

Rotating metrics admitting non-perfect fluids in General Relativity

Ng. Ibohal

Department of Mathematics, University of Manipur

Imphal-795003, Manipur, India

E-mail: (i) ngibohal@rediffmail.com

(ii) ngibohal@iucaa.ernet.in

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Abstract

In this paper an application of Newman-Janis algorithm in spherical symmetric metrics with the functions $M(u, r)$ and $e(u, r)$ has been discussed. After the transformation of the metric via this algorithm, these two functions $M(u, r)$ and $e(u, r)$ will be of the three variables u, r, θ . With these functions of three variables, all the Newman-Penrose (NP) spin coefficients, the Ricci as well as the Weyl scalars have been calculated from the Cartan's structure equations. From these NP quantities, a class of rotating solutions of Einstein's field equations can be obtained. These solutions include (a) Kerr-Newman, (b) rotating Vaidya solution, (c) rotating Vaidya-Bonnor solution, (d) rotating Husain's solution, (e) rotating Wang-Wu solutions. It is found that the technique developed by Wang and Wu can be used to generate new embedded solutions, that the Kerr-Newman solution can be combined smoothly with the rotating Vaidya solution to generate Kerr-Newman-Vaidya solution, and similarly, Kerr-Newman-Vaidya-Bonnor solution of the field equations. It has also shown that the embedded universes like Kerr-Newman de Sitter, rotating Vaidya-Bonnor-de Sitter, Kerr-Newman-Vaidya-de Sitter can be derived from the general solutions with Wang-Wu function. All rotating embedded solutions derived here can be written in Kerr-Schild forms, showing the extension of Xanthopoulos's theorem. It is also found that all the rotating solutions admit non-perfect fluids.

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1 Introduction

In an earlier paper [1] it is shown that Hawking's radiation [2] could be expressed in classical spacetime metrics, by considering the charge e to be function of the radial coordinate r of Reissner-Nordstrom as well as Kerr-Newman black holes. Since these two black holes describe the 'stationary' metrics, it has intended to search for 'non-stationary' rotating metrics in order to incorporate relativistic aspect of Hawking's radiation in general relativity. The non-rotating Vaidya metric is a non-stationary generalization of Schwarzschild vacuum solution, describing the gravitational field of a null radiating star. Many attempts have been made to generate non-stationary rotating metrics which describe the rotating external gravitational field of radiating bodies. To mention with, Vaidya and Patel [3] obtained a non-stationary rotating metric with mass $M = -m\{1 + u(b/a^2) - (b^2/4a^2) \cos^2 \theta\}^{3/2}$ having minus sign, where m , a , b are constant and u is the retarded time coordinate. They claimed that their metric recovers the Kerr metric when $b = 0$. However, it is well known that the Kerr metric has mass without negative sign. Carmeli and Kaye [4] have shown, by considering the mass M of Kerr metric directly as function of coordinate u , that the Kerr metric can be made a non-stationary rotating metric, called variable mass Kerr metric. However, Herrera and Martinez [5], Herrera et. al. [6] have discussed the interpretation of variable mass Kerr metric of Carmeli and Kaye. Gonzalez et.al.[7] have presented a non-stationary generalization of the Kerr-Newman metric, by allowing the three parameters a , m and e to be functions of coordinate u and shown that the variable $a(u)$ does not represent the rotating electromagnetic field of the Einstein-Maxwell equations and concluded that to take the parameters of a metric as functions of u does not generalize the solutions enough. Mallett [8] applied Newman-Janis algorithm to the Reissner-Nordstrom-de Sitter 'seed' solution to derive a Kerr-Newman-de Sitter solution and afterward he considered the mass and the charged to be the functions of the retarded time coordinate u in order to get non-stationary charged radiating metric. Xu [9] has discussed the nature of the field equations of Mallett's solution. Jing and Wang [10] have considered the mass $M(u)$ and the charge $e(u)$ unchanged after the application of Newman-Janis algorithm to the non-rotating Vaidya-Bonnor 'seed' solution with mass $M(u)$ and charge $e(u)$. In fact, after the application of the algorithm, the mass $M(u)$ and charge $e(u)$ should be functions $M(u, \theta)$, $e(u, \theta)$ of two variables u and θ .

Here we employ the Newman-Janis algorithm [11] to generate rotating non-stationary metrics from the spherically symmetric 'seed' metric with the functions $M(u, r)$ and $e(u, r)$, where u and r are the coordinates of the spacetime geometry. Newman-Janis algorithm [11] is a *complex coordinate transformation*, which has been introduced by Newman and Janis to obtain Kerr metric, a rotating Schwarzschild *vacuum* solu-

