Analytical approach to string-induced phase transition

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In an earlier work it was shown that gauge-theory strings present in the early Universe could have converted a potentially first-order phase transition into a second-order one. This demonstration was based on numerical integration of the relevant equations. In this paper we discuss the model employed there in a self-contained fashion and present analytical arguments to show that the previous results are of a more general validity, applicable to "seeding" in any supercooled medium. It is shown that in the presence of an appropriate seed, the supercooled state does not minimize the free energy (even locally) below a critical temperature. This contrasts with the phase transition in the absence of seeds, which is accompanied by extensive supercooling and which can be completed only by quantum tunneling.

I. INTRODUCTION

First-order phase transitions, involving supercooling, are expected to occur in the early Universe at temperatures comparable to mass scales of spontaneous symmetry breaking in grand unified theories (GUT's). Many GUT models also predict the existence of "strings," which are non-Abelian generalizations of the Nielsen-Olesen[1] magnetic flux tubes. (These are analogous to the Abrikosov[2] strings in superconductors.) It is therefore interesting to investigate the effects of such strings on the occurrence and dynamics of the GUT phase transition.

This equation was analyzed by one of us in a recent paper.[3] It was shown that if a GUT phase transition in the early Universe is ostensibly first order (i.e., involving supercooling) and if strings are present at that time, then they can act like seeds and precipitate the new phase without any supercooling. If T_{cr} is the temperature at which the onset of supercooling occurs, then at a definite temperature $T_1 < T_{cr}$, the supercooled state ceases to be a local minimum of the free energy. If strings were absent, the supercooled state could persist for temperatures much less than T_{cr}, whereas T_1 turns out to be comparable to T_{cr}.

In Ref. 3, this result was established by numerical calculations supported by plausibility arguments. We shall now present analytic arguments to explain these results. Our motivation is twofold. (i) The analytic arguments explain some salient features of the numerical analysis in a more transparent manner. (ii) The mathematical model of an induced phase transition presented here and in Ref. 3 should have a wider applicability, such as in ordinary matter. It therefore seems appropriate to verify the general validity of the numerical results using analytic arguments. We shall keep this paper self-contained as far as the description of the model is concerned.

Symmetry-breaking patterns in GUT's that can lead to the strings considered here are proposed, for instance, in Ref. 4. The role of strings (as well as monopoles) in inducing a phase transition has been considered earlier.[5] The scenario considered there is different. The strings (or monopoles) arise in a phase transition signaled by a particular scalar field. The same field then signals the subsequent, potentially first-order, phase transition. Our problem is formulated differently, as will be clear in the following.

The rest of this paper is organized as follows. In Sec. II we set up the formalism, state the results obtained previously, and point out the features which need explanation by analytic methods. In Sec. III we present the analytic arguments. Section IV contains the concluding remarks.

II. THE STATEMENT OF THE PROBLEM

Consider a particle-physics model based on some gauge group G. Let G be broken by the vacuum expectation value (VEV) of a scalar field ϕ at a mass scale m. Correspondingly, we expect a phase transition to occur[6,7] in the early Universe at a temperature $T \simeq m$. Such a phase transition can be described by using the temperature-dependence effective potential[7] for the scalar field. The minima of the effective potential correspond to different phases. We shall be interested in the case in which the phase transition is first order, i.e., the change in the VEV of ϕ is discontinuous. As the temperature drops below some critical temperature T_{cr}, a new VEV becomes energetically favorable, but remains separated from the existing VEV by a potential barrier. In such a case, small regions of the new phase originate by quantum tunneling, grow in size and fill the entire medium, thereby completing the phase transition. This process has been extensively investigated.[8,9] A typical effective potential that leads to such a behavior is

$$V^T(\phi) = \frac{\sigma}{4!}\phi^4 + \frac{\gamma}{3!}\phi^3 + \tfrac{1}{2}(m^2 + AT^2)\phi^2 + V_0 - \frac{\pi^2}{90}N(T)T^4 . \quad (1)$$

The scalar field is meant to be in a nontrivial representation of the gauge group but it is possible to reduce the problem to that of a single degree of freedom, represented

