Brane corresponding to the Nariai bulk

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Abstract

We consider the five-dimensional bulk spacetime with negative Λ described by the Nariai metric (which is not conformally flat) and match it with a vacuum brane satisfying the proper boundary conditions. It is shown that the brane metric corresponds to a cloud of string dust of constant energy density.

PACS: 04.50.+h, 11.10.Kk, 98.80.Hw

A fresh impetus to the old paradigm of spacetime with extra dimensions was recently given in [1], where it was suggested that compact extra dimensions may be macroscopic while our space-time is described as a lower-dimensional domain wall (brane) where all the matter is concentrated. A novel approach to higher-dimensional braneworld cosmology emerged after Randall and Sundrum postulated the existence of a *noncompact* spacelike fifth dimension [2]. According to this world-view, our perception of 'normal' four-dimensional gravity arises because we live on a domain wall (brane) embedded in or bounding a 'bulk' anti-de Sitter space (AdS). The metric describing the full (4+1)-dimensional space-time is non-factorizable, and the small value of the true five-dimensional Planck mass is related to its large effective four-dimensional value by the extremely large warp of the five-dimensional space. The novelty of the Randall–Sundrum (RS) model is to use the curvature of the bulk spacetime (with the Z_2 symmetry of reflection relative to the brane) to keep zero-mass gravitons localized on the brane.

This theory was studied in detail in the case of 5D anti de Sitter (AdS) bulk with flat or Schwarzschild vacuum brane and in the cosmological context. The bulk and brane solutions are matched by the Israel boundary conditions. The effective Einstein equation on the brane can be written [3] by using the Gauss–Codazzi relations. It would additionally involve square of the stress-energy tensor and projection of the bulk Weyl curvature tensor to the brane. The latter is trace-free and is known as the Weyl dark energy/radiation. In this sense, the system of equations on the brane is obviously not closed. It is therefore very difficult to find exact complete solutions with both bulk and brane metrics satisfying the proper boundary conditions. There exist only a few examples of complete solutions, among which the AdS bulk with flat or Schwarzschild brane and Schwarzschild–AdS bulk with FRW brane. Most of other solutions including black hole [4] and collapse [5,6] are solutions of only the brane equations without the corresponding solution in the bulk.

The purpose of this paper is to give one more simple example of complete solution. Specifically, the Nariai metric [7] offers an interesting case of the Einstein space which is not conformally flat. After the generalization of this metric to 5D case with negative Λ, the question of graviton confinement was studied for this conformally non-flat bulk spacetime [8], and it was shown that there exist no normalized modes for massless graviton (once again, there is a pointer to fine tuning of parameters inherent in the Randall–Sundrum (RS) model [9,10]). However, this was done with the bulk metric alone without any reference to the brane spacetime. In this paper, we complete the solution by finding the corresponding brane satisfying the proper boundary conditions.

The theory that we consider here is described by the following general action (see [11]):

$$S = M^3 \sum \left[\int_{\text{bulk}} (\mathcal{R} - 2\Lambda) - 2 \int_{\text{brane}} K \right] + \int_{\text{brane}} \left(m^2 R - 2\sigma \right) + \int_{\text{brane}} L\left(h_{\alpha\beta}, \phi\right) \tag{1}$$

in the standard notation, where the sum is taken over the bulk components bounded by the brane. We use the signature and sign conventions of [12]. The lagrangian $L\left(h_{\alpha\beta}, \phi\right)$ corresponds to the presence of matter fields ϕ on the brane and describes their dynamics, and the extrinsic curvature $K_{\alpha\beta}$ of the brane is defined with respect to the inner normal n^a, as it is done in [13]. Note that we have included the curvature term in the action for the brane which arises when one incorporates quantum effects generated by matter fields residing on the brane.

The equations in the bulk and on the brane are obtained from the variation of Eq. (1), which gives

$$\mathcal{G}_{ab} + \Lambda g_{ab} = 0\,, \tag{2}$$

$$m^2 G_{\alpha\beta} + \sigma h_{\alpha\beta} = \tau_{\alpha\beta} + M^3 \sum \left(K_{\alpha\beta} - h_{\alpha\beta} K \right)\,, \tag{3}$$

where $h_{\alpha\beta}$ is the induced metric on the brane, $\tau_{\alpha\beta}$ is the stress-energy tensor resulting from the Lagrangian $L\left(h_{\alpha\beta}, \phi\right)$, and the sum of the extrinsic curvatures on either side of the brane is taken.

