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PRINCIPLE OF GENERAL COVARIANCE AND QUANTUM THEORY

T. PADMANABHAN

Theoretical Astrophysics Group, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay-400005, India

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We emphasise the distinction between *formal* and *operational* notions of general covariance. Classically, formal covariance implies operational covariance. This is not true in quantum theory. Two observers may not agree on the results of measurement of a *tensorial* object like T_{ik} (stress tensor) in quantum theory. In particular, one observer might conclude that the measured value is zero while another might attribute non-zero value to it.

Classical general relativity describes spacetime as a Riemannian manifold M of 4 dimensions. Most of the physical observables in the theory can be represented as elements of direct products of the tangent space T_P at an event $P \in M$. For the sake of convenience, one may introduce a set of basis vectors for T_P, which—in turn—defines a natural basis on $T_P \otimes T_P \otimes \cdots$ and allows one to compute components of tensors of arbitrary rank. At this formal level, general covariance is tantamount to the statement that 'physics is independent of the choice of basis vectors for T_P.

Let us enquire about the operational significance of such a statement. Consider two observers S and S' who set up two co-ordinate systems x^i and $x^{i'}$ in some region $R \in M$. These observers use the natural basis $(\partial/\partial x^i)$ and $(\partial/\partial x^{i'})$ in the tangent space T_P. Consider, for example, a second rank tensor $T \in T_P \otimes T_P$, which will have the components T^{ik} and T'^{ik} in the two basis. Elementary differential geometry tells us that

$$T'^{ik} = \frac{\partial x^{i'}}{\partial x^a}\frac{\partial x^{k'}}{\partial x^b} T^{ab} . \tag{1}$$

The *physical* importance of general covariance stems from the following *assumption*: "If the observer S measures some physical observable $T \in T_P \otimes T_P$ and obtains the values T^{ab}, then *corresponding* measurements by S' will yield the values T'^{ik}, where T'^{ik} and T^{ab} are related by (1)". This is a powerful statement *about physics* and allows one to predict the results of measurements of a variable T provided it is known (or assumed) that $T \in T_P \otimes T_P$. (For example, if we assume that $(t, \boldsymbol{x})$ are components of a four-vector in special relativity, Eq. (1) allows us to derive the slowing down of uniformly moving clocks). One trivial consequence of (1) is that if T^{ab} is identically zero, all T'^{ik} will be zero.

It should be clear from the above that there are two aspects to the concept of general covariance: (i) Firstly, one has to decide whether the physical quantity we are interested in is a 'tensorial' object i.e. element of $T_P \otimes T_P \otimes \cdots$. (Clock readings, energy, or connection coefficients are not, while the curvature scalar, mass or the metric are.) (ii) Secondly, one should ensure that the physical theory guarantees the results of measurement by S and S' to be the components related by (1).

Normally, one does not make such a distinction between (i) and (ii). It is usual to assume that (i) "automatically" leads to (ii). In classical physics, this is indeed true if the equations of motion of the theory are formulated in terms of tensorial quantities. Consider, for example, the measurement of the electromagnetic field $F^{ik}(x)$ by the observer S. He can do this by measuring the accelerations induced on a set of charged particles via the equation

$$a^i = \frac{d^2x^i}{d\tau^2} = \frac{q}{m} F^{ik} u_k . \tag{2}$$

Since (2) is written in terms of tensorial quantities, it will read as

$$a^{i'} = \frac{d^2 \times^{i'}}{d\tau^2} = \frac{q}{m} F^{ik'} u'_k \tag{3}$$

in S'. Clearly if S and S' use similar procedures of measurement then they will 'automatically' obtain the numbers F^{ik} and F'^{ik} connected by tensorial transformation like (1). This argument hinges on the fact that variables appearing in the classical equations of motion are *directly measurable* c-numbers.

Quantum theory introduces several new aspects into the above discussion which we shall now investigate.

In quantum theory, physical observables are represented by operators acting on a Hilbert space H. We can obtain c-numbers by computing the "matrix elements" (i.e. components of the operator in a chosen basis in H). Consider, for example, the operator T^{ik} corresponding to the classical quantity T^{ik} in (1). We shall assume that (1) is valid for the operators used by S and S' as well. Then it follows that

$$\langle \psi | \hat{T}'_{ik} | \phi \rangle = \frac{\partial x^a}{\partial x^{i'}} \frac{\partial x^b}{\partial x^{k'}} \langle \psi | \hat{T}_{ab} | \phi \rangle \tag{4}$$

provided $|\phi\rangle' = |\phi\rangle$, etc. The Hilbert space H is assumed to be 'external' to the manifold M and hence same to the observers S and S'. Thus, formally, matrix elements transform covariantly. Similar relations can be written down for any other tensor-operator. We now ask: What do the results of measurements by S and S' give? Can they measure these matrix elements directly. It turns out that they cannot.

