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EXCELLENCE

Gentle giant

R. RAMACHANDRAN

John Willard Milnor, the wizard of higher dimensions, gets the Abel Prize, which is regarded as the "Mathematician's Nobel".

COURTESY: THE NORWEGIAN ACADEMY OF SCIENCE AND LETTERS



John Willard Milnor.The citation says: "All of Milnor's works display marks of great research: profound insights, vivid imagination, elements of surprise and supreme beauty."

AS the month of October is for Nobel Prizes, March has been, in the past eight years, for the prestigious Abel Prize in mathematics conferred by The Norwegian Academy of Science and Letters. One of the giants of modern mathematics, John Willard Milnor of the Institute for Mathematical Sciences, Stony Brook University, New York, United States, reputed for his work in differential topology, K-theory and dynamical systems, has been chosen by the Academy for

this year's Abel Prize. The decision was announced by the President of the Norwegian Academy, Oyvind Osterud, on March 23. Milnor will receive the award from His Majesty King Harald at a ceremony in Oslo on May 24.

The Abel Prize has widely come to be regarded as the “Mathematician's Nobel”. Instituted in 2001 to mark the 200th birth anniversary of the Norwegian mathematical genius Niels Henrik Abel (1802-1829), it is given in recognition of contributions of extraordinary depth and influence to mathematical sciences and has been awarded annually since 2003 (Frontline, April 20, 2007). The prize carries a cash award of six million Norwegian kroner, which is about €750,000 or $ 1 million, similar to the amount of a Nobel Prize. Unlike the other major award in mathematics, the Fields Medal, which is given once in four years at the International Congress of Mathematicians (ICM) to young mathematicians not over 40 years of age on January 1, the Abel Prize, just as the Nobel, has no age limit.

The past winners of the prize include such illustrious names as Jean-Pierre Serre (2003), Sir Michael Atiyah and Isadore M. Singer (2004), Peter D. Lax (2005), Lennart Carleson (2006), Srinivasa S.R. Varadhan (2007), John Griggs Thompson and Jacques Tits (2008), Mikhail Leonidovich Gromov (2009) and John Torrence Tate (2010). The Abel Prize winner is selected by a committee of five international mathematicians headed by Ragni Piene of the University of Oslo. The International Mathematical Union (IMU) and the European Mathematical Society (EMS) nominate members of the Abel Committee. Besides Piene, the committee for this year's award included Bjorn Engquist of the University of Texas at Austin, M.S. Raghunathan of the Tata Institute of Fundamental Research (TIFR), David Donoho of Stanford University and Hendrik W. Lenstra of Leiden University in the Netherlands.

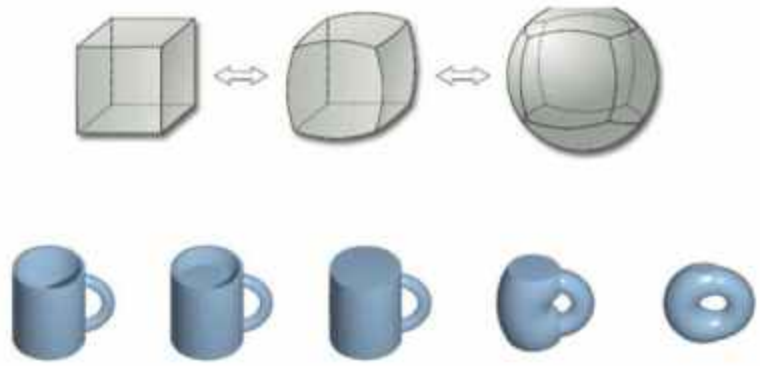
The 2011 award is given to Milnor, as the citation says, “for [his] pioneering discoveries in topology, geometry and algebra”. He has also made significant contributions in number theory. “All of Milnor's works,” the citation adds, “display marks of great research: profound insights, vivid imagination, elements of surprise and supreme beauty.” His profound ideas and fundamental discoveries have spawned new disciplines in mathematics and shaped to a great extent the mathematical landscape since the mid-20th century. Milnor has written tremendously influential books, loved in particular by graduate students and widely regarded as models of fine mathematical writing. In addition, given his affable personality, he has been called the “Gentle Giant of Mathematics”.

