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Time Symmetric Electrodynamics and the Arrow of Time in Cosmology
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Reviewed work(s):
Source: *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences*, Vol. 277, No. 1368 (Jan. 7, 1964), pp. 1-23
Published by: The Royal Society
Stable URL: http://www.jstor.org/stable/2414645
Accessed: 30/11/2011 06:22

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Time symmetric electrodynamics and the arrow of time in cosmology

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(*Received* 1 *May* 1963)

This paper seeks to establish a connexion between the local arrow of time given by the electromagnetic radiation and the cosmological arrow of time given by the expansion of the universe. The Wheeler–Feynman absorber theory of radiation is applied to the expanding cosmological models. First, it is shown that the Schwarzschild–Tetrode–Fokker principle of direct interparticle action can be extended to the general Riemannian space-time. This generalization is considerably simplified in the conformally flat spaces—as all the Robertson–Walker spaces are. In the application of the absorber theory to various cosmological models, the refractive index turns out to play a crucial part. The ambiguities connected with the sign of the imaginary part of the refractive index are resolved if two conditions are fulfilled: (i) a search is made for a self-consistent solution with *full* retarded (or advanced) solutions (ii) in an elementary theory the origin of the imaginary part of the refractive index is traced to the radiative reaction itself and not to the collisional damping considered by Hogarth. It is shown that full *retarded* solutions are consistent in the steady-state cosmology and full *advanced* solutions in the Einstein–de Sitter cosmology. Full advanced solutions are not consistent in the former and full retarded solutions in the latter. Some interesting implications of this result in the C-field approach to the steady-state cosmology are considered.

Introduction

It is well known that Maxwell's equations are time-symmetric. Observed electromagnetic phenomena, on the other hand, are asymmetric with respect to time. For example, an accelerated electric charge is observed to radiate energy and to suffer damping of its motion. The time-reversed situation in which energy converges on the charge from infinity is never observed.

The interesting suggestion has been made by Hogarth (1962) that the observed asymmetry of electromagnetic phenomena is a consequence of cosmology rather than of the basic electromagnetic equations. Indeed, an entirely time-symmetric theory, that of Wheeler & Feynman (1945, 1949), is used in Hogarth's work. The asymmetry is introduced by the expansion of the universe, which causes the 'future absorber' to be different from the 'past absorber'.

It is a feature of the time-symmetric theory that, instead of considering the usual initial-value problem, one seeks for a self-consistent solution of the equations. In a self-consistent solution, as explained in detail in § 2(*a*), the fields of all charged particles are prescribed and it is shown that the physical description is the same as that obtained by considering all the particle–particle interactions of the time-symmetric theory. The question evidently arises as to whether the observed retarded fields represent a self-consistent solution—it being understood that each particle radiates symmetrically. The remarkable result obtained by Wheeler & Feynman

(1945) was that, in a static universe of uniform density, a retarded solution is in fact self-consistent. However, an advanced solution is also self-consistent. (In a static universe, the two solutions, although formally different, could be identified with each other since they have the common property that radiation from a source propagates only in one sense along the light cone. It appears, however, that the static case also admits of an entirely time-symmetric self-consistent solution in which radiation propagates in both senses along the light cones, and this solution does not accord with observation.) The issue raised by Hogarth was whether in an expanding universe the retarded solution might still be self-consistent, but not the advanced solution. The answer to this question was found to depend on cosmology. It was found, for example, that the retarded solution was self-consistent in steady-state cosmology but not in the Einstein–de Sitter cosmology, while the situation was reversed for the advanced solution. The sense of time implied in the use of the terms 'retarded' and 'advanced' is that defined by the expansion of the universe.

The purpose of the present paper is to attempt to fill in two gaps that seem to us to exist in Hogarth's discussion. Our conclusions do not differ in any crucial respects, however. The issues we wish to discuss are the following.

(i) In the time-symmetric theory, Maxwell's equations are not immediately derived from an action principle. The electromagnetic field is an auxiliary quantity defined in terms of the world-lines of charged particles; and Maxwell's equations must be proved by a separate investigation outside the action principle. This was done by Wheeler & Feynman (1949) for the flat space-time. But this does not guarantee that Maxwell's equations are satisfied in general relativity. One cannot use the normal invariance, since normal invariance is derived from an action principle in which the field is given its own degrees of freedom, separate from the particles. To prove Maxwell's equations in the time-symmetric theory for the case of general relativity it is necessary, first, to generalize the Schwarzschild–Tetrode–Fokker action principle for the action at a distance, then to derive the field as an auxiliary quantity, and finally to prove the validity of the equations in a manner analogous to that used by Wheeler & Feynman. Although the required generalization is unexpectedly difficult in the most general case, the discussion is much simplified when the space-time is conformally flat, as it is for cosmologies with spatial homogeneity and isotropy.

(ii) In dealing with the 'response' of the universe to the field radiated by an accelerated charge we have used the concept of a refractive index, following the work of Wheeler & Feynman. We have preferred to consider an 'elementary' theory in which the imaginary part of the refractive index arises from radiative reaction, rather than from collisional damping, as in Hogarth's work. This permits one to deal more readily with an important question of sign. (We are indebted to Professor Feynman for a valuable discussion on this point.) A knowledge of the sign of the imaginary part of the refractive index is essential to establishing definitive conclusions.

This second issue explains why it is essential to work with time-symmetric electrodynamics and not with the usual field theory. The latter permits a particle to radiate any combination of advanced and retarded fields and leaves ambiguous

the sign of the radiative reaction. In time-symmetric electrodynamics, on the other hand, the sign of the radiative reaction is specified as soon as a self-consistent solution is obtained. For example, if one seeks for a self-consistent retarded solution the radiative reaction damps the motion of the particle. Oppositely, in an advanced solution the radiative reaction accelerates the motion while in an entirely symmetric situation there is no radiative reaction.

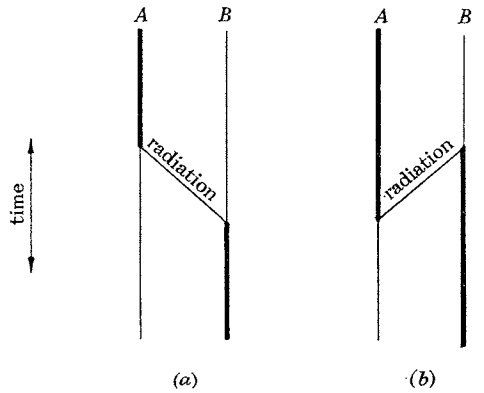


FIGURE 1

These considerations are related to a query raised by Gold (1962) and illustrated in figure 1. Energy passes between a system A and a system B. The heavy part of the world line of A is the part in which A possesses the energy, and similarly for B. The question is why the situation of figure 1 (a) is observed, but not of figure 1 (b). The sense in which time is taken to advance is irrelevant since both parts of the figure are invariant against rotation through 180°. If, for definiteness, we consider time to increase in the upward sense then the radiation is 'retarded' in both (a) and (b). (a) has the radiative reaction of the same sign as that given by time-symmetric electrodynamics. Case (b) cannot be consistent in the time-symmetric theory, and is therefore ruled out if one adopts this theory.

1. TIME-SYMMETRIC ELECTRODYNAMICS IN RIEMANNIAN SPACE-TIME

(a) *Electromagnetic action in flat space-time*

It will be convenient to begin with a brief review of the time-symmetric theory of action at a distance in flat space-time. Accordingly, we consider space-time to be given by co-ordinates x^i and by the line-element

$$\mathrm{d}s^2 = \eta_{ik}\,\mathrm{d}x^i\,\mathrm{d}x^k, \tag{1}$$

where $\eta_{ik} = \mathrm{diag}\,(-1,-1,-1,+1)$. The charges are labelled by letters $a, b, c, \ldots$. The ath particle has co-ordinates a^i, mass m_a, charge e_a and proper time a given by

$$\mathrm{d}a^2 = \eta_{ik}\,\mathrm{d}a^i\,\mathrm{d}a^k. \tag{2}$$

We have chosen the velocity of light to be unity so that time units are the same as space units. The Schwarzschild–Tetrode–Fokker action function is then defined by

$$J = -\sum_a m_a \int \mathrm{d}a - \sum_{a\neq b}\sum \tfrac{1}{2} e_a e_b \iint \delta(ab_i\, ab^i)\,\eta_{lm}\,\mathrm{d}a^l\,\mathrm{d}b^m, \tag{3}$$

where
$$(ab_i\, ab^i) = \eta_{ik}(a^i - b^i)\,(a^k - b^k). \tag{4}$$

The equations of motion can be obtained from (3) by requiring that J be stationary with respect to variations of the world lines of particles. If we define the 4-potential of b at a point X by the function

$$A_m^{(b)}(x) = \int e_b\, \delta(xb_i\, xb^i)\, \eta_{mk}\, \mathrm{d}b^k \tag{5}$$

the equations of motion take the form

$$m_a \frac{\mathrm{d}^2 a^k}{\mathrm{d}a^2} = e_a \sum_{b \neq a} F^{k(b)}{}_l \frac{\mathrm{d}a^l}{\mathrm{d}a}, \tag{6}$$

where
$$F_{kl}^{(b)} = \partial A_l^{(b)}(x)/\partial x^k - \partial A_k^{(b)}(x)/\partial x^l \tag{7}$$

represents the 'field' of charge b at point X.