To see this, consider a model system consisting of test charges coupled to the electromagnetic potential $A^i(x)$ via the interaction Hamiltonian,[1,2]

$$H_I = \int d^4x\sqrt{-g}J_iA^i \tag{5}$$

where J_i is the current vector of the test charges ("detector"). Let the quantum state of the field be $|I\rangle$ and the state of the detector be the energy eigenstate $|E_i\rangle$ at $t \to -\infty$. If the detector is found in a state $|E_f\rangle$ at $t \to +\infty$ with $E_f > E_i$, then we may claim that some energy has been transferred from the field to the detector. In the semi-classical limit, such a process would have been interpreted as detection of the electromagnetic field. If, for example, the J^i was due to a heavy charged particle then E_i and E_f would correspond to the kinetic energies $(P_i^2/2m)$ and $(P_f^2/2m)$ and the increase in the energy would be interpreted as due to the work done by the electric field.

What is observed in the full quantum theory, however, is the transition probability[2]

$$P \cong \sum_{\text{all}|F\rangle} |\langle F; E_f|H_I|E_i; I\rangle|^2 . \tag{6}$$

Straightforward calculation allows us to write this as

$$P = \int \sqrt{-g(x)}d^4x \int \sqrt{-g(y)}\, d^4y\, \mu_{ik}(x, y)\, \langle I|A^i(y)A^k(x)|I\rangle \tag{7}$$

where $\mu_{ik}(x, y)$ is entirely determined by the nature of the detector *and choice of states* $|E_i\rangle, |E_f\rangle$

$$\mu_{ik}(x, y) = \langle E_f|J_i(x)|E_i\rangle\langle E_i|J_k(y)|E_f\rangle . \tag{8}$$

From (7), it follows that measurements can only provide us with information about the two point functions like $\langle I/A^i(x)A^k(y)/I\rangle$. (In the semi-classical limit, this quantity will be proportional to the square of the amplitude of the electromagnetic field. Thus this measurement carries over to the measurement of the field.)

Consider now the observer S' who makes the same measurement. He will use a "similar" detector with energy levels $|E'_i\rangle$ and $|E'_f\rangle$. These states are defined as eigenstates of the detector Hamiltonian H' and will—in general—be different from the states $|E_i\rangle$ and $|E_f\rangle$ of the observer S. Mathematically speaking, this difference arises due to the existence of more than one timelike Killing vector field in part of the manifold.[3,4] Let $\xi^{\mathrm{I}}(=\partial/\partial t)$ and $\xi^{\mathrm{II}}(=\partial/\partial t')$ be two Killing vector fields corresponding to the time co-ordinates t and t' used by S and S'. The Hamiltonians used by S and S' will be H^{I} and H^{II} where

$$H^{\mathrm{I,II}} \equiv \int T^{ik}\xi_i^{\mathrm{I,II}} d\Sigma_k . \tag{9}$$

(Note that $H' \equiv H^{\mathrm{II}}$ is *not* obtained by transforming $H \equiv H^{\mathrm{I}}$ from S to S'. In fact, the H's defined by (9) are generally covariant scalars.) The states $|E\rangle'$ are eigenstates of H^{II} while the states $|E\rangle$ are eigenstates of H^{I}. Thus the energies defined by S and S' will not bear any simple relation with each other. In other words, μ_{ik} will *not* be related to μ'_{ik} by a tensorial transformation. Note that $J_i(x)$ is tensorial; it is the difference in the choice of states $|E_i\rangle$ used by S and S' which makes the difference in P.) Therefore $P' \neq P$. In particular, P can be zero while P' is not.

Several such examples are known in the literature, when S is an inertial observer and S' is a uniformly accelerated observer. If the detector is coupled to A^i then $P = 0$ and $P' \neq 0$. Recently Padmanabhan and Singh[5] have considered detectors which couple to the energy momentum tensor T_{ik} of a scalar field. It is again seen that $P = 0$ while $P' \neq 0$.

When $P = 0$ with $P' \neq 0$, the operational notion of general covariance breaks down. Since S and S' have used detectors constructed in similar manner, each has a right to interpret their observations in a similar manner. S will claim the absence of A^i (or T^{ik}) while S' will claim the existence of A^i (or T^{ik}). Though A^i (or T^{ik}) transforms tensorially (see Ref. 4), the results of measurement do not coincide with the results obtained by tensorial transformation. Formal covariance remains but operational covariance is lost.

It is often claimed in the literature that the number operator N in field theory is not covariant while operators like T_{ik} (stress tensor) are. Such a statement should be interpreted cautiously. The number operator N requires for its definition a choice of time co-ordinate. Hence it is not even a (formally covariant) tensorial object. (It is covariant under Lorentz transformation but not under general co-ordinate transformations.) On the other hand, T_{ik} is formally covariant in the sense of Eq. (4). But T_{ik} is not operationally covariant. Any detector capable of measuring T_{ik} will give contradictory results in S and S'! Thus the operational notion of general covariance is violated at a very fundamental level in quantum theory.

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