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by ϕ. The parameters σ and m^2 are positive and γ negative with $|\gamma| \sim m$. For perturbative renormalizability, σ is assumed to be less that unity. The constant A is between 1 and 10 in a typical grand unified theory, $N(T)$ accounts for the number of particle species present at temperature T and $V_0(>0)$ is to be chosen so as to make the value of V_T zero at its absolute minimum at $T=0$. The potential is shown in Fig. 1, where we have also defined f_1, f_2, and T_{cr}. (In order to emphasize more important features, the effect of the T^4 term has been ignored in this diagram.) Defining m_{cr}, ϵ, and f by

$$\frac{\sigma}{4!}m_{cr}^2 \equiv \tfrac{1}{2}(m^2 + AT_{cr}^2) \equiv \frac{1}{3!}\left|\frac{\gamma^2}{\sigma}\right|,$$
$$\epsilon \equiv \frac{4!AT_{cr}(T-T_{cr})}{\sigma m_{cr}^2} \equiv \frac{4!AT_{cr}\Delta T}{\sigma m_{cr}^2}, \tag{2}$$
$$f \equiv \frac{\langle\phi\rangle}{m_{cr}},$$

we can rewrite V^T as

$$V^T(f) = \frac{\sigma}{4!}f^2(f-1)^2 + \frac{\sigma}{4!}\epsilon f^2 + V_0 - \frac{\pi^2}{90}N(T)T^4. \tag{3}$$

We next turn to the effect of strings on this phase transition. We shall be interested in strings formed in the course of an earlier phase transition at a temperature $T' \simeq \mu \gg T_{cr} \simeq m_{cr}$ (where μ denotes another mass). They are expected to occur as defect lines due to the mismatch between equivalent but distinct vacua in different domains.[10] Such vortices are expected to have a diameter of the order of μ^{-1} and energy per unit length of the order of μ^2. They are topologically stable nontrivial field configurations involving the gauge fields and some charged scalar fields. For our purpose, strings are some background-field configurations confined to filamental regions of radius $\mu^{-1} \ll m_{cr}^{-1}$. We shall further assume that a nontrivial coupling exists between ϕ and these background fields. The details of the nature of this coupling are explained in Refs. 3 and 11. As a result of such coupling, ϕ is expected to have an expectation value of the order of μ in the core of the strings at temperature $T \lesssim \mu$, whereas in a region with no strings nearby, $\langle\phi\rangle$ should vanish. Outside the core region of radius μ^{-1}, ϕ decouples from the string and its configuration is then determined by its self-coupling.

Consider the vicinity of a single long straight string. We set up cylindrical coordinates r,θ,z. We do not require any detailed knowledge about the form of the string field configuration. We shall merely assume that at some radius r_0, $\mu^{-1} \ll r_0 \ll m_{cr}^{-1}$, f has a value f_0 with $\mu/m_{cr} \gg f_0 \gg 1$. Physically the radius r_0 characterizes a length scale outside of which ϕ has decoupled from the string field. As $r \to \infty$, we expect f to reach its constant equilibrium value, which around $T = T_{cr}$ should be f_1 or f_2 depending on which phase is favorable. We are interested in determining the transition from f_1 to f_2.

This behavior of f is governed by the action

$$S^T(f) = \int [\tfrac{1}{2}\partial_\mu f \partial^\mu f - V^T(f)]d^2r\,dz\,dt. \tag{4}$$

For the lowest-energy configuration we expect $\partial f/\partial t = \partial f/\partial\theta = \partial f/\partial z = 0$, so that we get the field equation

$$\frac{d^2f}{dr^2} + \frac{1}{r}\frac{df}{dr} - \frac{\sigma}{6}f^3 + \tfrac{1}{4}\sigma f^2 - \frac{\sigma}{12}(1+\epsilon)f = 0. \tag{5}$$

(Here we have rescaled r in units of m_{cr}^{-1}.) We are looking for solutions with $f = f_0$ at $r = r_0$. At $r = \infty$, two boundary possibilities exist. Either $f_\infty \equiv f(r=\infty) = f_1$ or f_2. Figure 2, to be explained below, shows such functions. At temperatures $T > T_{cr}$, f_∞ should be f_1. However, even for $T \lesssim T_{cr}$, f_∞ is likely to persist at f_1 because