tion of Einstein's field equations from the non-rotating Schwarzschild 'seed' solution. In another paper Newman et.al. [12] again applied the same transformation to the non-rotating charged Reissner-Nordstrom solution to get rotating charged Reissner-Nordstrom solution, which is now commonly known as Kerr-Newman black hole solution in General Relativity. So this complex coordinate transformation can be used to derive rotating solutions from the non-rotating 'seed' solutions of Einstein's equations of spherical symmetric metrics. Herrera and Jimenez [13] applied the same Newman-Janis algorithm to an interior non-rotating spherically symmetric seed metric and the resulting rotating interior was tried to match with the exterior Kerr metric on the boundary of the source. Drake and Turolla [14] have generated a class of metrics as possible sources for the Kerr metric by applying the same algorithm to any static spherically symmetric 'seed' metric. Drake and Szekeres [15] have shown the uniqueness of this algorithm in generating the Kerr-Newman metric and proved that the only electrovac Petrov type D spacetime generated by the algorithm with a vanishing Ricci scalar Λ is the Kerr-Newman space-time. Yazadjiev [16] has also shown that Sen's rotating dilation-axiom black-hole solution [17] can be derived from the static spherically symmetric dilation black hole solution via this algorithm too.

The purposes of this paper are

1. to apply the Newman-Janis algorithm to the spherical symmetric 'seed' metric with the functions $M(u, r)$ and $e(u, r)$ of two variables u, r ,
2. to calculate all the Newman-Penrose (NP) spin coefficients, the Ricci as well as the Weyl scalars [18] in general,
3. to give examples of rotating solutions, published and unpublished, of Einstein's field equations from these NP quantities.

The spherically symmetric metric with the functions of two variables u, r has been transformed via Newman-Janis algorithm [11] to get rotating metrics. After the transformation, M and e will be functions of three variables u, r, θ . Then we calculate all the Newman-Penrose (NP) spin coefficients, the Ricci as well as the Weyl scalars in general. Accordingly, the Einstein's tensors as well as the energy momentum tensors (EMT) of the matter fields have been presented in terms of complex null tetrad vectors. From this EMT one observes the description of having two fluids system in the field equations. To visualize the two fluid system we rewrite the Einstein tensors in terms of one unit time-like and three unit space-like vectors constructed from the complex null vectors.

Thus, we can generate rotating solutions mentioned in the abstract above from the Ricci as well as the Weyl scalars of the transformed metric. Consequently, some

of the results are cited for ready reference in the form of theorems based on rotating solutions discussed here.

Theorem 1 *If g_{ab}^{KN} is the Kerr-Newman solution of Einstein's field equations and ℓ_a is geodesic, shear free, rotating and expanding null vector and one of the double repeated principal null directions of the Weyl tensor of g_{ab}^{KN} , then $g_{ab}^{\text{KNV}} = g_{ab}^{\text{KN}} + 2Q(u, r, \theta) \ell_a \ell_b$ will be a rotating Kerr-Newman-Vaidya solution with $Q(u, r, \theta) = -r f(u) R^{-2}$, where $f(u)$ is the mass function of rotating Vaidya solution.*

Theorem 2 *If g_{ab}^{dS} is the rotating de Sitter solution of Einstein's field equations and ℓ_a is geodesic, shear free, rotating and expanding null vector and one of the double repeated principal null directions of the Weyl tensor of g_{ab}^{dS} , then $g_{ab}^{\text{KNdS}} = g_{ab}^{\text{dS}} + 2Q(r, \theta) \ell_a \ell_b$ will be a Kerr-Newman-de Sitter solution with $Q(r, \theta) = -(r m - e^2/2) R^{-2}$, where m and e are constant and represent the mass and the charge of Kerr-Newman black hole.*

Theorem 3 *All rotating stationary spherically symmetric solutions based on Newman-Janis algorithm are Petrov type D, whose one of the repeated null vectors, ℓ_a is geodesic, shear free, expanding as well as non-zero twist.*

Theorem 4 *All rotating non-stationary spherically symmetric solutions, derivable from the application of Newman-Janis algorithm and possessing a geodesic, shear free, expanding and rotating null vector ℓ_a , are algebraically special in the Petrov classification.*

The Kerr-Schild ansatz of Theorem 1 can also be written in another form as $g_{ab}^{\text{KNV}} = g_{ab}^{\text{V}} + 2Q(r, \theta) \ell_a \ell_b$ with $Q(r, \theta) = -(r m - e^2/2) R^{-2}$, where m and e are constant and represent the mass and the charge of Kerr-Newman black hole and g_{ab}^{V} is a rotating Vaidya solution of Einstein's field equations and ℓ_a is geodesic, shear free, rotating and expanding null vector of g_{ab}^{V} . This ansatz can be interpreted as the Kerr-Newman black hole embedded into the rotating Vaidya null radiating background, describing Kerr-Newman-Vaidya black hole. Similarly, theorem 2 states that Kerr-Newman black hole can also be embedded into the de Sitter cosmological background, describing Kerr-Newman-de Sitter black hole. Its alternative form will be the ansatz $g_{ab}^{\text{KNdS}} = g_{ab}^{\text{KN}} + 2Q(r, \theta) \ell_a \ell_b$ with $Q(r, \theta) = -(\Lambda^* r^4/6) R^{-2}$, where Λ^* is the cosmological constant. The extension of the theorem 2 in non-stationary version can be stated in the case of rotating Vaidya-Bonnor-de Sitter solution. Theorems 3 and 4 follow from the stationary as well as non-stationary rotating solutions to be discussed in the next sections.

