Space-Time Singularities and Conformal Gravity.

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We report here some recent work in the conformally invariant gravitation theory of Hoyle and Narlikar ([1]), which has a bearing on the nature of space-time singularities in general relativity. We find that a wide class of singularities in relativity arise from the choice of an « unphysical » conformal frame in the conformal theory.

The conformal gravity theory was first proposed as a Machian action at a distance theory. Although the action at a distance format is more natural and elegant, we describe the theory here in the corresponding field form ([2]).

We start with a Riemannian space-time manifold $\mathscr{M}$ with a metric tensor g_{ik} and a scalar mass function $m(x)$ representing the inertia of a typical particle at x. The mass field satisfies the wave equation

$$\Box m + \tfrac{1}{6} R m = N \,, \tag{1}$$

where R = scalar curvature and N = invariant particle number density. The gravitational-field equations are given by

$$\tfrac{1}{6} m^2 (R_{ik} - \tfrac{1}{2} g_{ik} R) = - T_{ik} + \tfrac{1}{3} m \{ g_{ik} \Box m - m_{;ik} \} + \tfrac{2}{3} \{ m_{;i} m_{;k} - \tfrac{1}{4} g_{ik} m_{;l} m^{;l} \} \,, \tag{2}$$

where T_{ik} = matter energy tensor. Equation (1) gives expression to Mach's idea that inertia owes its origin to the presence of other matter in the Universe.

The field equations (1) and (2) are conformally invariant. Hence if we consider another manifold $\mathscr{M}^*$ obtained from $\mathscr{M}$ by a conformal transformation

$$g^*_{ik} = \Omega^2 g_{ik} \,, \qquad m^* = m \Omega^{-1} \,, \tag{3}$$

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([1]) F. HOYLE and J. V. NARLIKAR: *Action at a Distance in Physics and Cosmology* (San Francisco Cal., 1974).

([2]) J. V. NARLIKAR: *Proc. Camb. Phil. Soc.*, **64**, 1971 (1968).

then g^*_{ik} and m^* also satisfy (1) and (2). The transformation $m \to m^*$ is mathematically valid provided Ω does not vanish (or become infinite). We will assume that Ω is a twice differentiable function of space-time co-ordinates with

$$0 < \Omega < \infty . \tag{4}$$

If these conditions are violated we will regard the conformal transformation « unphysical ».

Equations (1) and (2) take a simpler form in a special conformal frame in which $m = \text{constant} = m_0$. If we write $\Omega = m/m_0$ in (3) it is evident that the field equations become those of general relativity with the gravitational coupling constant $\varkappa = 6/m_0^2$. We will call this special conformal frame the « Einstein frame ». In the corresponding $\mathscr{M}^*$ the equations of general relativity hold. However, as mentioned above the conformal transformation to $\mathscr{M}^*$ will be unphysical if (4) is violated. This can happen, for example, if we insist on using the Einstein frame in a region where $m = 0$. We now show that this unphysical procedure leads to the space-time singularities of general relativity.

Consider first the simplest cosmological solution of (1) and (2) describing a dust filled homogeneous and isotropic universe. The line element is given by

$$ds^2 = d\tau^2 - dx^2 - dy^2 - dz^2 , \tag{5}$$

with the mass function

$$m = \lambda\tau^2 , \qquad \lambda = \text{constant} . \tag{6}$$

Thus $\mathscr{M}$ is a Minkowski manifold in which particle masses increase with epoch τ for $\tau > 0$. If we now use the corresponding Einstein frame with $\Omega \propto \tau^2$, we should get a standard Friedmann manifold. Indeed a transformation $\tau \propto t^{\frac{1}{3}}$ shows that $\mathscr{M}^*$ describes the Einstein-de Sitter model

$$ds^{*2} = dt^2 - Bt^{\frac{4}{3}}(dx^2 + dy^2 + dz^2) \tag{7}$$

with B a positive constant. Notice, however that while the transformation from (5) to (7) is valid for $t > 0$, $\tau > 0$, it becomes unphysical at $t = 0, \tau = 0$ where $\Omega = 0$. And this is precisely where the space-time singularity appears in the relativistic model. The manifold $\mathscr{M}^*$ is geodesically incomplete while the original manifold $\mathscr{M}$ is not. The world lines of the fundamental observers extend from $\tau = -\infty$ to $\tau = +\infty$ in $\mathscr{M}$ whereas they begin abruptly at $\tau = 0$ in $\mathscr{M}^*$.

