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The Schwarzschild Solution: Some Conceptual Difficulties

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It is shown that inconsistencies arise when we look upon the Schwarzschild solution as the space-time arising from a localized point singularity. The notion of black holes is critically examined, and it is argued that, since black hole formation never takes place within the past light cone of a typical external observer, the discussion of physical behavior of black holes, classical or quantum, is only of academic interest. It is suggested that problems related to the source could be avoided if the event horizon did not form and that the universe only contained quasi-black holes.

1. INTRODUCTION

The classic solution obtained by Schwarzschild[(1)] soon after Einstein proposed his general theory of relativity has played a pivotal role in modern discussions of gravitation. The solution, for example, inspired the weak-field tests of general relativity; it led to the concept of black hole, offered an explicit example of space-time singularity, and also features in the understanding of quantum field theory in curved space-time. Nevertheless there are several conceptual difficulties associated with this simple and elegant solution that are usually ignored because of its manifest usefulness. Our purpose in this article is to highlight these problems since we feel that their eventual resolution will advance our understanding of the complex basic interaction of gravitation.

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2. THE SOURCE PROBLEM

To understand this problem, let us consider first its Newtonian analogue. What is the gravitational potential outside a spherical distribution of gravitational mass M?

To answer this question, we begin with Poisson's equation relating potential ϕ to matter density ρ:

$$\nabla^2\phi = -4\pi G\rho \tag{1}$$

where G is the gravitational constant. Outside the matter distribution, $\rho = 0$ and the spherical symmetry allows us to write ϕ as a function of the radial coordinate r only; thus Eq. (1) reduces to

$$\frac{d^2\phi}{dr^2} + \frac{2}{r}\frac{d\phi}{dr} = 0 \tag{2}$$

This equation has the general solution in terms of arbitrary constants A and B:

$$\phi = A + \frac{B}{r} \tag{3}$$

Without loss of generality, we can take $\phi \to 0$ as $r \to \infty$ and thus set $A = 0$. How do we determine B? We need, evidently, to make use of the fact that the source has gravitational mass M. Applying the gravitational Gauss' theorem to the surface ∂V of the source whose volume is V, we get from (1)

$$\int_{\partial V} \nabla\phi \cdot \mathbf{n}\, dV = \int_V \nabla^2\phi\, dV = \int_V -4\pi G\rho\, dV = -4\pi GM \tag{4}$$

Therefore from (3) we find

$$B = GM \tag{5}$$

Notice that in (4) we have used the relation

$$M = \int_V \rho\, dV \tag{6}$$

If we were dealing with a point mass, we would have defined ρ in terms of the three-dimensional delta function

$$\rho = M\delta(\mathbf{r}) \tag{7}$$

and recovered (5). In fact our solution

$$\phi = \frac{GM}{r} \tag{8}$$

for the point mass or a spherical mass distribution has exact analogous in electrostatics for point charge or spherical charge distributions.

We now repeat this exercise for general relativity and solve Einstein's equations

$$R_{ik} - \tfrac{1}{2} g_{ik} R = -\kappa T_{ik}, \qquad \kappa = \text{const} \tag{9}$$

for the spherically symmetric line element

$$ds^2 = c^2 e^{\nu}\, dt^2 - e^{\lambda}\, dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{10}$$

outside a spherical mass distribution. Here $x^0 = t$, $x^1 = r$, $x^2 = \theta$, $x^3 = \phi$ and the mass distribution is confined to $r \leqslant R$, say.

In the empty space outside the mass we get from (9)

$$\frac{\partial \lambda}{\partial t} = 0, \qquad \frac{\partial}{\partial r}(\nu + \lambda) = 0 \tag{11}$$

from which, without loss of generality, we deduce

$$\lambda + \nu = 0 \tag{12}$$

The equation satisfied by λ is then

$$\frac{d}{dr}[r(1 - e^{-\lambda})] = 0 \tag{13}$$

which integrates to

$$e^{-\lambda} = 1 - \frac{2B}{c^2 r}, \qquad B = \text{const} \tag{14}$$

Now we encounter a difficulty: how to determine B. One method is to use the requirement that, for weak fields, general relativity should reduce to Newtonian theory, and so, at $r \to \infty$, the constant B in (14) should have the same value as B in (5). For practical purposes this procedure appears satisfactory, but conceptually it leaves much to be desired. Why should general relativity have to refer to another theory (Newton's) in order to fix its constants? Can then constant B not be determined from the source distribution entirely within the framework of general relativity?

