ik}^{(1)}(\tau), \quad (8)$$

it can be easily shown that

$$f(t) = \left(\frac{1}{2\sinh(k\pi)}\right)^{1/2}[e^{-(1/2)k\pi}g(t) + e^{-(1/2)k\pi}g^*(t)]. \quad (9)$$

The Boglubov coefficients can be read off from this equation:

$$\alpha = \left(\frac{1}{2\sinh(k\pi)}\right)^{1/2} e^{-(1/2)k\pi}, \qquad \beta = \left(\frac{1}{2\sinh(k\pi)}\right)^{1/2} e^{-(1/2)k\pi}. \quad (10)$$

We conclude that the number density of the particles "produced" is

$$n_{\mathbf{k}} = |\beta|^2 = \frac{1}{e^{(2\pi/g)|k_x|} - 1} \quad (11)$$

which corresponds to a Planck spectrum in the longitudinal momentum.

The same region of spacetime can also be represented in the static Minkowski coordinates with the modes $\Phi^{\pm} \sim \exp[\mp i(\omega T - \mathbf{K}\cdot\mathbf{X})]$. In this gauge, of course, there is no particle creation and the vacuum state remains a vacuum state at all times. It follows that the particle concepts in the Minkowski and Milne coordinates are

inequivalent. The Minkowski vacuum will contain Milne particles. We can obtain this particle content by evaluating the scalar products (Φ^-,f^+) and (Φ^-,g^+). It turns out that[3]

$$|(\Phi^-,f^+)|^2=\frac{1}{e^{2\pi|k_x|/g}-1},\quad |(\Phi^-,g^+)|^2=0. \tag{12}$$

In other words, the particle definitions in the Minkowski and Milne coordinates agree in the asymptotic future. However, the Minkowski vacuum will contain a thermal spectrum of particles in the asymptotic past. A similar coordinate system can be introduced in the "bottom quadrant" $(-T>|X|)$ of the Minkowski spacetime. The calculations in $\mathcal{L}$ lead to similar results. In this region, the particle concepts match in the asymptotic past but not in the future.

This is the situation we are led to if we try to quantize the field in the Milne coordinates. We will now attempt to understand these results.

Even though (11) corresponds to a temperature of $g/2\pi$, the result is very different from the standard result obtained in the Rindler frame for two reasons: (i) We are working in the upper and lower quadrants, while the Rindler coordinates exist only in the right and left quadrants. This makes the entire situation quite different. (ii) There is no "particle creation" in the Rindler coordinates (τ',ρ,y,z). The Rindler mode functions behave as $\exp(\pm i\omega\tau')$ for all times. The conventional result only says that these mode functions are connected to the Minkowski modes by a Bogoliubov transformation with an off-diagonal term which leads to a result similar to that in (11). In contrast, we are now working with a nonstatic background; the positive-frequency mode in the infinite past *does* get mixed up with positive- *and* negative-frequency modes of the infinite future.

The last feature is somewhat disturbing; it shows that, if we had no prior knowledge that we are dealing with flat spacetime, we would have accepted the result in (11) as genuine particle creation. In fact, the procedure we have followed is identical to the one usually adopted to study field theory in expanding universes. Our result suggests that particle creation can be a spurious effect even in a curved spacetime; it certainly will be coordinate dependent. *We have to produce a sensible criterion which will distinguish particle creation due to spacetime curvature from effects due to the choice of coordinates. Without such a criterion, it is meaningless to talk of quantum field theory in curved space.*

There does not seem to be any simple way of arriving at such a criterion. The procedure we have followed is mathematically sound and completely conventional. In fact, the analysis above has a perfect analogy with the quantization of a charged scalar field in the presence of an external, constant electric field.[4] If we represent the constant electric field in the gauge $A_i=(0,-ET,0,0)$, then the solutions to the Klein-Gordon equation

$$[(i\partial_k-qA_k)^2-m^2]\phi(T,\mathbf{X})=0 \tag{13}$$

can be expressed in the form $f_{\mathbf{K}}(T)\exp(i\mathbf{K}\cdot\mathbf{X})$, where $f_{\mathbf{K}}(T)$ satisfies the equation

$$\frac{d^2f_{\mathbf{K}}}{dT^2}+[(K_x+gET)^2+K_y^2+K_z^2+m^2]f_{\mathbf{K}}=0 \tag{14}$$