This paper is organized as follows: Section 2 presents a brief application of Newman-Janis algorithm to a spherically symmetric ‘seed’ metric with the functions $M(u, r)$ and $e(u, r)$. A general expressions of NP quantities with $M(u, r, \theta)$ and $e(u, r, \theta)$ are calculated from the Cartan’s structure equations and cited for further use in section 3. The general properties of the rotating spherically symmetric metric is discussed after observing the nature of the NP quantities. Section 4 discusses the energy conditions of the energy momentum tensor for a time like observer with its four velocity vector. Section 5 gives examples of rotating stationary as well as non-stationary solutions from the NP quantities. In section 6 we introduce the Wang-Wu function in the rotating solutions, and it is shown that the general rotating solutions with Wang-Wu function can be used to generate rotating new embedded solutions like Kerr-Newman-de Sitter, Kerr-Newman-Vaidya, Kerr-Newman-Vaidya-de Sitter, rotating Vaidya-Bonnor-de Sitter. The conclusion of the paper is cited in section 7 with suggestions and remarks of the solutions discussed in the earlier sections.

The presentation of this paper is essentially based on the Newman-Penrose (NP) spin-coefficient formalism [18]. The NP quantities are calculated from Cartan’s structure equations written in NP formalism by McIntosh and Hickman [19] in $(+, -, -, -)$ signature.

2 Newman-Janis algorithm

For application of Newman-Janis algorithm, we start with a spherical symmetric ‘seed’ metric written in the form

$$ds^2 = e^{2\phi} du^2 + 2du dr - r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (2.1)$$

where $e^{2\phi} = 1 - 2M(u, r)/r + e^2(u, r)/r^2$ and the coordinate chosen are $\{x^1, x^2, x^3, x^4\} = \{u, r, \theta, \phi\}$. The u -coordinate is related to the retarded time in flat space-time. So u -constant surfaces are null cones open to the future. The r -constant is null coordinate. The θ and ϕ are usual angle coordinates. The retarded time coordinate are used to evaluate the radiating (or outgoing) energy momentum tensor around the astronomical body [9]. Here M and e are the functions of the retarded time coordinate u and the radial coordinate r . Initially, when M, e are constant, this metric provides the Reissner-Nordstrom solution and also when both M, e are functions of u , it becomes the non-rotating Vaidya-Bonnor solution. The contravariant components of the metric (2.1) are

$$g^{ab} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -e^{2\phi}/r^2 & 0 & 0 \\ 0 & 0 & -1/r^2 & 0 \\ 0 & 0 & 0 & -1/r^2 \sin^2\theta \end{pmatrix}, \quad a, b = 1, 2, 3, 4. \quad (2.2)$$

These metric components can be expressed in terms of complex null tetrad [18]

$$\begin{aligned}\ell^a &= \delta_2^a, \\ n^a &= \delta_1^a - \frac{1}{2}\left(1 - \frac{2M}{r} + \frac{e^2}{r^2}\right)\delta_2^a,\end{aligned}\tag{2.3}$$

$$m^a = \frac{1}{\sqrt{2r}}\left(\delta_3^a + \frac{i}{\sin\theta}\delta_4^a\right)$$

$$\bar{m}^a = \frac{1}{\sqrt{2r}}\left(\delta_3^a - \frac{i}{\sin\theta}\delta_4^a\right).$$

Then the metric tensor g^{ab} of the line element (2.1) is expressed in these null tetrad vectors as

$$g^{ab} = \ell^a n^b + n^a \ell^b - m^a \bar{m}^b - \bar{m}^a m^b.\tag{2.4}$$

Here the null vectors ℓ^a and n^a are real, and m^a and \bar{m}^a are complex conjugates of each other. According to Newman and Janis [11], one can complexify the coordinate r and u by the following transformation

$$r = r' - i a \cos\theta, \quad u = u' + i a \cos\theta, \quad \theta = \theta', \quad \phi = \phi'.\tag{2.5}$$

This complexification can only be done by considering r' and u' real. Thus, from (2.5) we have the relations

$$dr = dr' + i a \sin\theta d\theta, \quad du = du' - i a \sin\theta d\theta, \quad d\theta = d\theta', \quad \text{and } d\phi = d\phi'.\tag{2.6}$$

These relations can be utilized in (2.1) to obtain the rotating metrics:

$$\begin{aligned}ds^2 &= e^{2\phi} du^2 + 2du dr + 2a \sin^2\theta(1 - e^{2\phi}) du d\phi \\ &\quad - 2a \sin^2\theta dr d\phi - R^2 d\theta^2 - \{R^2 - a^2 \sin^2\theta (e^{2\phi} - 2)\} \sin^2\theta d\phi^2,\end{aligned}\tag{2.7}$$

where $e^{2\phi} = 1 - 2rM(u, r, \theta)/R^2 + e^2(u, r, \theta)/R^2$ and $R^2 = r^2 + a^2 \cos^2\theta$. All the primes are being dropped for convenience of notation. According to Newman and Janis [11], we have also used a suitable substitution $d\theta = -i \sin\theta d\phi$. Then the covariant complex null tetrad vectors take the forms