The answer to this question is usually given by referring to the (time, time) component of (9) including the source distribution. Then (13) becomes

$$\frac{d}{dr}[r(1-e^{-\lambda})] = \kappa T_0^0 r^2 c^2 = \kappa r^2 \rho c^4 \tag{15}$$

where ρc^2 is the energy density.

Hence if we *define*

$$M = \int_0^R 4\pi r^2 \rho(r)\, dr \tag{16}$$

we get

$$B = \frac{c^4 \kappa}{8\pi} M \tag{17}$$

By a Cavendish-type experiment we can *identify* B with GM, where $G = c^4\kappa/8\pi$ and M is the gravitational mass.

However, this definition of M, as given by (16), is not as natural as it looks. First note that the volume element in (16) is $4\pi r^2\, dr$ whereas the proper volume element for the static ($t = \text{const}$) hypersurface should be $4\pi r^2 e^{\lambda/2}\, dr$. Secondly, the limit to point mass, at $r = 0$, of (16) leads to fresh difficulties. Suppose we define in this case

$$\rho = M\delta(\mathbf{r}) \tag{18}$$

so that (17) can apparently still hold for the point particle. However, the trace of the field equations

$$R = \kappa T \tag{19}$$

leads to the differential equation

$$\frac{d^2x}{dr^2} + \frac{4}{r}\frac{2x}{r^2} = -kc^2 T, \qquad x = e^{-\lambda} - 1 \tag{20}$$

This equation integrates to our earlier solution for λ if

$$T = M\delta(\mathbf{r})\, c^2 \tag{21}$$

Now, in the Schwarzschild coordinates, the source particle is supposed to be at rest at $\mathbf{r} = \mathbf{0}$, so that T_0^0 is the only nonzero component. So apparently, (21) is consistent with (18). But the condition $\nu + \lambda = 0$ that was obtained from (11) and (12) requires $T^1_1 = T^0_0$. Thus we have arrived at an inconsistency at $\mathbf{r} = \mathbf{0}$.

It could be argued that a point source at $\mathbf{r} = \mathbf{0}$ is unrealistic and that the Schwarzschild solution works for a distributed source only. This way out is unfortunately ruled out by the phenomenon of gravitational collapse that inevitably results in all the matter converging to $\mathbf{r} = \mathbf{0}$ in finite comoving time.

Thus we have to argue that the inconsistency described above is the artifact of space-time singularity that implies a breakdown of the coordinate system at $r = 0$. But since the source itself is located at the singularity, how is its influence conveyed outside? This conundrum remains unresolved.

We will return to the singularity at $r = 0$ in Section 4 after considering another conceptual problem associated with the actual process of gravitational collapse.

3. ARE BLACK HOLES RELEVANT TO PHYSICS?

Black holes are generally believed to have formed by gravitational collapse of massive bodies. In a strict sense what is a black hole (BH)? It is an object surrounded by an event horizon. By contrast we may call an object a *quasi*-black hole (QBH) if it is highly collapsed and very dim but still outside its event horizon. It is well known that as an object collapses toward its event horizon the intensity of its radiation as received by an external Schwarzschild observer (with trajectory $r = \text{const}$) rapidly falls. Thus a QBH will be invisible for all practical purposes although still *outside* the event horizon.

Most astrophysical scenarios using black holes are concerned with QBH's only. However, the laws of black hole physics apply to BHs, as does the notion of an evaporating BH. Our question is: are the BH phenomena really relevant to physics?

It is agreed that for a scientific hypothesis to be taken seriously, it must be testable—if not in practice (owing to limitations of technology available) at least in principle. So far as a BH is concerned, it is supposed to come into existence only when its outer surface *enters* the event horizon. As illustrated in Fig. 1, an outside observer cannot ever see this happening.