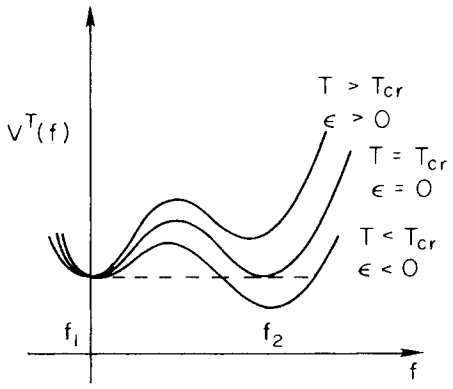

FIG. 1. Temperature-dependent effective potential for a scalar field signaling a first-order phase transition.

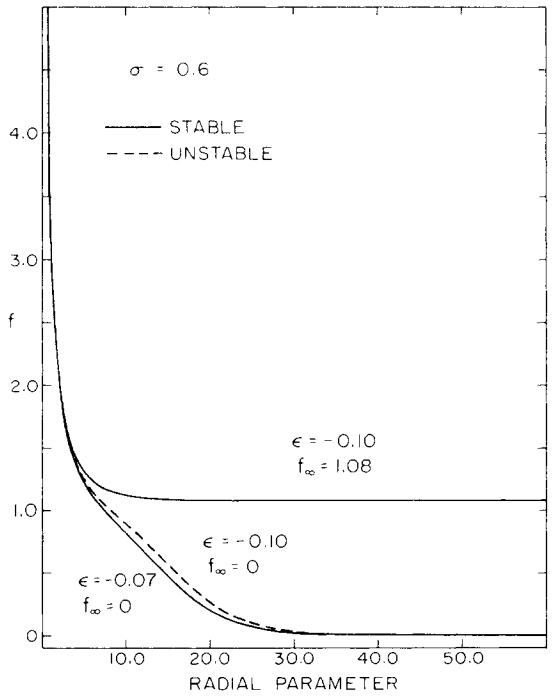

FIG. 2. Numerical solutions to Eq. (5).

of the potential barrier existing between f_1 and f_2. However, in the nontrivial configuration we have for f, there must be some value r_1 of r, presumably close to $r=r_0$, at which $f(r_1)=f_2$. We expect that as ϵ becomes negative and f_2 becomes a favorable minimum of energy, it becomes increasingly unfavorable for $f(r)$ to pass through this value and then to cross over the energy hump (such as in Fig. 1) to asymptotically reach f_1. Numerical calculations showed that this is indeed the case. Below a certain negative value of ϵ, no solution to (5) can be found with $f_\infty=f_1$.

We shall summarize the numerical results in the rest of this section. Equation (5) was solved by discretizing the r axis into points r_i and converting (5) into a set of simultaneous cubic equations for the variables $f_i \equiv f(r_i)$. The solution to this set was then found using the IMSL[12] routine ZSPOW, which implements a generalization of Newton's method for finding roots of polynomials. Since the latter method leads to a solution accessible from a given trial, it can be used to obtain the two different solutions with $f_\infty=0$ or $f_\infty=f_2$. To obtain the solution with $f_\infty=0$, we give a trial solution with all $f_i=0$ for r greater than a value such as 3 or 4 (in units of $m_{\rm cr}^{-1}$). It is then found that even for $\epsilon<0$, a solution to (5) exists with $f_\infty=0$. However, this is not true below some critical value of ϵ, which we shall denote as $\bar{\epsilon}$. For $\epsilon<\bar{\epsilon}$, no solution exists close to the given trial. If on the other hand, we supply a trial solution with all $f_i=f_2$ for large r, a solution is always found. We thus use ZSPOW as an existence prover (or disprover) of solutions to (5) with desired boundary conditions. In Fig. 2, taken from Ref. 3 we have presented typical examples. [The boundary condition $f(r_0)=f_0$ is arbitrarily taken to be $f(r=0.1)=10$. Showing that the phenomenon we are interested in is really insensitive to this choice at r_0 is one of the aims of this paper.] For $\epsilon \lesssim \bar{\epsilon}$, ZSPOW declares that no solution exists, but nevertheless produces a configuration which satisfies the equation as closely as possible, and which is close in values to the "stable" solutions that exist for $\epsilon>\bar{\epsilon}$. This is also given in Fig. 2.