The Nariai metric in the bulk as given in Ref. [8] reads as

$$ds_5^2 = e^{-2k|y|} \left(-dt^2 + dr^2 \right) + dy^2 + \frac{1}{2k^2} \left(d\theta^2 + \sinh^2\theta d\phi^2 \right)\,. \tag{4}$$

It is a solution of the bulk equation (2) with $\Lambda = -3k^2$. Since the Weyl curvature is non-zero for this metric and hence its projection, $E_{\mu\nu} := C_{\mu a \nu b}\, n^a\, n^b$ on the brane $y = \text{const}$, would be non-zero.

Now we consider the brane located at $y = 0$ which has induced metric $h_{\alpha\beta}$ given by the line element

$$ds_4^2 = -dt^2 + dr^2 + \frac{1}{2k^2} \left(d\theta^2 + \sinh^2\theta d\phi^2 \right)\,. \tag{5}$$

This is a spacetime having the structure of the product of a flat 2-dimensional space and a 2-sphere of constant curvature [14]. Further, it can be shown that the stress-energy tensor corresponding to this metric describes a cloud of string dust [15,16].

The stress-energy tensor for a string-dust distribution is given by [15,16],

$$T^{\mu\nu}_{\text{string}} = \rho \Sigma^{\mu\beta} \Sigma^{\nu}_{\beta}\,, \tag{6}$$

where ρ is the proper energy density of the cloud, and $\Sigma^{\mu\nu}$ is the bivector associated with this world-sheet: $\Sigma^{\mu\nu} = \epsilon^{AB} \frac{\partial x^\mu}{\partial \xi^A} \frac{\partial x^\nu}{\partial \xi^B}$. Here, ϵ^{AB} is the 2D Levi-Civita tensor (normalized so that $\epsilon^{AB}\epsilon_{AB} = 2$) and $\xi^A = (\xi^0, \xi^1)$ are the coordinates on the string world-sheet. Following Refs. [15,16], we readily conclude that the stress-energy tensor corresponding to the above brane metric accord with the string-dust stress-energy tensor, which satisfies the equation of state $T^0{}_0 + T^i{}_i = 0$, a typical of topological defects like cosmic string and global monopole.

The components of the Einstein tensor of this metric in the coordinates (t, r, θ, ϕ), are given by

$$G^\alpha{}_\beta = \operatorname{diag}\left(2k^2, 2k^2, 0, 0\right) . \tag{7}$$

Clearly, the above-mentioned equation of state for the string-dust distribution is satisfied and the string dust has constant negative energy density $\rho = -2k^2$.

The extrinsic curvature on either side of the brane is given by

$$K^\alpha{}_\beta = \operatorname{diag}(-k, -k, 0, 0)\,, \quad K = -2k\,. \tag{8}$$

Substituting it into Eq. (3), we obtain the system of equations for the vacuum brane ($\tau_{\alpha\beta} = 0$)

$$2m^2k^2 + \sigma = 2M^3k\,, \quad \sigma = 4M^3k\,, \tag{9}$$

whence we get the conditions of "fine tuning"

$$M^3 = -m^2k < 0\,, \quad \sigma = -4m^2k^2 < 0\,. \tag{10}$$

The brane metric thus describes a cloud of string dust of constant negative energy density. Furthermore, we see that the solution (4) with brane located at $y = 0$ requires both the five-dimensional Planck mass M and brane tension σ also to be *negative*. According to the fine-tuning conditions (10), only two of the four fundamental constants in this theory are independent; for example, the constants M and σ can be expressed in terms of m and Λ via Eq. (10). In contrast to the original RS model, our solution requires $m \neq 0$, i.e., it requires the presence of the induced curvature term in the action for the brane.

The Nariai bulk and the corresponding string-dust brane may not appear to be of much cosmological and astrophysical importance. However, they undoubtedly represent an interesting spacetime solution, and the Nariai metric has already seen a good bit of application in the context of black hole and quantum-gravity considerations [17]. Our aim was, in view of the paucity of complete exact solutions for the bulk-brane system, to present one more example involving very simple spacetimes.

Acknowledgments: Yu. S. would like to thank IUCAA for warm hospitality and acknowledges partial support from the INTAS grant for project No. 2000-334.

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