$$\ell_a = \delta_a^1 - a \sin^2\theta \delta_a^4,$$

$$n_a = \frac{1}{2} H(u, r, \theta) \delta_a^1 + \delta_a^2 - \frac{1}{2} H(u, r, \theta) a \sin^2\theta \delta_a^4,$$

$$m_a = -\frac{1}{\sqrt{2R}} \left\{ -ia \sin \theta \delta_a^1 + R^2 \delta_a^3 + i(r^2 + a^2) \sin \theta \delta_a^4 \right\}, \quad (2.8)$$

$$\bar{m}_a = -\frac{1}{\sqrt{2\bar{R}}} \left\{ ia \sin \theta \delta_a^1 + R^2 \delta_a^3 - i(r^2 + a^2) \sin \theta \delta_a^4 \right\}.$$

where $R = r + ia \cos \theta$. The null tetrad vectors chosen here are different from those chosen in [12], but are similar to those given in Chandrasekhar [20]. Now, after the transformation (2.5), the functions M and e must be of the three variables u, r, θ , however the old ones had explicitly u and r dependence. That is,

$$H(u, r, \theta) = 1 - \frac{2rM(u, r, \theta)}{R^2} + \frac{e^2(u, r, \theta)}{R^2} + \frac{a^2 \sin^2 \theta}{R^2}. \quad (2.9)$$

Then the line element has the covariant components of the metric tensor g_{ab} :

$$g_{ab} = \begin{pmatrix} e^{2\phi} & 1 & 0 & a \sin^2 \theta (1 - e^{2\phi}) \\ 1 & 0 & 0 & -a \sin^2 \theta \\ 0 & 0 & -R^2 & 0 \\ a \sin^2 \theta (1 - e^{2\phi}) & -a \sin^2 \theta & 0 & -\{R^2 - a^2 \sin^2 \theta (e^{2\phi} - 2)\} \sin^2 \theta \end{pmatrix} \quad (2.10)$$

$a, b = 1, 2, 3, 4$. This completes the application of Newman-Janis algorithm to the spherically symmetric ‘seed’ metric (2.1). The usefulness of this transformed metric (2.7) will be discussed in the following sections.

3 NP quantities for the rotating metric

In this section we shall derive the general NP spin coefficients, the Ricci scalars and the Weyl scalars for the metric (2.7) and present the general properties of the metric after observing the conditions of these NP quantities. First, the basis one-form of the tetrad vectors (2.8) are given below:

$$\theta^1 \equiv n_a dx^a = \frac{1}{2} H du + dr - \frac{1}{2} a H \sin^2 \theta d\phi,$$

$$\theta^2 \equiv \ell_a dx^a = du - a \sin^2 \theta d\phi,$$

$$\theta^3 \equiv -\bar{m}_a dx^a = \frac{1}{\sqrt{2\bar{R}}} \left\{ ia \sin \theta du + R^2 d\theta - i(r^2 + a^2) d\phi \right\}, \quad (3.1)$$

$$\theta^4 \equiv -m_a dx^a = \frac{1}{\sqrt{2R}} \left\{ -ia \sin \theta du + R^2 d\theta + i(r^2 + a^2) d\phi \right\}.$$

The intrinsic derivative operators for the metric (2.7) take the following forms:

$$\begin{aligned}
D &\equiv \ell^a \partial_a = \partial_r, \\
\Delta &\equiv n^a \partial_a = \frac{r^2 + a^2}{R^2} \partial_u - \frac{H}{2} \partial_r + \frac{a}{R^2} \partial_\phi, \\
\delta &\equiv m^a \partial_a = \frac{1}{\sqrt{2}R} \left\{ ia \sin \theta \partial_u + \partial_\theta + \frac{i}{\sin \theta} \partial_\phi \right\}, \\
\bar{\delta} &\equiv \bar{m}^a \partial_a = \frac{1}{\sqrt{2}\bar{R}} \left\{ -ia \sin \theta \partial_u + \partial_\theta - \frac{i}{\sin \theta} \partial_\phi \right\}.
\end{aligned} \tag{3.2}$$

By taking the exterior derivative of basis one-forms (3.1), we calculate the spin coefficients from the Cartan's equations of structure written in Newman-Penrose spin coefficients [19]:

$$\begin{aligned}
\kappa &= \sigma = \lambda = \epsilon = 0, \\
\rho &= -\frac{1}{\bar{R}}, \quad \mu = -\frac{H(u, r, \theta)}{2\bar{R}}, \\
\alpha &= \frac{(2ai - R \cos \theta)}{2\sqrt{2}\bar{R}\bar{R} \sin \theta}, \quad \beta = \frac{\cot \theta}{2\sqrt{2}R}, \\
\pi &= \frac{ia \sin \theta}{\sqrt{2}\bar{R}\bar{R}}, \quad \tau = -\frac{ia \sin \theta}{\sqrt{2}R^2},
\end{aligned} \tag{3.3}$$