Making use of Dirac's identity

$$\eta^{ik} \frac{\partial^2}{\partial x^i\, \partial x^k}\, \delta(xb_l\, xb^l) = -4\pi\delta(x^1 - b^1)\, \delta(x^2 - b^2)\, \delta(x^3 - b^3)\, \delta(x^4 - b^4) \tag{8}$$

we get
$$\eta^{ik} \frac{\partial^2}{\partial x^i\, \partial x^k} A_l^{(b)}(x) = -4\pi j_l^{(b)}(x), \tag{9}$$

where the charge-current vector $j_l^{(b)}(x)$ is given by

$$j_l^{(b)}(x) = e_b \int_{-\infty}^{\infty} \delta(x^1 - b^1)\, \delta(x^2 - b^2)\, \delta(x^3 - b^3)\, \delta(x^4 - b^4)\, \eta_{lm} \frac{\mathrm{d}b^m}{\mathrm{d}b}\, \mathrm{d}b. \tag{10}$$

Further, it can be shown that $A_m^{(b)}(x)$ satisfies the gauge condition

$$\partial A^{m(b)}(x)/\partial x^m = 0. \tag{11}$$

Therefore (9) gives
$$\partial F^{m(b)}{}_n(x)/\partial x^m = -4\pi j_n^{(b)}(x). \tag{12}$$

Also, from (7) we get
$$\frac{\partial F_{kl}^{(b)}}{\partial x^m} + \frac{\partial F_{lm}^{(b)}}{\partial x^k} + \frac{\partial F_{mk}^{(b)}}{\partial x^l} = 0. \tag{13}$$

Thus, formally, all Maxwell's equations, as well as the Lorentz force equation, are derivable from the action principle.

The time-symmetric nature of the theory is emphasized when we consider the form of $A_m^{(b)}(x)$. Simple integration shows that

$$A_m^{(b)}(x) = \tfrac{1}{2} R_m^{(b)}(x) + \tfrac{1}{2} S_m^{(b)}(x), \tag{14}$$

where $R_m^{(b)}(x)$ and $S_m^{(b)}(x)$ are the retarded and advanced Lienard–Wiechert potentials. Thus

$$R_m^{(b)}(x) = \frac{e_b\, \eta_{mn}\, (\mathrm{d}b^n/\mathrm{d}b)}{\eta_{ik}(x^k - b^k)\, \mathrm{d}b^i/\mathrm{d}b}, \tag{15}$$

evaluated at that point on the world line of b where the past light cone from X intersects it. $S_m^{(b)}(x)$ is given by a similar expression but evaluated at the point where the future light cone from X intersects the world-line of b. The theory therefore treats advanced and retarded potentials in an entirely symmetric way.

(*b*) *Generalized electromagnetic action in Riemannian space-time*

We consider a general Riemannian space-time with metric tensor and a line element

$$ds^2 = g_{ik}\,dx^i\,dx^k. \tag{16}$$

As before a^i denote the co-ordinates of particle a with proper time a given by

$$da^2 = g_{ik}\,da^i\,da^k. \tag{17}$$

The usual physical convention of taking $da^2 > 0$ will be followed.

We assume that the world-lines of particles are not closed. This excludes universes of the type considered by Gödel (1949). And we shall consider a situation where, given two points A, B in space-time, there is a unique geodesic Γ_{AB} joining them. This requirement simplifies the discussion though it will be shown later not to be a necessary condition.

Consider a vector X^{i_A}at A ($i_A = 1, 2, 3, 4$). From this vector we can generate a vector field by parallel displacement along all the geodesics from A. The vector at an arbitrary point B will be denoted by $\bar{X}^{i_B}$ ($i_B = 1, 2, 3, 4$). We write

$$\bar{X}^{i_B} = \bar{g}^{i_B}_{i_A} X^{i_A}, \tag{18}$$

where the quantities $\bar{g}^{i_B}_{i_A}$ are two-point tensors—they behave as components of a contravariant vector with index i_B at B and a covariant vector with index i_A at A. Synge (1960) calls these quantities 'parallel propagators'. We do not propose to go into the detailed properties of parallel propagators. These and two-point tensors in general have been considered by Synge and others.

Given two points A, B the distance between them is defined by

$$S_{AB} = \int_{\Gamma_{AB}} ds. \tag{19}$$

For points connected by a null geodesic $S^2_{AB} = 0$, for points connected by a time-like geodesic $S^2_{AB} > 0$, while for points connected by a space-like geodesic $S^2_{AB} < 0$.

To understand the form of the action function required it is instructive to look for 'elementary solutions' of the scalar and vector wave equations

$$g^{ik}\phi_{;ik} = 0, \tag{20}$$

$$g^{ik}A^{l}_{;ik} + R^{lm}A_m = 0. \tag{21}$$

These have been studied in detail by DeWitt & Brehme (1960). Following the work of Hadamard (1923), they obtain two-point Green functions for the equations (20) and (21). The scalar and vector Green functions for two points A, B are of the form

$$\bar{G} = (1/8\pi)\,[\Delta^{\frac{1}{2}}\delta(\sigma) - v\theta(-\sigma)], \tag{22}$$

$$\bar{G}_{i_A i_B} = (1/8\pi)\,[\Delta^{\frac{1}{2}}\bar{g}_{i_A i_B}\delta(\sigma) - v_{i_A i_B}\theta(-\sigma)], \tag{23}$$

respectively, where

$$\sigma = \tfrac{1}{2}S^2_{AB} \tag{24}$$

and

$$\Delta = -\det[\sigma_{;i_A i_B}]/\det[\bar{g}_{i_A i_B}]. \tag{25}$$

The function $\theta(x)$ is the Heaviside step function

$$\theta(x) = \begin{cases} 0 & (x < 0), \\ 1 & (x > 0). \end{cases} \tag{26}$$

The above definitions are symmetric with respect to A and B.

The first term in (23) represents parallel propagation of a disturbance from A to B, or vice versa, provided A and B are connected by a null geodesic. The second term arises from the curvature of space-time, which causes a disturbance to spread into the interior of the light cone. The functions v, $v_{i_A i_B}$ are uniquely determined once A and B are fixed.

By examining the behaviour of the Green functions in the limiting case when $A \to B$ and comparing it with the behaviour of the corresponding functions in flat space-time, it can be shown that they satisfy the inhomogeneous wave equations

$$g^{m_A k_A} \bar{G}_{;m_A k_A} = -(-\bar{g})^{-\frac{1}{2}} \delta^{(4)}_{(A,B)}, \tag{27}$$

$$g^{m_A k_A} \bar{G}_{i_A i_B; m_A k_A} + R^{m_A}_{i_A} \bar{G}_{m_A i_B} = -(-\bar{g})^{-\frac{1}{2}} \bar{g}_{i_A i_B} \delta^{(4)}_{(A,B)}, \tag{28}$$

where

$$\bar{g} = \det [\bar{g}_{i_A i_B}] \tag{29}$$

and $\delta^{(4)}_{(A,B)}$ is the 4-dimensional delta-function given by

$$\delta^{(4)}_{(A,B)} = \delta(a^1 - b^1)\, \delta(a^2 - b^2)\, \delta(a^3 - b^3)\, \delta(a^4 - b^4). \tag{30}$$

Further, $\bar{G}$ and $\bar{G}_{i_A i_B}$ are related by

$$\bar{G}^{i_A}_{i_B;\, i_A} = -\bar{G}_{;\, i_B}. \tag{31}$$

Courant & Hilbert (1962) have also considered these 'elementary solutions' in their general discussion of hyperbolic partial differential equations.

It is now possible to generalize the action function (3) to the form

$$J = -\sum_a m_a \int \mathrm{d}a - \sum_{a \neq b} \sum \tfrac{1}{2} e_a e_b \iint 4\pi \bar{G}_{i_A i_B}\, \mathrm{d}a^{i_A}\, \mathrm{d}b^{i_B}. \tag{32}$$

In flat space-time $v_{i_A i_B} = 0$, $\bar{g}_{i_A i_B} = \eta_{i_A i_B}$, $\Delta = 1$ and $\sigma = \frac{1}{2}(ab_i\, ab^i)$. Since $\delta(\frac{1}{2}x) = 2\delta(x)$ we see that (32) reduces to (3) in flat space-time. The problem that remains is to derive the equations of motion of the particles, to define the 'field' in a manner analogous to (5), and to prove that the field satisfies Maxwell's equations for Riemannian space-time.