Describing the work of Milnor at the time of the announcement of the prize, William Timothy Gowers of Cambridge University said: “There are many mathematicians with extraordinary achievements to their names…. But even in this illustrious company, John Milnor stands out as quite exceptional. It is not just that he has proved several famous theorems: it is also that he has made fundamental contributions to many areas of mathematics, apparently very different from each other, and that he is renowned as a quite exceptionally gifted expositor. As a result, his influence can be felt all over mathematics.”

“If the impact of a mathematician is to be measured not only by his own fantastic results but by the great results of others that grow out of one's work, then Milnor is certainly one of the greats

of the last half of the 20th century," says Kapil Hari Paranjape, an algebraic geometer and topologist at the Indian Institute of Science Education and Research (IISER), Mohali.

COURTESY: ABEL PRIZE WEBSITE



"The first time I came across the name of Milnor," Paranjape adds, "was when I heard that the only dimensions in which one can do algebra with division is 1, 2, 4 and 8. I was told that an 'easy proof' was based on Characteristic Classes on which Milnor had written a nice book. In later years, I read a number of his other books, such as Topology from a Differential Viewpoint, Morse Theory, Isolated Singularities of Complex Hypersurfaces and Algebraic K-Theory. These books not only explained the results and definitions, but laid the foundations of my geometric intuition. The same is probably true for many others in my generation."

Early years

Milnor was born on February 20, 1931, in New Jersey, and joined Princeton University in 1948 for his undergraduation. When he was barely 18 he proved what is known as the Fary-Milnor Theorem in knot theory. Even as an undergraduate in 1949, he was named a Henry Putnam Fellow. In 1953, before completing his doctoral work, he was appointed to the faculty position in Princeton. In 1954, he obtained his doctorate for his thesis Isotopy of Links in knot theory under Ralph Fox, whom he regards as having been closest to him during his early years. After his PhD, he continued to work at Princeton where he was Alfred P. Sloan Fellow from 1955 to 1959. In 1960, he was made a full professor, and in 1962, he was appointed to the Putnam Chair at Princeton.

Describing his early days at Princeton in his now famous talk titled "Growing Up in the Old Fine Hall" given in the late 1990s, Milnor says, "In the fifties I was very much interested in the fundamental problem of understanding the topology of higher dimensional manifolds…. [However] I thought that the good things to study were smooth manifolds and well behaved cell complexes; the good methods were from algebraic topology and differential geometry. I don't mean to say that these are not good things to study: I love them still and they are extremely important. But at that time I was completely uninterested in other parts of topology, for example, the study of complicated decompositions of 3-space, or infinite dimensional spaces, or nasty sets like indecomposable continua. I thought these were totally boring and not worth studying. Yet later some of the basic problems in which I was very much interested came to be solved by such

methods.” Indeed, this constant engagement with higher dimensional manifolds runs through much of his vast body of work.

Milnor has received all the major awards in mathematics. He was awarded the Fields Medal at the ICM in Stockholm in 1962 and the Wolf Prize in 1989, and is the only person to have won all the three Steele prizes of the American Mathematical Society (AMS), in 1982, 2004 and early this year, for seminal contribution to research, for mathematical exposition and for lifetime achievement respectively. He also received the U.S. National Medal of Science in 1967. And now, the coveted Abel Prize. The Fields Medal was for his most celebrated result, which is his proof in 1956 of the existence of seven-dimensional spheres with non-standard differential structure.