$$\begin{aligned}
\gamma &= \frac{1}{\sqrt{2}\bar{R}R^2} \left[(r - M - r M_{,r} + e e_{,r}) \bar{R} - \Delta^* \right], \\
\nu &= \frac{1}{\sqrt{2}\bar{R}R^2} \left[ia \sin \theta (r M_{,u} - e e_{,u}) - (r M_{,\theta} - e e_{,\theta}) \right],
\end{aligned}$$

where $\Delta^* = r^2 - 2rM(u, r, \theta) + a^2 + e^2(u, r, \theta)$ and the function $H(u, r, \theta)$ is given in (2.9). From these NP spin coefficients we conclude that the transformed metric (2.7) with the functions M and e of three variables u, r, θ possesses, in general, a geodesic ($\kappa = \epsilon = 0$), shear free ($\sigma = 0$), expanding ($\hat{\theta} \neq 0$) and rotating ($\omega^{*2} \neq 0$) null vector ℓ_a [20] where

$$\hat{\theta} \equiv -\frac{1}{2}(\rho + \bar{\rho}) = \frac{r}{R^2}, \tag{3.4}$$

$$\omega^{*2} \equiv -\frac{1}{4}(\rho - \bar{\rho})^2 = -\frac{a^2 \cos^2 \theta}{R^2 \bar{R}^2}. \quad (3.5)$$

Further we calculate the Weyl scalars:

$$\psi_0 = \psi_1 = 0,$$

$$\psi_2 = \frac{1}{\bar{R} \bar{R} R^2} \left[(-RM + e^2) + \bar{R} (rM_{,r} - ee_{,r}) + \frac{1}{6} \bar{R} \bar{R} (-2M_{,r} - rM_{,rr} + e_{,r}^2 + ee_{,rr}) \right],$$

$$\begin{aligned} \psi_3 = & \frac{-1}{2\sqrt{2}\bar{R}\bar{R}R^2} \left[4\{i a \sin \theta (rM_{,u} - ee_{,u}) - (rM_{,\theta} - ee_{,\theta})\} \right. \\ & \left. + \bar{R} \{i a \sin \theta (rM_{,u} - ee_{,u})_{,r} - (rM_{,\theta} - ee_{,\theta})_{,r}\} \right], \end{aligned} \quad (3.6)$$

$$\begin{aligned} \psi_4 = & \frac{1}{2\bar{R}\bar{R}R^2} \left[a^2 \sin^2 \theta (rM_{,u} - ee_{,u})_{,u} + 2ia \sin \theta (rM_{,u} - ee_{,u})_{,\theta} - (rM_{,\theta} - ee_{,\theta})_{,\theta} \right] \\ & - \frac{r a^2 \sin^2 \theta}{\bar{R} \bar{R} R^2 R^2} (rM_{,u} - ee_{,u}) - \frac{2 r a \sin^2 \theta - R^2 \cos \theta}{2\bar{R} \bar{R} R^2 R^2 \sin \theta} (rM_{,\theta} - ee_{,\theta}). \end{aligned}$$

The non-vanishing of the Weyl scalars ($\psi_2 \neq \psi_3 \neq \psi_4 \neq 0$) means that the metric (2.7) is an algebraically special in the Petrov classification. In the expression of ψ_2 it is found in general that there is no differential terms of $M(u, r, \theta)$ and $e(u, r, \theta)$ with respect to u and θ . This leads that for a static rotating metric with the mass $M(r)$ and the charge $e(r)$, the spacetime metric will be a Petrov type D ($\psi_2 \neq 0, \psi_3 = \psi_4 = 0$), whose one of the repeated principal null vectors is a geodesic, shear free, expanding (3.4) and rotating (3.5) vector ℓ_a .

The Ricci scalars of the metric (2.7) are obtained as follows:

$$\phi_{00} = \phi_{01} = \phi_{10} = \phi_{20} = \phi_{02} = 0,$$

$$\phi_{11} = \frac{1}{4R^2 \bar{R}^2} \left[2e^2 + 4r(rM_{,r} - ee_{,r}) + R^2(-2M_{,r} - rM_{,rr} + e_{,r}^2 + ee_{,rr}) \right],$$

$$\begin{aligned} \phi_{12} = & \frac{1}{2\sqrt{2}R^2 \bar{R}^2} \left[i a \sin \theta \{ (RM_{,u} - 2ee_{,u}) - (rM_{,r} - ee_{,r})_{,u} \bar{R} \} \right. \\ & \left. + \{ (RM_{,\theta} - 2ee_{,\theta}) - (rM_{,r} - ee_{,r})_{,\theta} \bar{R} \} \right], \end{aligned}$$

$$\begin{aligned} \phi_{21} = & -\frac{1}{2\sqrt{2}R^2 \bar{R}^2} \left[i a \sin \theta \{ (\bar{R}M_{,u} - 2ee_{,u}) - (rM_{,r} - ee_{,r})_{,u} R \} \right. \\ & \left. + \{ (\bar{R}M_{,\theta} - 2ee_{,\theta}) - (rM_{,r} - ee_{,r})_{,\theta} R \} \right], \end{aligned} \quad (3.7)$$