As reported in Ref. 3, the most interesting feature to emerge from these numerical calculations is the fact that $|\bar{\epsilon}| \simeq |\Delta T|/T_{\rm cr}$ is small, and is independent of σ as well as the arbitrarily chosen r_0 and f_0. $\bar{\epsilon}$ was found to be in the range -0.07 to -0.09. This has the important implication that due to the presence of the nontrivial configuration, the supercooled state ceases to be even a local minimum at a temperature *comparable to* $T_{\rm cr}$. That the result is independent of r_0 and f_0 also means that the details of the localized background solution (the vortex configuration) are irrelevant, as long as f becomes large enough to cross through the value f_2 at some value of r. In the following section we show that these numerically obtained results are of general validity.

In Ref. 3 it was also shown that as the temperature continues to drop, so that $\epsilon<\bar{\epsilon}$, the function $f(r)$ really becomes time dependent. The subsequent evolution can be determined by considering the time-dependent equation for f. From the configuration with $f_\infty=0$, the function evolves into one with $f_\infty=f_2$. This aspect of the problem will not be discussed here.

III. ANALYTICAL APPROACH

As long as there are two local minima of $V(f)$, it is in principle possible that two solutions exist to (5), both with $f(r_0)=f_0$, but one with $f_\infty=0$ and another with $f_\infty=f_2$. This means that these two configurations are position-dependent local minima of $V(f)$. The numerical calculations then show that for $\epsilon \lesssim \bar{\epsilon}$, the $f_\infty=0$ configuration ceases to be a local minimum. In order to estimate this $\bar{\epsilon}$, we analyze the linear stability of the solution with $f_\infty=0$. If small oscillations around the solution have only real frequencies the configuration should be stable, and a local minimum. We shall therefore look for the value of ϵ that makes at least one of the modes of oscillation possess an imaginary frequency.

Let us restore time dependence in Eq. (5), so that we have a term $-d^2f/dt^2$ on the left-hand side (with t rescaled to $m_{\rm cr}t$). If $\tilde{f}(r)$ is a time-independent solution of (5), we write $f(r,t)=\tilde{f}(r)+\bar{p}(r,t)$ where $\bar{p}$ is assumed to be everywhere small compared to unity. Decomposing the time dependence as $\bar{p}(r,t)=p(r)e^{i\omega t}$ and linearizing the equation in $p(r)$, we get

$$\omega^2 p+\frac{d^2p}{dr^2}+\frac{1}{r}\frac{dp}{dr}=\frac{\delta^2V(f)}{\delta f^2}\Big|_{f=\tilde{f}}p \equiv u(r)p \ , \tag{6}$$

where

$$\frac{\delta^2V}{\delta f^2}=\frac{\sigma}{2}[f^2-f+\tfrac{1}{6}(1+\epsilon)] \tag{7}$$

or, on substituting $p(r)=r^{1/2}q(r)$, we see that q satisfies the equation

$$\omega^2 q=-\frac{d^2q}{dr^2}+\left[u(r)-\frac{1}{4r^2}\right]q \ . \tag{8}$$

This is the one-dimensional Schrödinger equation with potential $v(r)=u(r)-\frac{1}{4}r^2$ for $r>0$ and infinite for $r<0$. We are looking for the condition under which a negative-energy bound state is possible for this potential.

Our first observation is that since $u(r;\sigma)$ is of the form $\sigma y(r)$, σ can be scaled away by rescaling $r\to\sigma^{1/2}r$. This only rescales the energy $\omega^2\to\omega^2/\sigma$, and if $\omega^2<0$, so is ω^2/σ. Thus, the occurrence of a negative-energy bound state (equivalently the critical value $\bar{\epsilon}$) is independent of σ.