We define the 4-potential of b at X by

$$A^{(b)}_{m_X}(x) = \int 4\pi e_b \bar{G}_{m_X i_B}\, \mathrm{d}b^{i_B}. \tag{33}$$

Consider a variation of the world-line of a. The change in the action as a result of this is given by

$$\delta J = -m_a \delta \int \mathrm{d}a - \delta \sum_{b \neq a} e_a \int A^{(b)}_{i_A}(a) \frac{\mathrm{d}a^{i_A}}{\mathrm{d}a}\, \mathrm{d}a. \tag{34}$$

Since we are now concerned with tensors having indices at the same point, we may drop the suffix A. Integrating by parts in the usual way and taking the variations

to vanish at the end-points of any finite section of the world-line of particle a and setting $\delta J = 0$ gives the equation of motion of particle a in the form

$$m_a \left[\frac{\mathrm{d}^2 a^i}{\mathrm{d}a^2} + \Gamma^i_{kl}\frac{\mathrm{d}a^k}{\mathrm{d}a}\frac{\mathrm{d}a^l}{\mathrm{d}a}\right] = e_a \sum_{b \neq a} F^{i(b)}_k(a)\frac{\mathrm{d}a^k}{\mathrm{d}a}, \tag{35}$$

where

$$F^{(b)}_{ik}(x) = A^{(b)}_k(x)_{;i} - A^{(b)}_i(x)_{;k}. \tag{36}$$

Equation (35) is the familiar form of the Lorentz equation in curved space-time.

To derive Maxwell's equations, we first note that $F^{(b)}_{kl}$ defined by (36) satisfies the equation

$$F^{(b)}_{kl;m} + F^{(b)}_{lm;k} + F^{(b)}_{mk;l} = 0. \tag{37}$$

This is a generalization of (13).

Next, from (31) and (33) we get

$$\begin{aligned} A^{m_X(b)}_{;m_X} &= \int 4\pi e_b \bar{G}^{m_X}_{i_B;m_X}\frac{\mathrm{d}b^{i_B}}{\mathrm{d}b}\,\mathrm{d}b \\ &= \int -4\pi e_b \bar{G}_{;i_B}\frac{\mathrm{d}b^{i_B}}{\mathrm{d}b}\,\mathrm{d}b \\ &= \int -4\pi e_b \frac{\mathrm{d}\bar{G}}{\mathrm{d}b}\,\mathrm{d}b = 0, \end{aligned} \tag{38}$$

which is the gauge condition. It is relevant here that the world-line of B is open. The finiteness of the two-point scalar v at an infinite distance from the light cone defined by the point X is used in the last step of (38).

Lastly, because the function $\bar{G}_{i_A i_B}$ satisfies the inhomogeneous wave equation (28), it follows that the 4-potentials $A^{m_X(b)}$ satisfy the equation

$$g^{i_X k_X} A^{m_X(b)}_{;i_X k_X} + R^{m_X}_{l_X} A^{l_X(b)} = -4\pi j^{m_X(b)}, \tag{39}$$

where

$$j^{m_X(b)} = \int e_b \bar{g}^{m_X}_{i_B}(-\bar{g})^{-\frac{1}{2}}\delta^4_{(X,B)}\frac{\mathrm{d}b^{i_B}}{\mathrm{d}b}\,\mathrm{d}b \tag{40}$$

is the charge-current vector due to particle b. Clearly, equation (40) is a generalization of the flat-space expression given by (10).

Dropping the suffix X and summing with respect to b we obtain from (36) to (39),

$$\left.\begin{aligned} F_{ik} &= A_{k;i} - A_{i;k}, \quad F_{kl;m} + F_{lm;k} + F_{mk;l} = 0, \\ A^m_{;m} &= 0, \quad g^{ik}A^m_{;ik} + R^m_l A^l = -4\pi j^m. \end{aligned}\right\} \tag{41}$$

By differentiating the field tensor F_{ik} and using (41) we obtain

$$F^{mn}_{;m} = -4\pi j^n, \tag{42}$$

which is the generalization of (12).

Hence from a generalized action principle with J given by (32) it is possible to obtain the Maxwell and Lorentz equations of classical electromagnetism. As in the case of flat space-time, the difference between the present theory and the usual theory lies in the equal status given to advanced and retarded solutions. We still use (14) for $A^{(b)}_m(x)$ but $R^{(b)}_m(x)$, $S^{(b)}_m(x)$ have modified forms in curved space-time. Thus

$$R^{(b)}_{m_X}(x) = \frac{e_b \Delta^{\frac{1}{2}}\bar{g}_{m_X i_B}(\mathrm{d}b^{i_B}/\mathrm{d}b)}{(\sigma_{;k_B}(\mathrm{d}b^{i_B}/\mathrm{d}b))} - e_b\int_{-\infty}^{b(-)} v_{m_X i_B}\frac{\mathrm{d}b^{i_B}}{\mathrm{d}b}\,\mathrm{d}b, \tag{43}$$

where $b_{(-)}$ denotes the value of b at the point on the world-line of b where the past light cone from X meets it. $S^{(b)}_{m_X}$ is similarly defined. It will be seen that the function $v_{m_X i_B}$ gives a contribution from the whole past section of the world-line of b up to $b = b_{(-)}$. This property, not present in flat space-time, is interpreted by DeWitt & Brehme as a 'scattering' produced by the curvature of space-time.

It now remains to consider the modifications required in the above arguments in cases where there are two (or more) null geodesics joining two given points in space-time. Such a situation can arise, for example, where a dense region of matter acts as a 'lens' focusing null rays coming from one point onto another. And the lens analogy gives the clue to the solution of the problem. The action function in (32) must in such cases include the sum of Green functions over all distinct geodesics joining any two points on the world lines of charged particles. With this understanding, all the subsequent formulae remain unchanged.

Similarly, there may be no null rays connecting a given point A with the world-line of a given particle b. This can happen, for example, in the steady state universe where b may be beyond the 'observable' region of A. In such cases, there is no contribution to the action.

(c) *Conformally flat universes*

The expressions obtained above become much easier to handle in the case of a conformally flat space. Such spaces have line-elements of the form

$$ds^2 = e^{2\zeta}\eta_{ik}\,dx^i\,dx^k, \tag{44}$$

where ζ is a function of x^i. All homogeneous and isotropic cosmologies possess line elements that can be written in the form of (44). Thus starting from the well-known Robertson–Walker line element

$$ds^2 = dt^2 - S^2(t)\left[\frac{dr^2}{1-kr^2} - r^2(d\theta^2 + \sin^2\theta\,d\phi^2)\right] \quad (k = 0, \pm 1) \tag{45}$$

it was shown by Infeld & Schild (1945) that (45) can be put in the form (44) by means of the following transformations

$$\int_{t_0}^{t}\frac{dz}{S(z)} = \int_0^{\frac{1}{2}(\tau+\rho)}\frac{dx}{1+kx^2} + \int_0^{\frac{1}{2}(\tau-\rho)}\frac{dx}{1+kx^2},$$

$$\int_0^r \frac{du}{\sqrt{(1-ku^2)}} = \int_0^{\frac{1}{2}(\tau+\rho)}\frac{dx}{1+kx^2} - \int_0^{\frac{1}{2}(\tau-\rho)}\frac{dx}{1+kx^2},$$

$$ds^2 = e^{2\zeta}[d\tau^2 - d\rho^2 - \rho^2(d\theta^2 + \sin^2\theta\,d\phi^2)],$$

$$e^{2\zeta} = \frac{S^2(t)}{[1+\frac{1}{4}k(\tau+\rho)^2][1+\frac{1}{4}k(\tau-\rho)^2]}. \tag{46}$$

The quantity t_0 is arbitrary, and can be taken without loss of generality as zero.

In the case of the steady-state cosmology, $S(t) = e^{Ht}$, $k = 0$, H being a constant independent of t. We have $H\tau = 1 - e^{-Ht}$. It is convenient to redefine the zero of τ so that $H\tau = -e^{-Ht}$ and to define the zero of t as the present. Then $-\infty < \tau < -1/H$ in the past and $-1/H < \tau < 0$ in the future. We have

$$\exp\zeta(\tau) = -1/H\tau. \tag{47}$$

For the Einstein–de Sitter cosmology, $S(t) = (\frac{3}{2}Ht)^{\frac{2}{3}}$, $k = 0$ where H is the present value of Hubble's constant. Evidently,

$$\tau = (2/H)\,(\tfrac{3}{2}Ht)^{\frac{1}{2}}, \quad \exp\zeta(\tau) = (\tfrac{1}{2}H\tau)^2. \tag{48}$$

The present value of t is $\frac{2}{3}H^{-1}$ and of τ is $2H^{-1}$. The past is $0 < \tau < 2H^{-1}$ and the future is $2H^{-1} < \tau < \infty$. These formulae will be needed at a later stage.