$$\phi_{22} = -\frac{1}{2R^2R^2} \left[2r(rM_{,u} - ee_{,u}) - \cot\theta(rM_{,\theta} - ee_{,\theta}) + a^2 \sin^2\theta(rM_{,u} - ee_{,u})_{,u} - (rM_{,\theta} - ee_{,\theta})_{,\theta} \right],$$

$$\Lambda = \frac{1}{12R^2} (2M_{,r} + rM_{,rr} - e_{,r}^2 - ee_{,rr}).$$

Here we observe that the general expressions of ϕ_{11} and Λ do not involve any differential terms with respect to u and θ , although the functions M and e of the metric (2.7) are functions of three variables u, r, θ . The vanishing of ϕ_{00} suggests the possibility that the transformed metric (2.7) does not possess a perfect fluid $T_{ab} = (\rho^* + p)u_a u_b - p g_{ab}$ as $\phi_{00} = 2\phi_{11} = \phi_{22} = -K(\rho^* + p)/4$, $\Lambda = K(3p - \rho^*)/24$ with a time-like vector $u^a = (\ell^a + n^a)/\sqrt{2}$ [21,22]. It is also worth mentioning that for a static rotating metric with $M(r)$ and $e(r)$, the Ricci scalars ϕ_{12} and ϕ_{22} will be vanished.

Then the Einstein's tensor is computed from these Ricci scalars (3.7) as follows

$$G_{ab} = -2\phi_{22}\ell_a\ell_b - 4\phi_{11}\{\ell_{(a}n_{b)} + m_{(a}\bar{m}_{b)}\} - 6\Lambda g_{ab} + 4\phi_{12}\ell_{(a}\bar{m}_{b)} + 4\phi_{21}\ell_{(a}m_{b)}, \quad (3.8)$$

where $2\ell_{(a}n_{b)} = \ell_a n_b + n_a \ell_b$. For non-rotating fields ($a = 0$), the Ricci scalars ϕ_{12} , ϕ_{21} will vanish and this Einstein's tensor will reduce to that presented by Glass and Krisch [23]. From the Einstein's equations:

$$G_{ab} \equiv R_{ab} - \frac{1}{2}Rg_{ab} = -K T_{ab} \quad (3.9)$$

we obtain the null density μ^* , the matter density ρ^* , the pressure p as well as the rotation function ω as

$$K\mu^* = 2\phi_{22}, \quad K\omega = -2\phi_{12}, \quad K\rho^* = 2\phi_{11} + 6\Lambda, \quad Kp = 2\phi_{11} - 6\Lambda \quad (3.10)$$

where the Ricci scalars ϕ_{11} , ϕ_{12} , ϕ_{22} , Λ are given in (3.7).

To have the two-rotating fluid description we shall introduce a time-like unit vector u^a and three unit space-like vectors v^a , w^a , z^a such that

$$u_a = \frac{1}{\sqrt{2}}(\ell_a + n_a),$$

$$v_a = \frac{1}{\sqrt{2}}(\ell_a - n_a),$$

$$w_a = \frac{1}{\sqrt{2}}(m_a + \bar{m}_a), \quad z_a = -\frac{i}{\sqrt{2}}(m_a - \bar{m}_a) \quad (3.11)$$

with the normalization conditions $u_a u^a = 1$, $v_a v^a = w_a w^a = z_a z^a = -1$. Then the explicit forms of these unit vectors are as follows

$$\begin{aligned} u_a &= \frac{1}{\sqrt{2}} \left\{ \left(1 + \frac{1}{2} H\right) \delta_a^1 + \delta_a^2 - \left(1 + \frac{1}{2} H\right) a \sin^2 \theta \delta_a^4 \right\}, \\ v_a &= \frac{1}{\sqrt{2}} \left\{ \left(1 - \frac{1}{2} H\right) \delta_a^1 - \delta_a^2 - \left(1 - \frac{1}{2} H\right) a \sin^2 \theta \delta_a^4 \right\}, \\ w_a &= -\frac{1}{R^2} \left\{ -a^2 \sin \theta \cos \theta \delta_a^1 + r R^2 \delta_a^3 + a (r^2 + a^2) \sin \theta \cos \theta \delta_a^4 \right\}, \\ z_a &= \frac{1}{R^2} \left\{ a r \sin \theta \delta_a^1 + a \cos \theta R^2 \delta_a^3 - r (r^2 + a^2) \sin \theta \delta_a^4 \right\}, \end{aligned} \quad (3.12)$$