In terms of the Schwarzschild line element, a light ray leaving the surface of the collapsing object at (R, T) reaches an outside stationary observer $r = \text{const}$ at time t, where

$$c(t - T) = r - R + \frac{2GM}{c^2} \ln \left\{ \frac{r - 2GM/c^2}{r - R} \right\}, \qquad R_s = \frac{2GM}{c^2} \tag{22}$$

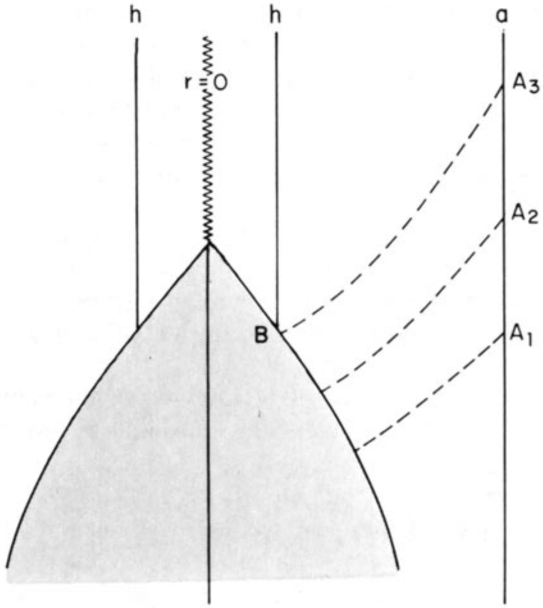

Fig. 1. Space-time diagram showing the gravitational collapse of a massive body (shaded region) to a singularity (jagged line). The world line of an external observer A is shown by *a*, with world points A_1, A_2,..., etc. The event horizon is shown by *h*. The body becomes a black hole only after it enters *h*. The dashed lines schematically describe the past light cones of world points on *a*. Prior to and close to point B the body is a quasi-black hole. No measurement can convey to A that a black hole is formed, no matter how far into the future *a* is extended.

Clearly, as $R \to R_s$, $t \to \infty$. This result can be generalized without significant alteration to any outside observer who always stays *beyond* the event horizon. Thus our argument can be summarized as follows.

For the detection of any object by whatever means, it must come within the observer's past light cone. This does not ever happen for a BH. So none of the laws describing the behavior of BHs (as opposed to the QBHs) are in principle detectable or testable by the class of observers who stay outside their event horizons. Since most observers (including those on the Earth) are of this type, to them the BH's are not relevant as physical objects.

It might be argued that we should not restrict our discussion to outside observers only. However, as we discuss below, fresh difficulties arise when we include the experiences of observers inside the event horizon.

4. OBSERVERS INSIDE THE EVENT HORIZON

We begin our discussion by returning to the source problem of Section 2. To settle whether the Schwarzschild solution corresponds to a point source at $\mathbf{r} = \mathbf{0}$, we have to exaine the behavior of the gravitational field near the origin. In an analogy, note that the behavior of $\nabla^2\phi$ near $\mathbf{r} = \mathbf{0}$ in Newtonian gravity is capable of determining the nature of the source. Since the outside observers have no access to the region inside the event horizon, it is necessary to consider the observers inside the Schwarzschild radius $R_s = 2GM/c^2$.

Let us consider the following situation: A spherical star of mass M, made of dust, starts to collapse at $t = 0$ from a radius $r = R_0$. An observer, equipped with a rocketship and a dying curiosity (no pun involved!) to know what happens at $r < R_s$, follows the dust ball. He starts at $r = R_0 + L$ at $t = 0$ and falls freely up to $r = R_s$. Once he is just inside the black hole ($r = R_s - \varepsilon$ with ε infinitesimal), he uses the rocket to travel along the timelike trajectory AB [shown in Fig. 2], which can be expressed in Kruskal coordinates (u, v) by

$$u = \beta(v - u_0) + u_0 \tag{23}$$

where β and u_0 are constants. This trajectory will intersect the singularity $r = 0$ at the coordinate time

$$v_f = \frac{k\beta + \sqrt{k^2 + 1 - \beta^2}}{(1 - \beta^2)} \tag{24}$$

with $k = u_0(1 - \beta)$. One can make v_f as large as one wants by taking β close to unity (this should also be clear from Fig. 2). Thus the observer can exist inside the $r = R_s$ surface for an arbitrarily large coordinate time interval.

Can such an "inside observer" gather any physical information about the singularity that is supposed to be formed at $r = 0$?