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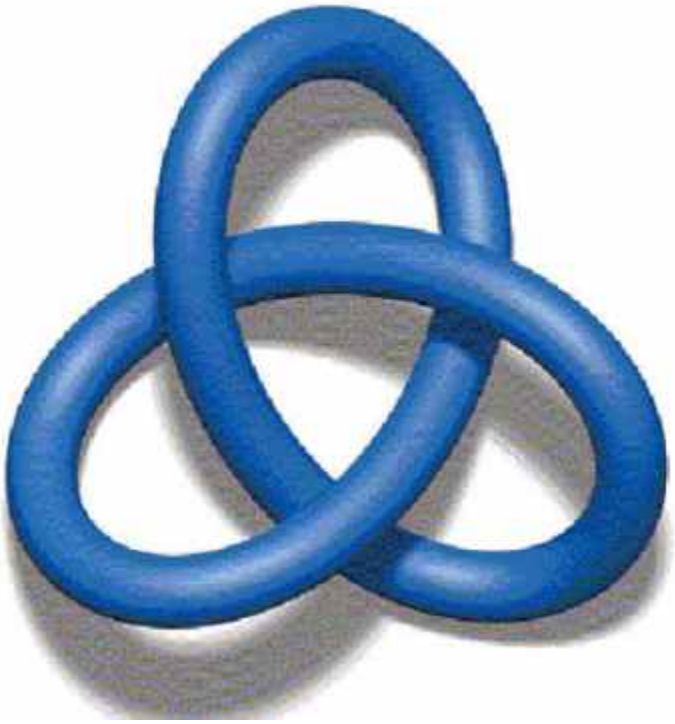

We know what a sphere is: it is the collection of all points equidistant from a point called the centre. Thus a circle on a plane is a one-dimensional sphere in a two-dimensional space; a ball is a two-dimensional sphere, or a 2-sphere in three-dimensional space. As mathematicians are wont to, this description is easily abstracted to higher dimensions and talk of n-spheres (in n+1 dimensions). Spheres are, in fact, among the most basic spaces in topology, the branch of mathematics that studies the properties of objects under their “continuous deformations” – deformations that allow you to stretch the shape at will but not tear or punch a hole in it. If you started with a basketball and deflated it or started with a cube and inflated it to look a like a sphere (Figure 1), all these objects are the same from the point of view of a topologist. But topologically, a sphere is different from a tea cup or a doughnut because you cannot turn a sphere into a doughnut by any amount of continuous deformations without puncturing it.

Mathematicians also talk of differentiable structures, which can be defined on any geometrical object or space. When the space is differentiable, topological deformations of it are “smoother” than a continuous one is required to be – that is, without any folds, corners, sharp edges or kinks – so that one can do differential calculus, the language in which physical theories and equations are formulated, on different spaces, or manifolds as mathematicians call them. (A manifold is a curved surface, say the surface of a sphere or a doughnut, small pieces of which look roughly like small pieces of a Euclidean space.)

An alternative way of describing a differentiable structure is as follows. Defining a differentiable structure over a small patch of a sphere is like drawing a map of that area. In defining it for the whole space, say a sphere, is to ensure that the pieces of maps fit together and cover the whole sphere. Intuitively, this does not seem to be a difficult proposition, and, in fact, for an ordinary sphere it is not. Our experience with two and three dimensions, in fact, intuitively suggests that it should be possible to render all “continuous deformations” of a standard sphere to be smooth or differentiable as well (and this can be proved mathematically as well for lower dimensions). That is, all topologically equivalent spheres will also be smoothly equivalent. But as we go higher up in dimensions, things begin to get mathematically complicated and become counterintuitive. You seem to have many choices of how to put the maps together.