Consider now Maxwell's equations in a conformally flat space. For a single particle b we have

$$\partial(\sqrt{(-g)}\,F^{mn(b)})/\partial x^m = -4\pi j^{n(b)}\sqrt{(-g)}. \tag{49}$$

For the metric $g_{ik} = \mathrm{e}^{2\zeta}\eta_{ik}$, $\sqrt{(-g)} = \mathrm{e}^{4\zeta}$ and $g^{ik} = \eta^{ik}\,\mathrm{e}^{-2\zeta}$ so that (49) can be written in the form

$$\partial(\eta^{ml}\eta^{nk}F^{(b)}_{lk})/\partial x^m = -4\pi j^{n(b)}\sqrt{(-g)}. \tag{50}$$

Inserting for the current vector from (40) we get

$$\frac{\partial}{\partial x^m}(\eta^{ml}\eta^{nk}F^{(b)}_{lk}) = -4\pi e_b\int_{-\infty}^{\infty}\delta(x^1-b^1)\,\delta(x^2-b^2)\,\delta(x^3-b^3)\,\delta(x^4-b^4)\frac{\mathrm{d}b^n}{\mathrm{d}b}\,\mathrm{d}b. \tag{51}$$

Comparing (51) with (12) and (10) we find that the solution $F^{(b)}_{lk}$ of (51) is formally the same as the solution of (12) and (10) in a flat space. We can therefore use (14) and (15) to determine $F^{(b)}_{lk}$ in a conformally flat space. It must be remembered, however, that the vector A^k determined by (14) and (15) is not identical with the vector given by (43). This apparent paradox is resolved by the fact that although the A^k determined by (14) and (15) give the correct field tensor F_{lk} for conformally flat space—i.e. a field identical to that given by (43)—it does not have the right gauge. If (14) and (15) are used for A^k in conformal space then $A^k_{\ ;k}$ is not zero, as it is if (43) is used.

This property makes the case of a conformally flat space a particularly simple one. The formulae available in flat space-time can be used to determine the electromagnetic fields. In a general Riemannian metric, on the other hand, the formulae given above are of considerable complexity, owing to the second term in (43). It is fortunate for our purpose that we have only to deal with conformally flat spaces, as we shall do in the remainder of the paper. Since we shall only be concerned with determining the field F_{lk} it is of no consequence that we shall be making use of flat-space formulae that do not satisfy $A^k_{\ ;k} = 0$.

2. Absorber theory of radiation in an expanding universe

(*a*) *Self-consistent solutions*

We shall follow the treatment of Wheeler & Feynman by considering the effect of a particle a experiencing a prescribed acceleration. In accordance with the time-symmetry of the theory the field radiated by the particle can be written as $\frac{1}{2}(F^{(a)}_{\text{ret.}} + F^{(a)}_{\text{adv.}})$ where $F^{(a)}_{\text{ret.}}$ is the retarded field of the usual theory, and $F^{(a)}_{\text{adv.}}$ is the advanced field. (It is not necessary for the present purpose to specify a particular component of the field, so component indices are omitted here.) The field from a will produce accelerations of other particles, which in turn will radiate advanced and retarded fields, and the total field can be written as

$$\tfrac{1}{2}(F^{(a)}_{\text{ret.}} + F^{(a)}_{\text{adv.}}) + \tfrac{1}{2}\sum_{b\neq a}(F^{(b)}_{\text{ret.}} + F^{(b)}_{\text{adv.}}), \tag{52}$$

the summation with respect to b being the 'response' of the rest of the universe to the prescribed acceleration of a. While there is nothing to distinguish any arrow of time so far as the field of a is concerned, it is possible for an arrow to emerge from the response which depends on the motions or distribution of the particles b. This is the case in an expanding universe where the expansion introduces an asymmetry between the 'past' and the 'future'.

The important question is whether (52) can yield the fully retarded field of the usual theory. To decide whether this is possible Wheeler & Feynman begin with the assumption that such a result is indeed consistent with the equations. On this assumption, the acceleration of b can be calculated as in the usual theory. With the acceleration of b known, the field $\frac{1}{2}(F^{(b)}_{\text{ret.}} + F^{(b)}_{\text{adv.}})$ is determined by the time symmetric theory. Finally, if (52) yields the full retarded field assumed initially, then a full retarded field is a consistent solution of the time-symmetric theory. While such a demonstration does not rule out other consistent solutions, it is a significant issue in any cosmology that the fully retarded field of experience should be permitted.

It is implicit in the above discussion that we distinguish between the *total* acceleration of b and the part of the acceleration which arises from the field of a. In practice charges may be affected by non-electromagnetic forces as well and in general b will possess an additional acceleration. The field arising from such an additional acceleration is not included in (52) when making the check for self-consistency. This separation is based on the linearity of the equations. Now, although Maxwell's equations are linear in electromagnetic quantities, they are not linear in the metric tensor. Hence the above separation is not strictly valid unless the metric tensor is the same for all solutions in question. We expect the changes in the metric due to changes in the electromagnetic fields to be small—of the order of the ratio of gravitational to electrical forces. This ratio is $\sim 10^{-40}$ in the case of an electron.

Wheeler & Feynman (1945) show that the problem of consistency, for example, for the case of a totally retarded field, can be considered either from a general point of view or as a matter of explicit calculation. The general view makes use of the result that a solution of the homogeneous wave equation that tends to zero at infinity faster than $1/r$ at all times must be zero everywhere. A similar situation holds in the present case and it may be useful to outline the argument. We consider a charge a that remains close to $\rho = 0$, the charge being oscillated for a finite time about the line $\rho = 0$. Suppose it can be shown that the normal retarded field—the outgoing wave—is attenuated, say by electrons in space, according to an exponential factor that $\to 0$ as ρ increases. This is *in addition* to the $1/\rho$ dependence for the vacuum. Then the normal retarded field can be considered as effectively zero everywhere except inside a tube of finite radius containing the line $\rho = 0$.

Now consider the total field given by (52). This for consistency must be just $\Sigma F_{\text{ret.}}$. As $\rho \to \infty$ at any value of τ, the field $\frac{1}{2}[F^{(a)}_{\text{ret.}} + \sum_{b \neq a} F^{(b)}_{\text{ret.}}] = \frac{1}{2}\Sigma F_{\text{ret.}}$ tends to zero faster than $1/\rho$, from the attenuation property mentioned above. The advanced part of the total field (52), $[\frac{1}{2}F^{(a)}_{\text{adv.}} + \sum_{b \neq a} F^{(b)}_{\text{adv.}}] = \frac{1}{2}\Sigma F_{\text{adv.}}$, also tends to zero as $\rho \to \infty$ for the following reason. The only effective contribution to $\Sigma F_{\text{adv.}}$ comes from the tube of finite radius containing $\rho = 0$. The contribution comes in fact from the

points on the world-lines of particles that lie on the future light cone of the point at $\rho \to \infty$, and such points lie in the infinite future. The acceleration of a has ceased, however, after a finite time, so that particles at finite ρ do not experience acceleration in the infinite future. This is illustrated in figure 2.

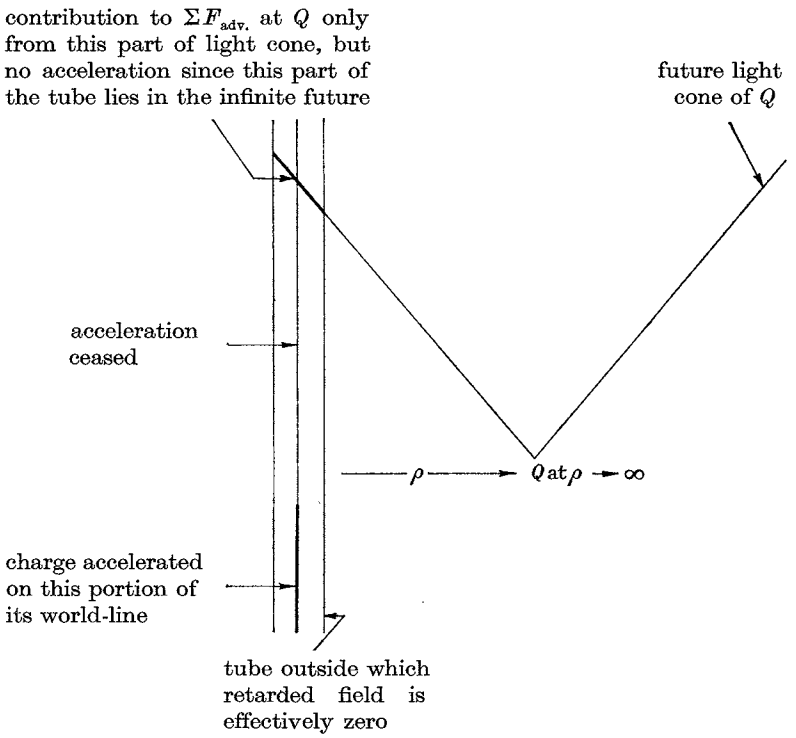


FIGURE 2

Therefore $\Sigma(F_{\text{ret.}} - F_{\text{adv.}})$ tends to zero faster than ρ^{-1} as $\rho \to \infty$. Since this field, being sourceless, satisfies the homogeneous wave equation, it must vanish everywhere, i.e.

$$F^{(a)}_{\text{ret.}} + \sum_{b \neq a} F^{(b)}_{\text{ret.}} = \sum_{b \neq a} F^{(b)}_{\text{adv.}} + F^{(a)}_{\text{adv.}} \tag{53}$$

so that (52) is just $F^{(a)}_{\text{ret.}} + \sum_{b \neq a} F^{(b)}_{\text{ret.}} = \Sigma F_{\text{ret.}}$, the fully retarded field. We see therefore that the fully retarded field is self-consistent provided it is suitably attenuated with increasing ρ. Similarly, the fully advanced field will be self-consistent if it is suitably attenuated. To proceed further it is evidently necessary to consider attenuation properties.