[compare with (3)]. It is possible to express the solutions to (14) in terms of parabolic cylinder functions. Following *exactly the same procedure* which we adopted for the Milne coordinates, we can get the result[4]

$$f_K^+(T)=c_1(K)g_K^+(T)+c_2(K)g_K^-(T), \tag{15}$$

where

$$c_1(K)=\frac{\sqrt{2\pi}}{\Gamma[\frac{1}{2}(1-i\lambda)]}\exp\left[-\frac{\pi}{4}(\lambda-i)\right],$$
$$c_2(K)=\exp\left[-\frac{\pi}{2}(\lambda+i)\right], \tag{16}$$

and

$$\lambda=(qE)^{-1}(K_y^2+K_z^2+m^2)=(qE)^{-1}(K_\perp^2+m^2)$$

[compare with (9)]. The mean number of particles created in the mode $\mathbf{K}$ is

$$n_{\mathbf{K}}=|c_2|^2=\exp\left[-\frac{\pi}{qE}(K_\perp^2+m^2)\right] \tag{17}$$

[compare with (11)]. The probability for a vacuum to remain a vacuum and the imaginary part of the effective Lagrangian can be computed from (17); detailed calculation shows[4] that the results agree with those due to Schwinger.[5] Thus our procedure has an exact parallel in the electromagnetic case.

This parallel also suggests a possible way out of our paradox. To see this, notice that there is some difficulty in interpreting the particle creation even in electrodynamics.[6] A constant electric field can also be expressed in a static gauge $A_i'=(-EX,0,0,0)$. In this gauge, the mode functions remain $\exp(\pm i\omega t)$ at all times and there can be no particle creation. Thus we are faced with a gauge dependence of the particle creation in electromagnetism which is in complete analogy with the problem of coordinate dependence of particle creation in gravity. If we could establish the gauge invariance of the electromagnetic result, we may be able to use an analogous procedure in gravity.

The crucial point, regarding electrodynamics, is the following: In the classical theory, we are allowed to make the gauge transformations $A_i\rightarrow A_i+\partial_i\chi$ with any sufficiently smooth χ. However, this is not the case in the quantum theory; the allowed class of gauge transformations is now only those which can be implemented as uni-

tary transformations in the Hilbert space of states. This will necessarily impose some constraints on the global, asymptotic behavior of the allowed set of functions χ which induces gauge transformations. It can be shown that the gauge transformation[6] connecting A_i and A_i' does not belong to this set. Thus the theories based on gauges A_i and A_i' live in different Hilbert spaces; there is no unitary transformation connecting them.

We suspect that the results in the case of gravity are of similar origin. The coordinate transformations which connect Minkowski spacetime with Milne coordinates (or, for that matter, the Rindler coordinates) spoil the asymptotic behavior of original metric. It is doubtful whether such transformations can be unitarily implemented in the Hilbert space of a full quantum theory of gravity.[7] It is very likely that quantum gravity is coordinate invariant (just as QED is gauge invariant) as long as one restricts oneself to transformations which are unitarily implementable. In such a full theory, the particle concept will be as much coordinate invariant as it is gauge invariant in QED.

The above discussion stresses the fact that the quantum theory remains meaningfully invariant only under a subset of classically allowed transformations. This subset is characterized by sensible boundary conditions at large distances. The same conclusion can be arrived at by a strictly operational approach to the problem: Any physically realizable electric field has to be confined in space and time; it can be shown that there is no ambiguity in the particle creation for such fields. Similarly, any physically realizable coordinate system can differ from the Minkowski coordinates only in a finite region of spacetime. (This excludes Milne, Rindler, and a host of other coordinate systems as physically unrealizable.) Under such transformations, which leaves the asymptotic domain unchanged, the standard concepts of field theory will be coordinate invariant. We feel that the results obtained in other cases are not of practical significance.

The above results may be summarized as follows: Given a particular region of a spacetime manifold, it is possible to introduce local coordinate charts in which the metric is time dependent. If one attempts to carry over the standard physical interpretation of the quantum field theory to such coordinate systems, one is inevitably led to effects which must be interpreted as particle creation. Such effects occur even in flat spacetime. To avoid these disturbing results it seems essential to disallow—in quantum theory—a large class of coordinate transformations which are classically admissible. This restriction in the deployment of coordinate charts will have important consequences for quantum gravity.

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