Exotic spheres

In 1956, Milnor discovered an extraordinary mathematical object: a seven-dimensional space that is topologically equivalent to a 7-sphere (in eight-dimensional space) but with a differential structure distinct from the standard 7-sphere. That is, besides the standard 7-sphere, there is another 7-sphere that cannot be continuously transformed into the standard one without causing kinks or edges to it. That is, physically speaking, the two 7-spheres represent two inequivalent worlds. When Milnor discovered it, it was a completely unexpected and surprising result to mathematicians, including Milnor himself. “When I first came upon such an example in the mid-50s, I was puzzled and did not know what to make of it,” Milnor has said. He called such n-spheres with differential structures distinct from the standard n-sphere “exotic spheres”.

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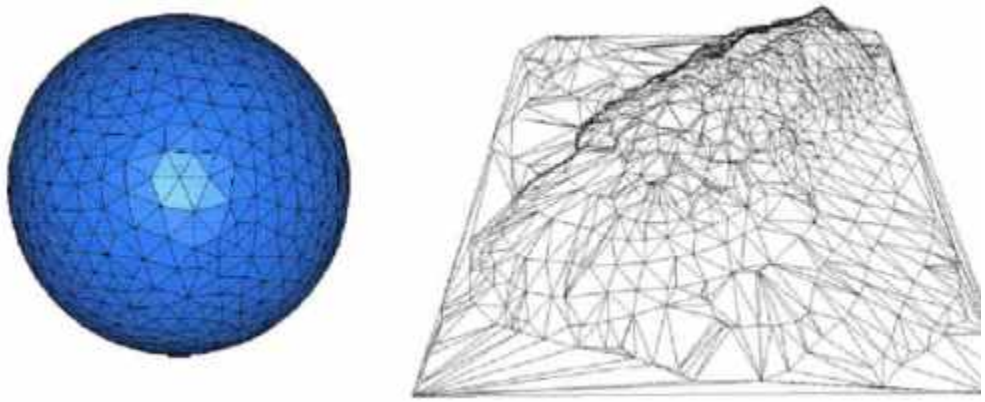
NIELS HENRIK ABEL, in whose memory the prize was instituted.

After Milnor's discovery, the question naturally arose as to how many such genuinely distinct structures were there in each dimension. With Michel Kervaire, Milnor worked this out for several dimensions. For seven dimensions he showed that there were 28 distinct differential structures – 27 besides the standard 7-sphere. The apparently patternless sequence of numbers 1, 1, 1, ?, 1, 1, 28, 2, 8, 6, 992, 1, 3, 2, 165256, 2, 16, 16, 523264, 24… denotes the number of differentiable structures topologically equivalent to the standard n-sphere as n increases from 1 to 20. Interestingly, this sequence of numbers is actually related to other topological phenomena (of how spheres of various dimensions can wrap around each other) and also to the seemingly very different field of number theory. The status for four dimensions is still unresolved. It is not known if there is one, more than one or infinitely many smooth (differentiable) structures topologically equivalent to the 4-sphere. The claim that there exists precisely one is known as “the smooth Poincaré Conjecture in dimension 4”. Thus Milnor's exotic spheres show that the smooth Poincaré Conjecture is actually wrong in dimension 7 (and in higher dimensions as in the above sequence). His work, in fact, led to the opening up of an entirely new field, called differential topology, and an explosion of related work by a generation of brilliant mathematicians that has continued to date.

K-theory

How was Milnor able to show that his exotic spheres were really exotic? To demonstrate such things conclusively, mathematicians resort to the idea of finding an ‘invariant' associated with the structures to differentiate between them. That is, one can calculate something associated with the object in question that remains invariant or same if the objects being compared are equivalent, and different if they are distinct. A very important class of invariants was developed in the 1950s by Alexander Grothendieck in the context of algebraic geometry and later by Michael Atiyah and Friedrich Hirzebruch. This led to the area of mathematics known as K-theory (where K stands for Klasse in German meaning class).