The nonsingularity of $\mathscr{M}$ is advantageous in many respects. HOYLE (3) has shown how the astrophysics in the $\tau < 0$ region can lead to the observed microwave background whose origin is postulated *ad hoc* in the conventional treatments confined to $\mathscr{M}^*$. If masses transform as (3) it is easy to show that the Dirac equation for massive spin-$\frac{1}{2}$ particles is conformally invariant. Since Maxwell's equations are also conformally invariant it is in principle possible to do quantum electrodynamics in $\mathscr{M}$ near $\tau = 0$. This is not possible in $\mathscr{M}^*$ which becomes singular at $\tau = 0$.

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(3) F. HOYLE: *Astrophys. Journ.*, **198**, 661 (1975).
(4) S. W. HAWKING and G. F. R. ELLIS: *The Large Scale Structure of Space-Time* (Cambridge, Mass., 1973).

To what extent is the simple example discussed above typical of the general situation? Recently we have investigated a number of more complicated singularities of general relativity and we found that in all cases they appear to arise because the Einstein frame was chosen in regions where $m = 0$ in the original nonsingular $\mathcal{M}$. We briefly describe three such investigations below.

I) Consider the Bianchi type I universe with shear ([4]) given by the line element

$$\text{(8)} \qquad ds^{*2} = dt^2 - \left(\frac{9M}{2}\right)^{-\frac{4}{3}} t^2(t+\Sigma)^{-\frac{2}{3}} dx^2 - \left(\frac{9M}{2}\right)^{\frac{4}{3}} (t+\Sigma)^{\frac{4}{3}} (dy^2 + dz^2)\,,$$

where M and Σ are positive constants. This is obtained as a solution of Einstein's equations for homogeneous dust and has a pancake singularity at $t = 0$. It can be shown ([5]) that this manifold $\mathcal{M}^*$ has been obtained as a conformal transform of a nonsingular manifold $\mathcal{M}$ with

$$\Omega = \frac{t}{t^2 + 1}\,.$$

The manifold $\mathcal{M}$ is geodesically complete, has finite-curvature invariants at $t = 0$ including the magnitude of the Weyl tensor: $(C_{ijkl}C^{ijkl})^{\frac{1}{2}}$. The singularity of $\mathcal{M}^*$ can be seen to arise from the vanishing of Ω as $t = 0$.

II) A few year ago BELINSKY *et al.* ([6]) obtained what they claim to be the most general type of cosmological singularity in general relativity. By using a syncronous time co-ordinate t, they show how the singularity developes in an oscillatory manner as $t \to 0$. In this case it is possible to find a manifold $\mathcal{M}$ as a solution of (1) and (2) and a conformal function Ω with the following properties:

i) $\mathcal{M}$ is non singular, *i.e.* geodesically complete with finite-curvature invariants,

ii) $\mathcal{M}^*$ obtained from $\mathcal{M}$ by using the conformal function Ω, is the general solution of Belinsky *et al.* (op. cit),

iii) $\Omega \to 0$ as $t \to 0$.

Here, again, the singularity of $\mathcal{M}^*$ is due to the choice of an unphysical conformal frame.

III) The Taub-NUT space-time ([7]) is geodesically incomplete although it does not have infinite-curvature invariants or infinite matter density. For example, there are null geodesics in the Taub region which do not extend beyond a finite range of their affine parameter. In this case it is possible to construct a nonsingular geodesically complete space-time manifold $\mathcal{M}$ as a solution of (1) and (2) of which the Taub space-time $\mathcal{M}^*$ is a conformal transform. The conformal function, however, vanishes on the singular hypersurfaces of $\mathcal{M}^*$. Thus these hypersurfaces correspond to the $m = 0$ hypersurfaces of $\mathcal{M}$.

These examples generate our confidence that the physically meaningful (as opposed to mathematically pathological) singularities of general relativity arise from the unphysical choice of the Einstein frame at the $m = 0$ hypersurfaces of conformal gra-

([5]) A. K. KAMBHAVI: *Pramana*, **7**, 344 (1977).
([6]) V. A. BELINSKY, I. M. KHALATNIKOV and E. M. LIFSHITZ: *Adv. Phys.*, **19**, 525 (1970).
([7]) C. W. MISNER and A. H. TAUB: *Sov. Phys. JETP*, **28**, 122 (1969).

vity. Just as the existence of the horizon around a highly compact spherical mass distribution m exposed the inadequacy of the Schwarzschild co-ordinates at $R = 2m$, so we feel that the existence of space-time singularities in general relativity tells us the inadequacy of the Einstein conformal frame in dealing with physics close to the $m = 0$ hypersurfaces.

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