(*b*) *Advanced and retarded solutions and their attenuation properties*

It was seen before that the formulae giving electromagnetic fields in general Riemannian space-time become considerably simplified in conformally flat spaces. In order to simplify the explicit examples still further, we shall consider the Robertson–Walker spaces with $k = 0$. Both the steady state and Einstein–de Sitter models fall in this class and will be used to illustrate the results. The formulae in the cases $k = \pm 1$ are more cumbersome, but not different in principle.

In this section we shall work out the advanced and retarded solutions of the electromagnetic equations in the case of an oscillating charge. In view of the conformal invariance of Maxwell's equations we first consider the solutions in flat space-time.

Fourier analyze the acceleration $\mathbf{u}(t)$ of the charge and consider a single component with frequency ω_0. Writing $\mathbf{u} = \mathbf{u}_0 \exp(-i\omega_0 t)$ and choosing spherical polar co-ordinates θ, ϕ such that $\theta = 0$ corresponds to the direction of $\mathbf{u}$, gives the electric and magnetic fields several wavelengths away from the charge in the form

$$E_\theta = \frac{eu_0}{c^2 r} \sin\theta \exp[-i\omega_0(t+\epsilon r/c)] = \epsilon H_\phi, \tag{54}$$

where $\epsilon = +1$ for advanced solutions and $\epsilon = -1$ for retarded solutions.

Because of the conformal invariance of the solutions, the corresponding field in a space with line-element

$$ds^2 = \exp[2\zeta(\tau)][c^2 d\tau^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2)] \tag{55}$$

are given by

$$E_\theta = \frac{eu_0}{c^2 r} \sin\theta \exp[-i\omega_0(\tau+\epsilon r/c)] = \epsilon H_\phi, \tag{56}$$

the acceleration with respect to the (r, θ, ϕ, τ) co-ordinates being $\mathbf{u}_0 \exp(-i\omega_0\tau)$, with $\mathbf{u}_0$ again in the direction $\theta = 0$. [In the case $k = 0$, the equations (46) give $\rho = r$.]

The solutions (56) represent propagation into free space. However, in the cosmological case charges are present in space and the disturbance causes them to be set in motion. Fields arising from this motion modify the original field, the modification being represented by ascribing a refractive index $n - ik$ to the medium. In order to obtain a formula for $n - ik$ it is necessary to transform the field from the (r, θ, ϕ, τ) co-ordinates to a reference frame that is locally flat and in which the local charge distribution has no stream motion. The required transformation in the neighbourhood of a typical point P is of the form

$$\left.\begin{aligned} x &= r_P(\theta-\theta_P)\exp(\zeta_P) + o_2, \quad & y &= r_P \sin\theta_P(\phi-\phi_P)\exp(\zeta_P) + o_2, \\ z &= (r-r_P)\exp(\zeta_P) + o_2, & t &= (\tau-\tau_P)\exp(\zeta_P) + o_2. \end{aligned}\right\} \tag{57}$$

The suffix P above denotes co-ordinates of P. The quantities $r - r_P$, $\theta - \theta_P$, $\phi - \phi_P$, $\tau - \tau_P$ are small and o_2 stands for terms of second order in these quantities. The line-element in the neighbourhood of P is

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 - o_4 \tag{58}$$

provided the o_2-terms in (57) are suitably chosen. It is easy to verify that (x, y, z, t) co-ordinates have the required properties in the neighbourhood of P.

Transforming the electromagnetic tensor we find that the only non-zero components (to this order of approximation) are an electric field in the x-direction and a magnetic field in the y-direction. These can be written as $\mathscr{E}_x$, $\mathscr{H}_y$ respectively, where

$$\mathscr{E}_x = \frac{eu_0 \sin\theta_P}{c^2 r_P \exp(2\zeta_P)} \exp\{-i\omega_0[t\exp(-\zeta_P) + \tau_P + (\epsilon r_P/c)]\} = \epsilon\mathscr{H}_y. \tag{59}$$

The co-ordinate t measures proper time in the neighbourhood of P, so that an observer moving with the charge distribution at P finds the frequency of the field to be

$$\omega = \omega_0 \exp(-\zeta_P). \tag{60}$$

Retarded waves are therefore red-shifted and advanced waves blue-shifted. The Poynting vector contains the factor $(r_P \exp \zeta_P)^{-2} \times \exp(-2\zeta_P)$. The first of these factors is an inverse square effect, while the second factor is the well-known combination of energy loss by red-shift and the 'number effect'.

Turning now to the effect of the charge distribution, we note that a plane wave propagating in the z-direction in the locally flat space near P undergoes a change of phase between z and $z+\mathrm{d}z$ represented by a factor

$$\exp[-\mathrm{i}\epsilon(n-\mathrm{i}k-1)(\omega_0/c)\exp(-\zeta_P)\,\mathrm{d}z],$$

where $n-\mathrm{i}k$ is the complex refractive index determined by an observer using locally flat space and moving with the charge-distribution. Writing $\mathrm{d}r = \exp(-\zeta_P)\,\mathrm{d}z$ the phase factor is $\exp[-\mathrm{i}\epsilon(n-\mathrm{i}k-1)(\omega_0/c)\,\mathrm{d}r]$. The total modification over the range 0 to r is given by the factor

$$\eta = \exp\left[-\frac{\mathrm{i}\omega_0\epsilon}{c}\int_0^r (n-\mathrm{i}k-1)\,\mathrm{d}r\right]. \tag{61}$$

Thus the field in the (r, θ, ϕ, τ) co-ordinates is

$$E_\theta = \frac{eu_0 \sin\theta}{c^2 r}\,\eta \exp\{-\mathrm{i}\omega_0[\tau + (\epsilon r/c)]\} = \epsilon H_\phi. \tag{62}$$

The quantity η contains the real exponential factor $\exp\left[-\frac{\epsilon w_0}{c}\int_0^r k\,\mathrm{d}r\right]$. In order that this shall tend to zero as $r \to \infty$ we require*

$$\left.\begin{aligned} &\int_0^\infty k\,\mathrm{d}r \to -\infty \quad \text{for the retarded solution,} \\ &\int_0^\infty k\,\mathrm{d}r \to +\infty \quad \text{for the advanced solution.} \end{aligned}\right\} \tag{63}$$

It must be noted, however, that in certain spaces part of the range of r may not be physically accessible. For example, in the steady-state cosmology, the limits on r are $0 < r < c/H$ for a retarded solution and $0 < r < \infty$ for the advanced solution. Similarly in the Einstein–de Sitter universe $0 < r < \infty$ for the retarded solution and $0 < r < 2c/H$ for the advanced solution. The situation in the two cases may therefore be summarized as follows:

consistent solution	steady-state universe	Einstein–de Sitter universe
retarded	$\int_0^{c/H} k\,\mathrm{d}r = -\infty$	$\int_0^{\infty} k\,\mathrm{d}r = -\infty$
advanced	$\int_0^{\infty} k\,\mathrm{d}r = +\infty$	$\int_0^{2c/H} k\,\mathrm{d}r = +\infty$

* In order that the general theorem stated in § 2 (*a*) can be applied it is also necessary for $\mathrm{d}\eta/\mathrm{d}r$ to tend to zero at infinity. It is easily verified that $\mathrm{d}\eta/\mathrm{d}r \to 0$ as $r \to c/H$ in the steady-state universe.