In the 1960s it became clear to mathematicians that there ought to be an algebraic version of the K-theory that would be helpful to resolve questions arising in higher dimensional spaces. However, little was known about how to define the basic concepts to develop such a theory. It was Milnor who laid the basic foundations by defining the necessary fundamental mathematical entities for this theory, which was later completed by Daniel Quillen, another American mathematician. While developing the algebraic K-theory, Milnor had formulated a conjecture, which became sufficiently important and fetched Vladimir Voevodsky his Fields Medal in 2002 for proving it. This field of K-theory is replete with Fields medallists: Milnor, Grothendieck, Atiyah and Quillen. Besides differential topology, Milnor's “exotic spheres” opened up the new field of algebraic K-theory as well.



The beautiful theorem he proved as an undergraduate about “knot geometry” was already evidence of his brilliance. It answered the following question: If you take a curve in three dimensions that starts and ends at the same point and forms a knot, how curved must it be? Two knots are said to be equivalent if we can transform one into the other by pulling or pushing the branches of the thread, but not cutting or sticking. Consider the trefoil knot (Figure 2). It is obvious that for the curve to be knotted, unlike a circle or an unknotted closed curve, it has to double back on itself. The curve has to go round twice.

In 1929, in Copenhagen, the German mathematician Werner Fenchel showed that the total curvature of a closed curve is at least 2. The result was generalised to arbitrary dimensions by the Polish mathematician Karol Borsuk in 1949, when he also posed the question: If the curve is

knotted, is the total curvature greater than 4 ? Milnor heard of the problem first as a freshman in Princeton while attending Al Tucker's differential geometry class. A proof for this was provided by Milnor in 1949 (and the paper was published in 1950 after Fox asked him to send it for publication) and independently by the Hungarian mathematician Istvan Fary at about the same time. It is, therefore, called the Fary-Milnor theorem.

The curvature of a curve at any point is the reciprocal of the radius of the circle that best approximates the curve near the point. The smaller the circle at the point, the greater the curvature. The total curvature is obtained by adding (or, more precisely, integrating) the curvature at all points of the curve. For a simple unknotted circle of fixed radius R, the total curvature is the circumference (2R) multiplied by the constant curvature (1/R), which is 2 and consistent with Fenchel's result. (In fact, this is true for any closed curve in the plane, if we take the curvature as negative when the approximating circle lies on the ‘outside'.) Intuitively, of course, it is clear that for the curve to be knotted it has in some sense to “go round twice”. If we disregard the crossing in a knotted curve, the plane projection of the curve will have a total curvature of 4. But a knotted curve cannot lie in a single plane because when the curve crosses itself some branch has to go over another and there will be some curvature in the direction out of the paper. Thus one intuitively expects the total curvature to be greater than 4. The proof due to Milnor does not involve hard mathematics, but it was rather elegant. But it was a great surprise to the mathematics community that an undergraduate could prove such a theorem. A great mathematical talent was discovered in Milnor.

One of the techniques for studying curved geometrical shapes such as manifolds is to triangulate them. This has its origins in the 18th century from the method used to draw precise topographic maps of terrestrial regions with their hills and valleys. When triangulating a piece of land, you choose a set of reference points and compute the coordinates of nearby points with respect to these. These points are then joined by connecting lines and edges so that a web of triangles is created. The density of triangles is such that the curvature of the landscape inside the triangles is negligible. Naturally, if the landscape is hilly, the density of reference points chosen will be higher as compared to a flat landscape.