The surprising answer is "no"! Note that the "trajectory of a point particle at origin," $\mathbf{r} = 0$, is a spacelike trajectory in the standard Schwarzschild metric. Therefore the singularity exists to the future of all physical observers confined to the region $0 \leqslant r \leqslant R_s$. *In other words, no observers in the Schwarzschild metric (whether they stay outside $r = R_s$ or*

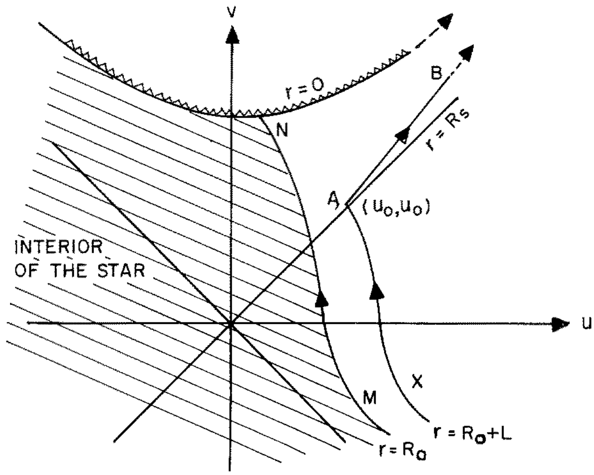

Fig. 2. Space-time diagram in Kruskal (u, v) coordinates describing the collapse of a star. MN represents the world line of the surface of the star and XAB represents the world line of an observer following the collapse by falling into the event horizon.

choose to fall inside) will ever be able to observe either the formation or the physical effects of a singularity at $r=0$. We leave it to the reader to decide whether a singularity that can *never* be observed and that can never affect any physical process "exists" in any sense of the word.

The fact that $\mathbf{r}=\mathbf{0}$ is a spacelike trajectory adds to the misgivings of Section 2. To interpret the Schwarzschild solution as the metric produced by the point particle, one has to satisfy two conditions: (i) Einstein's equations, with a point mass as the source, located at the origin, have to be satisfied by the metric and (ii) the source particle should follow a timelike trajectory. In Section 2 we pointed out that condition (i) is violated in the Schwarzschild metric. It is now clear that condition (ii) is also violated in any collapse that leads to a point source.

At $t \leqslant 0$ (before the start of the collapse all the dust particles making up the star were following timelike trajectories. If Einstein's equations are satisfied, then source particles will continue along geodesics which will be timelike. Thus the source can never be confined to the point singularity at $\mathbf{r}=0$ (which is spacelike) if Einstein's equations are always satisfied. This fact provides the connection between condition (i) and condition (ii). Given physically meaningful initial conditions, (i) implies (ii). It is the violation of (i) in the Schwarzschild metric (described in Section 2) that paves the way for the violation of (ii).

Lastly, consider a relabeling of (r, t) as (t', r') for $r < R_s$. Then the Schwarzschild metric can be written in a rather suggestive form for $r < R_s$:

$$ds^2 = \left(\frac{R_s}{t'} - 1\right) dt^2 - \left(\frac{R_s}{t'} - 1\right)^{-1} dr'^2 - t'^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

This represents a nonstatic space-time [the metric coefficients depend on the timelike coordinate t'] with a singularity at the *spacelike hypersurface* $t' = 0$! Such an interpretation, which should be as valid as the conventional interpretation, brings out the ambiguity inherent in the Schwarzschild solution.

We therefore find that considering observers inside the event horizon makes the problems of interpretation even more difficult, and we wonder whether nature allows gravitational collapse to continue inside the event horizon at all.

5. CONCLUSION

The Schwarzschild solution may be looked upon in three parts. Part I, describing the asymptotically flat region far away from the source, is observationally well tested for the Newtonian and post-Newtonian terms. Part II describes black holes and is concerned with how the space-time behaves when event horizons are formed. It is this region that explicitly features in the laws of black hole physics and the quantum aspects of black hole evaporation.

We have shown that in strict technical terms an outside observer never sees an event horizon forming, no matter how long he lives. As such this part is outside the domain of his physical experience. Those who believe that black holes (and not just the quasi-black holes) have physical relevance should produce a thought experiment to demonstrate to an external observer that a black hole (and not a QBH) has formed in a given region.

Part III of Schwarzschild's solution describes the interior of the event horizon. This is the region that contains the point source singularity that is the eventual outcome of gravitational collapse. Within the framework of general relativity this situation has a real status; yet we have shown how inconsistencies arise concerning the anture of the source when we take into consideration all the consequences of Einstein's equations. We have no solution to offer for this difficulty nor do we believe that one exists within the conventional classical framework. These problems can be avoided by introducing negative energy or stresses to reverse the gravitational collapse before the event horizon is formed.

It may also well be that quantum gravity—when it is perfected—will resolve these conundrums. Just as the notion of an isolated electric charge in classical electrodynamics was seriously modified in quantum electrodynamics, we may discover that the above problems arise because the Schwarzschild solution offers an oversimplified picture of the real situation.

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