The metric tensor g_{ab} can be expressed in these unit vectors

$$g_{ab} = u_a u_b - v_a v_b - w_a w_b - z_a z_b. \quad (3.13)$$

Thus the Einstein's equations are written in two-fluid system

$$\begin{aligned} G_{ab} &= -K[\mu^* \ell_a \ell_b + \rho^* (u_a u_b - v_a v_b) + p (w_a w_b + z_a z_b) \\ &\quad + (\omega + \bar{\omega}) \{u_{(a} w_b) + v_{(a} w_b)\} - i(\omega - \bar{\omega}) \{u_{(a} z_b) + v_{(a} z_b)\}], \end{aligned} \quad (3.14)$$

where μ^* , ρ^* , p and ω are related with the Ricci scalars given in (3.7) as:

$$\begin{aligned} K \mu^* &= -\frac{1}{R^2 R^2} \left\{ 2r(r M_{,u} - e e_{,u}) - \cot \theta (r M_{,\theta} - e e_{,\theta}) \right. \\ &\quad \left. + a^2 \sin^2 \theta (r M_{,u} - e e_{,u})_{,u} - (r M_{,\theta} - e e_{,\theta})_{,\theta} \right\} \\ K \rho^* &= \frac{1}{R^2 R^2} \left\{ e^2 + 2r(r M_{,r} - e e_{,r}) \right\} \\ K p &= \frac{1}{R^2 R^2} \left\{ e^2 + 2r(r M_{,r} - e e_{,r}) - R^2 (2M_{,r} + r M_{,rr} - e_{,r}^2 - e e_{,rr}) \right\}, \quad (3.15) \\ K \omega &= -\frac{1}{\sqrt{2} R^2 R^2} \left[i a \sin \theta \left\{ (R M_{,u} - 2 e e_{,u}) - (r M_{,r} - e e_{,r})_{,u} \bar{R} \right\} \right. \\ &\quad \left. + \left\{ (R M_{,\theta} - 2 e e_{,\theta}) - (r M_{,r} - e e_{,r})_{,\theta} \bar{R} \right\} \right]. \end{aligned}$$

The expression of null radiation density μ^* involves the derivative of the functions $M(u, r, \theta)$ and $e(u, r, \theta)$ with respect to u and θ . Those of ρ^* and p are with respect to

r only. However, the expression of the rotation function ω is involved the derivative of mass and the charge with respect to three variables u, r, θ . From the above equations, it is observed that the Einstein's tensor (3.14) of the rotating fluid reduce to those of Glass and Krisch [23] of *non-rotating* string fluid when $a = e = 0$ and $M = M(u, r)$. In the non-rotating Vaidya-type radiation null fluid, the null density μ^* takes the form $\mu^* = -2M_{,u}/Kr^2$. This shows that μ^* is always negative, since $\partial M/\partial u$ is positive [24].

4 Stress-energy tensor and energy conditions

From the Einstein tensor (3.8) and the relations (3.10) of Ricci scalars with μ^*, ρ^*, p, ω we can introduce the total energy momentum tensor (EMT) for a rotating fluid as follows:

$$\begin{aligned} T_{ab} &= T_{ab}^{(n)} + T_{ab}^{(m)} \\ &= \mu^* \ell_a \ell_b + 2\rho^* \ell_{(a} n_{b)} + 2p m_{(a} \bar{m}_{b)} \\ &\quad + 2\omega \ell_{(a} \bar{m}_{b)} + 2\bar{\omega} \ell_{(a} m_{b)} \end{aligned} \quad (4.1)$$

where the EMTs for the rotating null fluid as well as that of the rotating matter are given respectively below:

$$T_{ab}^{(n)} = \mu^* \ell_a \ell_b + \omega \ell_{(a} \bar{m}_{b)} + \bar{\omega} \ell_{(a} m_{b)} \quad (4.2)$$

$$T_{ab}^{(m)} = 2(\rho^* + p) \ell_{(a} n_{b)} - p g_{ab} + \omega \ell_{(a} \bar{m}_{b)} + \bar{\omega} \ell_{(a} m_{b)}, \quad (4.3)$$

where $\bar{\omega}$ is the complex conjugate of ω . When $\omega = 0$ initially, these EMTs is similar to those introduced by Husain [25] in the case of non-rotating fluid.

In General Relativity the stress-energy tensor represents the matter that describes the gravitation in the space-time geometry through Einstein's field equations. From the conditions of Ricci scalar $\phi_{00} = 0$, obtained above for the space-time geometry (2.7), it may be concluded that the stress-energy tensor given above does not, in general describe a perfect fluid, as for a perfect fluid the Ricci scalar $\phi_{00} = -K(\rho^* + p)/4$ must not vanish. Hence, it will be interesting to study the nature of the energy conditions for rotating non-perfect fluid given in (4.1). When the rotation factor ω vanishes in (4.2), this fluid may be thought of null radiation fluid of non-rotating Vaidya space-time. So we refer to this rotating ($\omega \neq 0$) null radiation fluid as rotating Vaidya type radiating fluid, shortly rotating Vaidya fluid (4.2).