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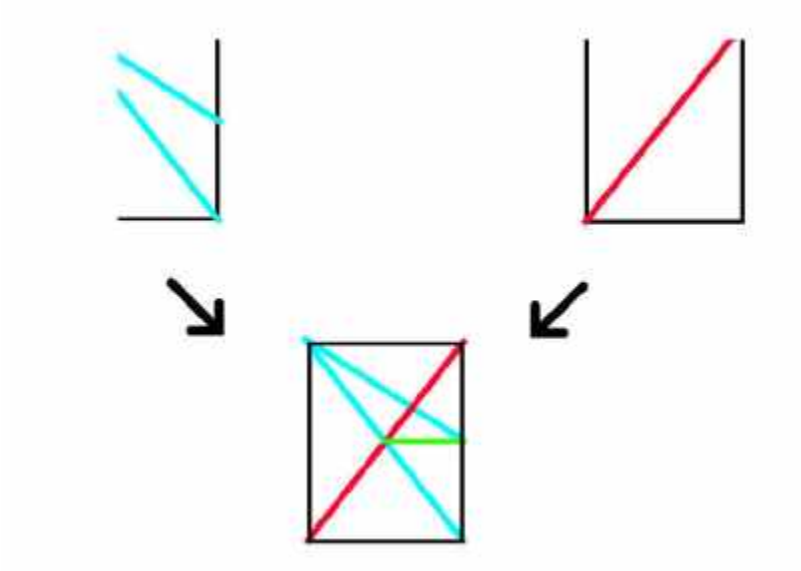

This method of triangulation, by which a piecewise linearisation of the surface is obtained, can be used for arbitrary mathematical surfaces (Figure 3). The idea being that one can study the manifold by studying how these little building blocks are put together. It is also obvious that every manifold can be triangulated by continuously deforming into a space of appropriate dimension that is built out of triangles. But how does one prove this? Also, a given manifold can be triangulated in different ways. How do we know that different triangulations give rise to the same conclusions about the manifold? One way to do that is to determine if two triangulations have a ‘common refinement'. A refinement of a triangulation is obtained by dividing the triangles into smaller triangles, which gives a new triangulation, a refinement of the original triangulation. It can be shown mathematically that a refined triangulation will have the same essential properties as the original. Thus, if you originally had two triangulations and there exists a new triangulation that refines both the original triangulations, then it would establish that the two triangulations describe the same manifold.

In two dimensions, triangulations and their common refinement always exist (Figure 4). Is this true in higher dimensions? Die Hauptvermutung, the ‘main conjecture' in German, of algebraic topology (known earlier as combinatorial topology), which was made by the German mathematician Ernst Steiniz and the Austrian mathematician Heinrich Tietze in 1908, states that it is always possible to find a common refinement to two triangulations of the same space of any dimension. The conjecture was proved in two dimensions by Tibor Rado in the 1920s and in three dimensions by Edwin E. Moise in the 1950s. This conjecture was disproved by Milnor for dimensions equal to six or higher. And in 1982, Michael Freedman discovered a four-dimensional manifold that cannot be triangulated at all.

Hairy Ball Theorem

The Hairy Ball Theorem of algebraic topology to which Milnor made important contributions is famously stated as “You cannot comb a hairy ball flat without creating a cowlick (that is, without any hair sticking up somewhere)” or “You cannot comb the hair on a coconut”. We know what a

tangent to a circle, a 1-sphere, at a point is. For a sphere in three-dimensional space, the 2-sphere, we can draw a tangent plane at a point, which can be roughly thought of as being the plane that best approximates the manifold at that point.

For higher dimensional manifolds, mathematicians talk of tangent spaces. On the tangent planes at every point of the 2-sphere, we can choose a system of x and y coordinates with the point as the origin of the two axes. We can visualise these axes as arrows pointing in the positive x-direction and positive y-direction. The question that arises is, can we choose our arrows at the various points in such a way that they vary continuously as the points on the 2-sphere vary continuously. The Hairy Ball Theorem states that you cannot do this on an even n-sphere without the tangent becoming zero at some point. The theorem was first stated in the late 19th century by Henri Poincaré for the 2-sphere case.

The 2-sphere case was proved by Luitzen Brouwer in 1912. In fact, it is not even possible to choose one axis in a continuous way. That is, there is no way of combing the hair of a hairy ball without having points where the hair bunches up as a cowlick. Milnor, building on the work of Raoul Bott, showed in 1958 that 1, 3, 5 and 7 were the only dimensions in which the hairy ball could be combed without a cowlick. This result has been described as “magical”, according to Gowers. This result was also obtained independently by Friedrich Hirzebruch, a German mathematician, and Kervaire. More significantly, Milnor had come up with a surprising and beautiful new proof later in 1978, which used multivariable calculus and not the tools of algebraic topology that had seemed essential to the proof until then (Figure 5).