In order to decide whether the problem of Eq. (8) admits a negative-energy bound state, we shall use a simple but physically transparent criterion.[13] Let the minimum value of $v(r)$ be $-v_0$ (so that v_0 would be the maximum kinetic energy of a zero-energy particle), and let an appropriately defined width of the well be w. A particle of mass $m=\frac{1}{2}$ confined within a region w will have a momentum $p\simeq\hbar/w\sim w^{-1}$ and hence a zero-point energy $p^2\simeq w^{-2}$. For the particle to be confined (i.e., for a bound state to exist) we must have $p^2<v_0$; or in other words, $w^2v_0>1$. This is only a sufficient condition and not a necessary one, but it will provide an estimate of the magnitude of $\bar{\epsilon}$. We only need now to estimate the width and depth of $u(r)$.

The $u(r)$ is determined by the time-independent solution $\tilde{f}(r)$. We therefore build a function that approxi-

mates $\tilde{f}(r)$. We have constrained $\tilde{f}(r)$ by the condition that $\tilde{f}(r_0)=f_0$. We also expect the solution for $r<r_0$ (near the origin) to be dominated by the vortex interaction. Therefore, we shall approximate $\tilde{f}(r)$ by the asymptotic forms

$$\tilde{f}(r)=\left[\frac{6}{\sigma}\right]^{1/2}\frac{1}{r-r_I} \quad (\text{as } r\to r_I,\ r>r_I) \tag{9}$$

$$=B\exp\left[-\left[\frac{(1+\epsilon)\sigma}{12}\right]^{1/2}r\right] \quad (\text{as } r\to\infty)\ . \tag{10}$$

Here B and r_I are constants. The value of r_I is related to r_0 and f_0 and we expect $r_I \ll 1$. The solution (9) is valid only when the terms $3f$, etc., are small compared to $2f^2$. This translates into the constraint

$$(r-r_I)^2<\frac{8}{3\sigma}\ . \tag{11}$$

We shall take this to be the domain of validity of the solution (9). We shall now determine B by directly matching (9) with (10) at $(r-r_I)=\sqrt{8/3\sigma}$. As we shall see, the value of B is not so important as its order of magnitude. Performing this matching, we find

$$B=\tfrac{3}{2}\exp\left\{\left[\frac{\sqrt{2}}{3}-\left[\frac{\sigma}{12}\right]^{1/2}r_I\right](1+\epsilon)\right\}\ . \tag{12}$$

Since $\sigma<1$ and $r_I\ll 1$, we find that when $\epsilon=0$, a good estimate for B is $B\simeq 2.40$.

In Fig. 3, we have sketched $u(r)$ for the $f_\infty=0$ solution, and also, the dashed line for the $u(r)$ that would result from the $f_\infty=f_2$ solution. These general forms can be deduced quite simply because u is the curvature of the graphs in Fig. 1, and we know the general form of $f(r)$ from the above analysis or directly from Fig. 2. We note that the $f_\infty=f_2$ solution is monotonic and bounded below by f_2, so that $u(r)$ remains monotonic and bounded below by $\sigma(1+\epsilon)/12$. On the other hand, the $f_\infty=0$ solution

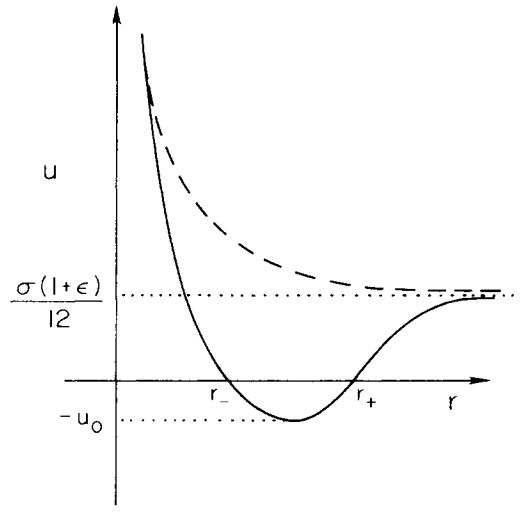

FIG. 3. The equivalent Schrödinger potential of Eq. (6) derived from the $f_\infty=0$ and the $f_\infty=f_2$ solutions of Eq. (5).

assumes the values of f where $\delta^2 v/\delta f^2=0$. These two zeros of $\partial^2/\partial f^2$ will be denoted by f_+ and f_-, with $f_->f_+$. These can be easily determined from Eq. (7):

$$\begin{aligned} f_- &= 0.789-\frac{\epsilon}{\sqrt{3}}+O(\epsilon^2)\ , \\ f_+ &= 0.211+\frac{\epsilon}{\sqrt{3}}+O(\epsilon^2)\ . \end{aligned} \tag{13}$$