(c) The refractive index

We consider first the consistency of retarded fields. At the upper limits of the integrals in (63) the proper frequency $\omega = \omega_0 \exp(-\zeta_P)$ tends to zero. We are therefore concerned with the form of k at low frequencies. Remembering that the problem must be considered in locally flat space we write the equation of motion of a particle of charge e mass m in the form

$$m\ddot{x} = e\mathscr{E}_x + \frac{2e^2}{3c^3}\dddot{x}. \tag{64}$$

As emphasized earlier, there is no ambiguity in the sign of the radiation damping term. A consistency of the retarded fields in time symmetric electrodynamics requires the radiation damping to be $\frac{1}{2}(F_{\text{ret.}} - F_{\text{adv.}})$. This is the formula given by Dirac (1938) and it reduces to the last term in (64) at low frequencies. The solution of (61) can be written in the form

$$x = -\frac{e\mathscr{E}_x}{m\omega^2}\left[1 - \frac{2ie^2}{3mc^3}\omega + O(\omega^2)\right]. \tag{65}$$

If N denotes the number of charged particles per unit volume, the above displacement gives the medium a polarization in the x-direction equal to Nex and hence a complex retractive index $n - ik$ given by

$$(n-ik)^2\mathscr{E}_x = \mathscr{E}_x + 4\pi Nex. \tag{66}$$

Use of (65) gives

$$1-(n-ik)^2 = \frac{4\pi Ne^2}{m\omega^2}\left[1 - \frac{2ie^2}{3mc^3}\omega + O(\omega^2)\right]. \tag{67}$$

The ambiguity in the sign of $n - ik$ is removed by requiring that the branch chosen must be the one that gives $n \to +1$ as $N \to 0$. With this physically necessary vacuum condition we obtain

$$k \to -\sqrt{\left(\frac{4\pi Ne^2}{m}\right)}\frac{1}{\omega} \quad (\omega \to 0), \tag{68}$$

in the case N independent of ω. This is the case in the steady-state cosmology in which continuous creation keeps N constant in spite of the expansion. Now, from (47), along the future light cone $r = c(\tau + 1/H)$ and $\exp(\zeta) = (1 - rH/c)^{-1}$. Therefore

$$\int_0^{c/H} k\,\mathrm{d}r = -\sqrt{\left(\frac{4\pi Ne^2}{m\omega_0^2}\right)}\int_0^{c/H}\frac{\mathrm{d}r}{1-rH/c} = -\infty. \tag{69}$$

The consistency condition for full retarded fields is therefore satisfied in the steady-state cosmology.

In the Einstein–de Sitter cosmology, and indeed in all evolutionary cosmologies, N decreases with τ. We have $N \propto \exp(-3\zeta) \propto \omega^3$. Referring to (67) we see that k tends to zero as ω^{-2}. The integral (63) in this case has the form

$$\int_0^{\infty} k\,\mathrm{d}r = -(\text{positive constant})\int_0^{\infty} \mathrm{e}^{-2\zeta}\,\mathrm{d}r. \tag{70}$$

Using (48) with $r = c(\tau - 2/H)$ on the future light cone, we get

$$-\int_0^{\infty}\mathrm{e}^{-2\zeta}\mathrm{d}r = -\int_0^{\infty}\left(1+\frac{Hr}{2c}\right)^{-4}\mathrm{d}r = -\frac{2c}{3H}. \tag{71}$$

The integral in (70) therefore does not diverge to $-\infty$. The consistency condition for retarded solutions is not therefore satisfied in the Einstein–de Sitter cosmology.

Turning now to the case of advanced solutions we have to consider the form of k when $\omega = \omega_0 \exp(-\zeta) \to \infty$. The situation now is not so clear since a classical theory must break down at $\omega > mc^3/e^2$. Indeed, quantum phenomena are important at much lower frequencies than this. However, if we suppose quantum cross-sections remain finite—as seems entirely reasonable—then a result of the form

$$1-(n-\mathrm{i}k)^2 = \frac{4\pi Ne^2}{m\omega^2}\,[1+O(1/\omega)] \tag{72}$$

will hold. That is to say

$$(n-\mathrm{i}k)^2 \to 1-\frac{4\pi Ne^2}{m\omega^2} \tag{73}$$

for sufficiently high frequency. (See, for example, Landau & Lifshitz, 1960.)

In the steady-state cosmology, the asymptotic form of k arises from the $O(1/\omega)$ term in (72) which requires k to tend to zero as ω^{-3}, i.e. as $\exp(3\zeta)$. The integral in (63) therefore becomes proportional to

$$\int_0^\infty \left(1+\frac{Hr}{c}\right)^{-3} \mathrm{d}r = \frac{c}{2H}, \tag{74}$$

i.e. it does not diverge. The advanced solutions are not consistent with the steady-state theory.

In Einstein–de Sitter cosmology, on the other hand, $N \propto \omega^3$ and (72) gives $k \propto \omega^{\frac{1}{2}}$ as $\omega \to \infty$. Accordingly

$$\int_0^{2c/H} k\,\mathrm{d}r \propto \int_0^{2c/H} \frac{\mathrm{d}r}{1-Hr/2c} = +\infty. \tag{75}$$

Full advanced solutions are therefore consistent in the Einstein–de Sitter cosmology, provided we accept the above statements on the form of the refractive index at high frequencies and provided we are correct in attributing a positive sign to k. The imaginary part of the refractive index arises from the radiative reaction, which we expect to have a sign opposite to that in the retarded case. The reason is of course that, to give a consistent advanced solution, the radiative reaction must be $\frac{1}{2}(F_{\text{adv.}} - F_{\text{ret.}})$. This is of opposite sign to the reaction $\frac{1}{2}(F_{\text{ret.}} - F_{\text{adv.}})$ in the retarded case.

It may be useful to express these results in general descriptive terms. The constancy of N in the steady-state cosmology causes the distant parts of the future light cone to behave as an enclosing 'ionosphere'. The red-shift reduces ω sufficiently for $4\pi Ne^2/m\omega^2$ to exceed unity. On the other hand, along the past light cone there is a blue-shift and conditions are essentially transparent. It is this asymmetry that permits a consistent fully retarded solution but not a consistent fully advanced solution.

In the Einstein–de Sitter cosmology, N decreases along the future light cone and increases along the past light cone. This effect overcompensates the effect of ω. Thus, along the future light cone N decreases so fast that $4\pi Ne^2/m\omega^2$ never

approaches unity and the future light cone is essentially transparent. Along the past light cone N increases fast enough to make $4\pi Ne^2/m\omega^2$ exceed unity, giving an enclosing ionosphere.

It must be emphasized, however, that although these simple considerations give insight into the results, care must be taken to ensure that the imaginary part of the refractive index takes its appropriate sign. The important point emerges that it is the imaginary part of the refractive index that plays a crucial role in deciding the nature of a consistent solution. Hence Wheeler & Feynman's use of the word 'absorber' theory of radiation to describe the situation.

(*d*) *An explicit example*

The above analysis treats the problem quite generally; it is, nevertheless, instructive to work out an explicit example. For this purpose we shall follow the method used by Wheeler & Feynman (1945) in their second derivation of the formula of radiative reaction to an oscillating charge. Wheeler & Feynman consider the case of such a charge O inside a spherical cavity of radius R. Inside the cavity the medium is of refractive index unity; outside it the refractive index is $n-ik$. In the cosmological case the situation is somewhat different. As a wave leaves the charge to travel into the future (or past), it undergoes red (blue) shift and goes through regions of varying refractive indexes and varying densities. The general solution of the wave equation under such circumstances is very complicated. For our purposes it is sufficient to consider it in a somewhat simplified form. This approach, to be described below, has the added advantage of demonstrating clearly the part played by the universe in contributing to the radiative reaction.

We shall assume that the changes in refractive index and particle densities, etc., take place in discrete steps, rather than continuously. It will be seen later that the final form of the answer can be put in a form where the steps can be made as small as we please, thus going over into the continuous case. Accordingly, we divide the medium into discrete shells, the outer radius of the lth shell being $r = R_l$. Let the refractive index be given by

$$n-ik = \begin{cases} n_0 - ik_0 = 1 & (0 < r < R_0), \\ n_l - ik_l & (R_{l-1} < r < R_l). \end{cases} \tag{76}$$

Let N_l denote the particle density in the lth shell. The cavity radius R is thus equal to R_0.

First we give expressions for the retarded field—i.e. the field that would be taken in the usual non-time-symmetric theory. As the electric field propagates outwards from O it undergoes reflexion and refraction at the boundary of each shell. The outward going wave in the empty space $0 < r < R_0$ is given by

$$H_\theta = \frac{eu_0}{c^2 r}\sin\theta \exp\left[-i\omega_0(\tau - r/c)\right], \quad H_\phi = -\frac{eu_0}{c^2 r}\sin\theta \exp\left[-i\omega_0(\tau - r/c)\right]; \tag{77}$$

where we are concerned with fields several wavelengths away from O. Since the reflexion and refraction at spherical boundaries produce components parallel to

the incident ones, we shall drop the suffixes θ, ϕ and write the electric and magnetic fields in the lth shell as

$$E(l) = E_l \sin\theta \frac{eu_0}{c^2 r} \exp(-i\omega_0 \tau) = \frac{eu_0}{c^2 r} \exp(-i\omega_0 \tau) \sin\theta \left[\frac{A_l}{n_l - ik_l} \exp[(i\omega_0 r/c)\overline{n_l - ik_l}] - \frac{B_l}{n_l - ik_l} \exp[-(i\omega_0 r/c)\overline{n_l - ik_l}] \right], \quad (78)$$

$$H(l) = H_l \sin\theta \frac{eu_0}{c^2 r} \exp(-i\omega_0 \tau) = \frac{eu_0}{c^2 r} \exp(-i\omega_0 \tau) \sin\theta [-A_l \exp[(i\omega_0 r/c)\overline{n_l - ik_l}] - B_l \exp[-(i\omega_0 r/c)\overline{n_l - ik_l}]] \quad (79)$$

where, from (77) we have

$$A_0 = 1. \quad (80)$$

The boundary conditions at $r = R_l$ require E, H to be continuous. This gives the following conditions for A_l, B_l

$$\frac{A_l}{n_l - ik_l} \exp[(i\omega_0 R_l/c)(n_l - ik_l)] - \frac{B_l}{n_l - ik_l} \exp[-(i\omega_0 R_l/c)(n_l - ik_l)] = \frac{A_{l+1}}{n_{l+1} - ik_{l+1}} \exp[(i\omega_0 R_l/c)(n_{l+1} - ik_{l+1})] - \frac{B_{l+1}}{n_{l+1} - ik_{l+1}} \exp[-(i\omega_0 R_l/c)(n_{l+1} - ik_{l+1})], \quad (81)$$

$$A_l \exp[(i\omega_0 R_l/c)(n_l - ik_l)] + B_l \exp[-(i\omega_0 R_l/c)(n_l - ik_l)] = A_{l+1} \exp[(i\omega_0 R_l/c)(n_{l+1} - ik_{l+1})] + B_l \exp[-(i\omega_0 R_l/c)(n_{l+1} - ik_{l+1})]. \quad (82)$$