In 1966, Marc Kac posed the question, an inverse mathematical problem: “Can you hear the shape of a drum?” More precisely, it means that whether one can deduce the shape of a plane region by knowing the frequencies at which it resonates (its spectrum), where, as in the case of a drum, the boundary is assumed to be held fixed. Long before Kac posed this question, mathematicians had been investigating the analogous question in higher dimensions: Is a Riemannian manifold (with boundary) determined by its spectrum? A Riemannian manifold is a manifold endowed with a way of measuring distances and angles.

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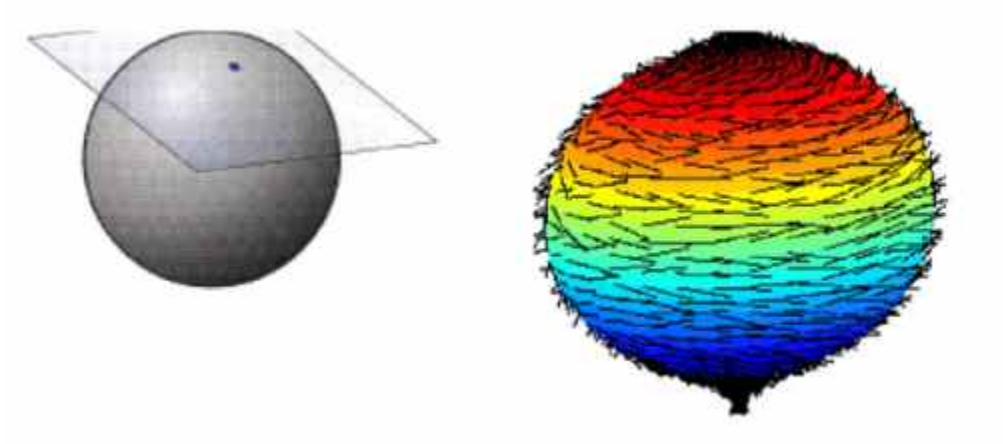

In 1964, Milnor had first settled the question in the negative for 16 dimensions. He found two distinct 16-dimensional manifolds with the same spectrum. But the problem for plane regions remained open until 1991, when Carolyn Gordon, David Webb and Scott Wolpert found examples of distinct plane drums which sound the same.

Numerous other mathematical concepts, results and conjectures are named after Milnor – Milnor fibration, Milnor number, Milnor map, and many others. More recently, Milnor turned his attention to the study of dynamical systems in low dimensions to which the Milnor-Thurston kneading theory is an important contribution.

“What I love most about the study of mathematics,” says Milnor in his Princeton University talk, “is its anarchy! There is no mathematical czar who tells us which direction we must work in, what we must be doing…. I like to picture the frontier of mathematics as a great ragged wall, with the unknown, the unsolved problems, to one side, and with thousands of mathematicians on the other side, each trying to nibble away at different parts of the problem using different approaches. Perhaps most of them don't get very far, but every now and then one of them breaks through and opens a new area of understanding. Then perhaps another one makes another breakthrough and opens another new area. Sometimes these breakthroughs come together, so that we have different parts of mathematics merging, giving us wide new perspectives…. I am making a plea for mathematical tolerance: Even if some branch of mathematics seems uninteresting today, it should not be given up completely. It is important to have people working in many different directions with many different points of view in order to attack the basic problem of understanding the mathematical world, and its applications.”

Milnor, perhaps more than anyone else in contemporary mathematics, has opened new windows in the “great ragged wall” of mathematics and given fresh perspectives to old problems as well as new unsolved problems over the last six decades.

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