We now proceed to determine the width and the depth of $u(r)$. [The full potential is $v(r)$. As we shall soon see the values of r for which $\tilde{f}(r)=f_\mp$ are large enough that $1/4r^2$ can be ignored.] The width (see Fig. 3) is determined by r_+ and r_-, the points at which $\tilde{f}=f_+$ and f_-, respectively. Looking at the solutions (9) and (10), and taking $B\simeq 2.4$, we conclude that the values in (13) fall within the domain of validity of (10). We then find

$$\left[\frac{(1+\epsilon)\sigma}{12}\right]^{1/2}(r_+-r_-)=\ln\left[\frac{f_-}{f_+}\right]\ , \tag{14}$$

so that the constant B disappears from the analysis, giving

$$\sqrt{\sigma}w=2\sqrt{3}[1.317-4.128\epsilon+O(\epsilon^2)]\ . \tag{15}$$

The depth of $u(r)$ is given by the value of $\delta^2 V/\delta f^2$ at the point at which $\delta^3 V/\delta f^3$ vanishes, which occurs at $f=\frac{1}{2}$. So,

$$u_0=-\frac{\sigma}{24}(1-2\epsilon)\ . \tag{16}$$

Now requiring that $w^2|u_0|>1$ for a bound state to be possible, we find

$$\bar{\epsilon}\lesssim -0.02 \tag{17}$$

which is the result we wanted.

Note that the disappearance of B from our final results implies that $\bar{\epsilon}$ is, if at all, only weakly dependent on the parameters r_0 and f_0. This also justifies our—rather cavalier—attitude towards the exact meaning and value of r_0, r_I, etc. As long as the broad inequalities used before are satisfied, the exact values of these parameters do not matter.

IV. CONCLUSION

We have shown that a small, negative value of ϵ saturates the uncertainty principle bound and a bound state becomes possible thereafter. We wish to emphasize that we have established *the existence and the smallness* of the $\bar{\epsilon}$. The prevention of extensive supercooling and the seeding of a rather quick phase transition follows from this result. Also, we found that $\bar{\epsilon}$ has to be negative. The above analysis repeated for $\epsilon\gtrsim 0$ shows that a bound state is not possible for $\epsilon\gtrsim 0$. Since this latter result is intuitively plausible, we are reassured about the credibility of our criterion.

The main weakness of our analysis is that the exponential form (10) may have been extrapolated too far back. This may seem especially unjustified for $\epsilon<0$ as can be seen in Fig. 2. We would offer the following arguments

to convince the reader that our result is not an accident.

A detailed inspection of our analysis reveals that the two important ingredients which have gone into our result are the following: (i) For $\epsilon=0$, $w^2\sigma/12$ should be less than 2 and (ii) to leading order in ϵ, w is a decreasing function of ϵ. For $\epsilon=0$, (9) and (10) are indeed a good approximation of $\tilde{f}$. So we believe the ϵ-independent part of our value of w to be close to the correct one. Furthermore, the reader can convince herself or himself that on making ϵ negative, r_+ increases but r_- does not alter greatly, so that w increases (with decreasing ϵ). Thus, both these features of our analysis reflect what we believe to be really the case and the result they lead to is not accidental. The width of the potential well seems to be decided essentially by the behavior of $\tilde{f}$ for large r and not by the conditions near the origin. We believe this feature of our analysis to be also of general validity.

In conclusion, we would like to emphasize the importance of this phenomenon to gauge field theory models of the early Universe. The most striking implication of spontaneously broken gauge theories to the early Universe is the occurrence of phase transitions. Investigations into these phase transitions,[14,15] including the original inflationary model of Guth, have shown that first-order phase transitions involving extensive supercooling are a disaster to cosmology because the true vacuum bubbles do not percolate, leading to a very inhomogeneous Universe. Also, first-order phase transitions are predicted by many GUT's for large ranges of natural parameter values. We are thus faced with the problem of searching for a mechanism to prevent extensive supercooling. The phenomenon discussed here shows that strings provide a possible solution.

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