Knowing A_0 from (80) and one more A or B determines all other A's and B's from the recurrence relations (81) and (82). For the usual retarded solution of the non-time-symmetric theory one puts $B_l \to 0$ as $l \to \infty$.

We investigate now whether this is a consistent solution of the time-symmetric theory. Consider the time-symmetric fields radiated by the particles lying in the lth shell with refractive index $n_l - ik_l$ and particle number density (for proper volumes) N_l. This can be written as the product of the following factors:

$E(l)$ the total electric field in the shell.

$\frac{e}{m} p_l(\omega) \exp\left[-\zeta\left(\tau_0 + \frac{r}{c}\right)\right]$ the factor which, when multiplied by the electric field gives the acceleration of the absorber particle in the (r, θ, ϕ, τ) frame.

$-\frac{e}{2c^2 r} \sin\theta$ the factor giving the half advanced field of the absorber particle (as a result of its acceleration) parallel to the source particle.

$\exp(-i\omega_0 r/c)$ phase correction factor.

$r^2 N_l \, dr \, d\Omega \exp[3\zeta(\tau_0 + r/c)]$ summation over all absorber particles lying in a volume element.

$\exp[-(i\omega_0 r)/c\{\cos\theta \cos\theta_1 + \sin\theta \sin\theta_1 \cos(\phi - \phi_1)\}]$ phase correction necessary for calculating the field at a point $(r_1, \theta_1, \phi_1, \tau_1)$ in the cavity. It is assumed that $c/\omega_0 \ll r_1 \ll R_0$.

The function $p_l(\omega)$ is related to $n_l - ik_l$ by the formula

$$1-(n_l-ik_l)^2 = \frac{4\pi N_l e^2}{m\omega^2} p_l(\omega), \tag{83}$$

where

$$\omega = \omega_0 \exp[-\zeta(\tau_0 + r/c)]. \tag{84}$$

τ_0 refers to the instant when the outgoing wave from O left O.

The angular integral can be evaluated in exactly the same way as by Wheeler & Feynman (1945) or by the method of stationary phase (Jeffreys & Jeffreys 1956). The radial integral is somewhat more complicated owing to the discontinuities at the shell boundaries. Separating the radial and angular parts, the contribution from the shell can be written in the form

$$\begin{aligned} F_l &= \int_\Omega \exp[-(i\omega_0 r_1/c)\{\cos\theta\cos\theta_1 + \sin\theta\sin\theta_1\cos(\phi-\phi_1)\}]\sin^2\theta\frac{d\Omega}{4\pi} \\ &\quad \times \int_{R_{l-1}}^{R_l} \frac{4\pi N_l e^2}{2mc^4} u_0 E_l p_l(\omega)\exp[2\zeta(\tau_0 + r/c)]\exp[-i\omega_0(\tau + r/c)]\,dr \\ &= f_l \times \frac{1}{2}\frac{eu_0}{c^2 r_1}\sin\theta_1[\exp[-i\omega_0(\tau - r_1/c)] - \exp[-i\omega_0(\tau + r_1/c)]] \end{aligned} \tag{85}$$

where

$$f_l = \int_{R_{l-1}}^{R_l} \frac{4\pi N_l e^2}{2m\omega_0^2} E_l p_l(\omega)\exp[2\zeta(\tau_0 + r/c)]\exp(-i\omega_0 r/c)(i\omega_0/c)\,dr. \tag{86}$$

Using (83) and (84) we get

$$\begin{aligned} f_l &= \int_{R_{l-1}}^{R_l} \tfrac{1}{2}E_l[1-(n_l-ik_l)^2]\exp(-i\omega_0 r/c)(i\omega_0/c)\,dr \\ &= \int_{R_{l-1}}^{R_l} \frac{A_l}{2(n_l-ik_l)}(1-n_l+ik_l)(1+n_l-ik_l)\exp[-(i\omega_0 r/c)(1-n_l+ik_l)](i\omega_0/c)\,dr \\ &\quad - \int_{R_{l-1}}^{R_l} \frac{B_l}{2(n_l-ik_l)}(1-n_l+ik_l)(1+n_l-ik_l)\exp[-(i\omega_0 r/c)(1+n_l-ik_l)](i\omega_0/c)\,dr \\ &= h_{l-1} - h_l, \end{aligned} \tag{87}$$

where, from use of (78), (81), (82), we get

$$\begin{aligned} h_l = \exp[-(i\omega_0 R_l/c)]\Bigg[&\frac{1+n_l-ik_l}{2(n_l-ik_l)} A_l \exp[(n_l-ik_l)\,i\omega_0 R_l/c] \\ &- \frac{1-n_l+ik_l}{2(n_l-ik_l)} B_l \exp[-(n_l-ik_l)\,i\omega_0 R_l/c]\Bigg]. \end{aligned} \tag{88}$$

The total contribution from the future absorber when evaluated near the source particle is therefore of the form

$$\sum_{l=0}^{\infty} F_l = \left(\sum_{l=0}^{\infty} f_l\right) \times \tfrac{1}{2}\{F^{(0)}_{\text{ret.}} - F^{(0)}_{\text{adv.}}\}. \tag{89}$$

For consistency,

$$\sum_{l=0}^{\infty} f_l = h_0 - \lim_{l\to\infty} h_l = 1 - \lim_{l\to\infty} h_l = 1,$$

i.e.

$$\lim_{l\to\infty} h_l = 0, \tag{90}$$

since from (80), $h_0 = A_0 = 1$.

We have already remarked that for the retarded field whose consistency we seek to establish there is no incoming wave at infinity in the future absorber. Therefore the second term in h_l goes to zero as $l \to \infty$. In the limit, therefore, as $l \to \infty$ we have, from (82)

$$A_{l+1} \exp\left[(n_{l+1} - \mathrm{i}k_{l+1})\, \mathrm{i}\omega_0 R_{l+1}/c\right] \sim A_l \exp\left[(n_l - \mathrm{i}k_l)\, \mathrm{i}\omega_0 R_l/c\right] \lambda_l, \tag{91}$$

where

$$\lambda_l = \exp\left[(\mathrm{i}\omega_0/c)\,(n_{l+1} - \mathrm{i}k_{l+1})\,(R_{l+1} - R_l)\right]. \tag{92}$$

The condition (90) is therefore equivalent to

$$\exp\left(\sum_{l=0}^{\infty} \frac{\mathrm{i}\omega_0}{c}\,(n_{l+1} - \mathrm{i}k_{l+1})\,(R_{l+1} - R_l)\right) = 0. \tag{93}$$

We can now replace the discrete changes of refractive index and particle density by continuous ones. The sum in (93) is replaced by an integral. Since k is negative in a universe with purely retarded solutions, the condition (93) is equivalent to

$$\int_0^{\infty} k\, \mathrm{d}r = -\infty. \tag{94}$$

This is precisely the condition (63) obtained from the general considerations before. Provided this is satisfied we have a consistent solution in the time-symmetric theory, with purely retarded solutions. The condition for purely advanced fields can be obtained by slight variations in the above argument.

3. The arrow of time in cosmology

(*a*) *Evolutionary cosmologies*

The above considerations show that the arrow of time cannot be explained in terms of the Einstein–de Sitter cosmology. Nor we believe can it be explained in other evolutionary cosmologies. The reason is that in the case $k = -1$ and the case $k = +1$ with a cosmological constant leading to unlimited expansion, the particle density N tends to zero as ω^3 in the future. In oscillating cosmologies with $k = +1$ we see no possibility of avoiding a mixture of advanced and retarded fields. These views should of course be checked by direct calculation, but we shall not do this in the present paper.

It can be argued that evolutionary cosmologies leave one no worse off than before —time's arrow does not lie in cosmology and is to be sought elsewhere. This is not our opinion, however. We have been much impressed by the astonishing result that the time symmetric electrodynamics *can* be made to work, in spite of the fact that its symmetric starting point—the action principle—seems at first sight quite hopeless as a basis for understanding the asymmetric world that we actually observe. Such a remarkable success is not to be expected from a 'wrong' theory. We feel that the structural ideas of the theory must be basically 'right', and that whatever further stages of sophistication are introduced (e.g. quantization) these ideas will survive. The implication we believe is that evolutionary cosmologies, at any rate those based on the Robertson–Walker line element, must be discarded. Our opinion is that these considerations carry more weight than the rather uncertain observational attempts that have been made to distinguish between different cosmologies.

(b) Steady-state cosmology

We wish to end the paper by enquiring whether the origin of the arrow of time in steady-state cosmology can be traced any further. This cannot be done if one argues from the perfect cosmological principle of Bondi & Gold (1948). The principle requires the universal proper density to remain constant and this leads, as shown above in § 2, to the usual sense of the arrow of time in electrodynamics. The argument cannot be taken beyond this. But in C-field cosmology an interesting further insight can be obtained. The equations of C-field cosmology have been given by Hoyle & Narlikar (1962). However, for consistency with the sign of the electromagnetic action used by Wheeler & Feynman, the sign of the non-electromagnetic part of the action must be reversed from that given by Hoyle & Narlikar, namely to

$$-\frac{1}{16\pi G}\int R\sqrt{(-g)}\,\mathrm{d}^4x-\Sigma m\int \mathrm{d}s-\tfrac{1}{2}f\int C_iC^i\sqrt{(-g)}\,\mathrm{d}^4x+\Sigma m\int C_i\frac{\mathrm{d}x^i}{\mathrm{d}s}\,\mathrm{d}s, \quad (95)$$

where $C_i=\partial C/\partial x^i$ and f is a positive coupling constant. As creation (or destruction) of matter is admitted, the cases of world lines beginning (or ending) in a given volume must be taken into account. For this reason the contribution from the last term in (95) need not be zero for a variation of C.

Variation of g_{ik} gives

$$R^{ik}-\tfrac{1}{2}g^{ik}R=-8\pi G[T^{ik}-f(C^iC^k-\tfrac{1}{2}g^{ik}C^lC_l)], \quad (96)$$

while the variation of C gives

$$C^i_{;i}=\frac{1}{f}j^i_{;i}, \quad j^i=\rho\frac{\mathrm{d}x^i}{\mathrm{d}s}, \quad (97)$$

where passage to a continuum of proper density ρ and stream velocity $v^i=\mathrm{d}x^i/\mathrm{d}s$ has been made.

In the homogeneous isotropic case C is a function only of the time co-ordinate. The case $k=0$ gives

$$3\frac{\dot{S}^2}{S^2}=8\pi G(\rho-\tfrac{1}{2}f\dot{C}^2), \quad (98)$$

$$2\frac{\ddot{S}}{S}+\frac{\dot{S}^2}{S^2}=4\pi Gf\dot{C}^2, \quad (99)$$

$$\ddot{C}+3\frac{\dot{S}}{S}\dot{C}=\frac{1}{f}\left(\dot{\rho}+3\frac{\dot{S}}{S}\rho\right). \quad (100)$$

Equation (100) integrates to

$$\dot{C}=\frac{\rho}{f}+\frac{A}{S^3}, \quad A \text{ a constant.} \quad (101)$$

Although (101) is not symmetric when t is changed to $-t$, this does not necessarily imply a time asymmetry. Equations (98), (99), (100) are unchanged if $-C$ is written for C and $-t$ for t. In fact the *symmetry* of the equations is shown by eliminating C. Defining

$$3H^2=4\pi Gf, \quad T=3Ht,$$

$$\xi=S^3, \qquad u=\left(\frac{\mathrm{d}\xi}{\mathrm{d}T}\right)^2+2A\xi, \quad (102)$$

the elimination yields
$$\frac{1}{4}\left(\frac{du}{d\xi}\right)^2 = \xi\frac{du}{d\xi} - u. \tag{103}$$

The quantity u is evidently unaffected by writing $-T$ for T. Equation (103) controls the expansion of the universe.

The zero of T can be chosen so that $\xi(T=0)=1$. Assigning $u(T=0)$ allows (103) to be solved for $du/d\xi$. One could then follow either branch by numerical integration. Evidently there is a single parameter family of solutions. This is given by

$$u = 2\lambda\xi - \lambda^2 \tag{104}$$

where λ is the parameter. Insertion of (104) in the definition of u gives a differential equation for ξ, which can be integrated to yield

$$2\xi = \frac{\lambda^2}{\lambda - A} + (\lambda - A)(T - T_0)^2. \tag{105}$$

Substitution in (99) shows that $\dot{C}$ is different from zero if $\lambda \neq 0$. The case $\lambda = 0$ is just the Einstein–de Sitter cosmology. The cases $\lambda \neq 0$ are analogous to the Einstein–de Sitter cosmology, except for the important difference that (105) has no singularity at $T = T_0$. Furthermore, $C^i_{;i} = 0$ and there is no creation (or destruction) of matter. It is an interesting point that even without creation of matter, the C-field can be used to prevent singularities developing in space-time.

To obtain the steady-state cosmology one must make the postulate that $C^i_{;i} \neq 0$ occurs. This rules out the class of solutions given by (104). There remains a singular solution of (103),
$$u = \xi^2. \tag{106}$$

This leads to the solution
$$\xi = A[1 + \cosh(T - T_0)]. \tag{107}$$

Without loss of generality the zero of T can be chosen so that

$$\xi = A[1 + \cosh T], \tag{108}$$

a time symmetric result. The question therefore arises of how an arrow of time can arise. The apparent paradox is resolved in the following interesting and remarkable way.

We first note that for the solution given by (108), $\dot{C} = 1$ and

$$\rho = f\cosh T/(1 + \cosh T). \tag{109}$$

As $T \to \pm\infty$, $\rho \to f = 3H^2/4\pi G$, while at $T = 0$, ρ takes a minimum value of $\frac{1}{2}f = 3H^2/8\pi G$. The system tends to a steady-state solution either at $T \to -\infty$ or $T \to +\infty$. Now differentiation of (108) shows that $\dot{S}$ is proportional to $\sinh T$, so that the universe contracts for $T < 0$ and expands for $T > 0$. The steady state at large negative T is maintained through a destruction of matter, while that at large positive T is maintained through creation of matter. However, the words expansion/contraction and creation/destruction can obviously be interchanged by writing $-T$ for T. What cannot be changed is that there must be expansion plus creation at one end of the time axis and contraction plus destruction at the other end. The

electromagnetic discussion of § 2 is switched at the contraction end, because red shifts and blue shifts must be interchanged. The full advanced field becomes the consistent solution. Hence electromagnetic signals propagate outward along the time axis, in the sense directed away from $T = 0$. This is the case both at large positive T and large negative T.

Our point of view is that the thermodynamic arrow of time follows the sense of propagation of electromagnetic signals, and therefore that the thermodynamic arrow is always directed in a sense away from $T = 0$. The latter is the moment at which the universe is momentarily at rest, and also the moment at which the proper density takes its least value $3H^2/8\pi G$. It is accordingly a physically well defined point of division of the time axis. In the neighbourhood of $T = 0$ we expect a mixture of advanced and retarded fields. However, the range over which there is effective propagation of signals along both branches of the light cone is confined to the range in which $|T|$ is of order unity. For $|T| \gg 1$, $\cosh T$ can be approximated as $\frac{1}{2}\exp|T|$, and the steady state may be considered to have been reached. Since a point placed at random will in general have $|T| \gg 1$, a random observer will be in one or other of the two steady-state situations. And if the observer decides his arrow of time from the sense of propagation of electromagnetic signals, or from the usual thermodynamic criteria, he will *in either case* choose as his positive direction of T the sense in which the universe must be described as expanding.

We end with a few general remarks. The action (95) treats the C-terms as a field. For consistency with the action-at-a-distance formulation of time-symmetric electrodynamics, the field term $\int C_i C^i \sqrt{(-g)}\,\mathrm{d}^4x$ should be replaced by a particle–particle interaction.

The chief weakness of C-field cosmology is that it treats matter simply as a fluid. It gives no information as to what type of particle is 'created'. Astrophysics requires created material to be largely hydrogen, although the hydrogen could arise indirectly—e.g. through the decay of neutrons. It also seems that on empirical grounds an equal creation of protons and antiprotons can be ruled out. The asymmetry between matter and anti-matter has analogy to time asymmetry. An apparent asymmetry can arise from an actual symmetry in the following way. Set the positive sense of the T-axis in accordance with the observed sense of electromagnetic propagation, and understand by 'matter' the stuff we find around us. Then an observer on the opposite side of $T = 0$ who used similar criteria would have 'anti-matter' created. The composition of the universe would be symmetric about $T = 0$, with matter on one side and 'anti-matter' on the other side.

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