As the T_{ab} does not include the perfect fluid, it seems that the stress-energy tensor can represent the interaction of rotating Vaidya fluid with rotating non-perfect fluid. Since there is a coupling term of the rotation scalar a with $\partial M/\partial u$ in the expression of ω appearing in T_{ab} , the energy condition of this T_{ab} satisfying Einstein's field equations

will be a new area to discuss in the classical General Relativity. For this purpose, we write the matter part $T_{ab}^{(m)}$ of (4.1) in terms of time-like u^a as well as space-like vectors v^a , w^a , z^a given in (3.12) as

$$T_{ab} = \mu^* \ell_a \ell_b + (\rho^* + p) (u_a u_b - v_a v_b) - p g_{ab} + (\omega + \bar{\omega}) \{u_{(a} w_{b)} + v_{(a} w_{b)}\} - i(\omega - \bar{\omega}) \{u_{(a} z_{b)} + v_{(a} z_{b)}\}, \quad (4.4)$$

and its trace is $T \equiv T_{ab} g^{ab} = 2(\rho^* - p)$, which is different from that of a perfect fluid. This trace will be vanished when $\rho^* = p$. This means that the matter part $T_{ab}^{(m)}$ of the stress-energy tensor may be that of electromagnetic field whose trace is zero. [The stress-energy tensor for a non-rotating perfect fluid is $T_{ab}^{(pf)} = (\rho^* + p)u_a u_b - p g_{ab}$ with unit time-like vector u_a and trace $T^{(pf)} = \rho^* - 3p$, which will be zero when $\rho^* = 3p$].

To study the energy conditions of the energy-momentum tensor, we consider a time-like observer with its four-velocity vector U_a [26,27]

$$U_a = \hat{\alpha} u_a + \hat{\beta} v_a + \hat{\gamma} w_a + \hat{\delta} z_a, \quad (4.5)$$

where $\hat{\alpha}$, $\hat{\beta}$, $\hat{\gamma}$ and $\hat{\delta}$ are arbitrary constants. The four-velocity vector U_a is subjected to the condition that

$$U^a U_a = \hat{\alpha}^2 - \hat{\beta}^2 - \hat{\gamma}^2 - \hat{\delta}^2 \geq 0. \quad (4.6)$$

Now, $T_{ab} U^a U^b$ will represent the energy density as measured by the time-like observer with the unit tangent vector U^a . Then the energy conditions [36] are the following:

(a) *Weak energy condition:* The energy momentum tensor obeys the inequality $T_{ab} U^a U^b \geq 0$ for any timelike vector U^a i.e., $T_{ab} U^a U^b \geq 0$ implies that

$$\begin{aligned} (i) \quad & \frac{\mu^*}{2} + \rho^* \geq 0, \quad (ii) \quad \pm \frac{\mu^*}{2} + \rho^* + p \geq 0, \quad (iii) \quad \frac{3\mu^*}{2} + \rho^* + p \geq 0, \\ (iv) \quad & \left(\frac{\mu^*}{2} + \rho^* + p\right) \pm (\omega + \bar{\omega}) \geq 0, \quad (v) \quad \left(\frac{\mu^*}{2} + \rho^* + p\right) \pm i(\omega - \bar{\omega}) \geq 0. \end{aligned} \quad (4.7)$$

(b) *Strong energy condition:* The Ricci tensor for T_{ab} (4.4) satisfies the inequality $R_{ab} U^a U^b \geq 0$ for all time-like vector U^a , i.e. $T_{ab} U^a U^b \geq \frac{1}{2}T$, which implies that

$$\begin{aligned} (i) \quad & \frac{\mu^*}{2} + p \geq 0, \quad (ii) \quad \pm \frac{\mu^*}{2} + \rho^* + p \geq 0, \quad (iii) \quad \frac{3\mu^*}{2} + \rho^* + p \geq 0, \\ (iv) \quad & \left(\frac{\mu^*}{2} + \rho^* + p\right) \pm (\omega + \bar{\omega}) \geq 0, \quad (v) \quad \left(\frac{\mu^*}{2} + \rho^* + p\right) \pm i(\omega - \bar{\omega}) \geq 0. \end{aligned} \quad (4.8)$$

(c) *Dominant energy condition:* For all future directed, time-like vector U^a , $T_{ab} U^b$ should be a future directed non-space like vector. This energy condition is equivalent to

$$(i) \quad \mu^* \rho^* + \rho^{*2} - (f^2 + g^2) \geq 0,$$

$$\begin{aligned}
(ii) \quad & -\mu^* \rho^* + \rho^{*2} - p^2 + (f^2 + g^2) \geq 0, \quad (iii) \quad 3\mu^* \rho^* + \rho^{*2} - p^2 - 3(f^2 + g^2) \geq 0, \\
(iv) \quad & \{\mu^* \rho^* + \rho^{*2} - p^2 - (f^2 + g^2)\} \pm 2f(\rho^* - p) \geq 0, \\
(v) \quad & \{\mu^* \rho^* + \rho^{*2} - p^2 - (f^2 + g^2)\} \pm 2g(\rho^* - p) \geq 0,
\end{aligned} \tag{4.9}$$

where $2f = \omega + \bar{\omega}$ and $2ig = \omega - \bar{\omega}$. These are the energy conditions satisfied by the energy-momentum tensor (4.4) in general. We find that the rotation function ω is involved in all the three energy conditions.

5 Rotating solutions recovered from the general solutions

In the above section we present the full expressions of NP spin coefficients (3.3) the Weyl scalars (3.6) and the Ricci scalars (3.7) with arbitrary functions M and e of three coordinate variables u, r, θ . These NP quantities are so transparent that these quantities can explain the nature of solutions of Einstein's field equations discussed in this paper. For example, the NP spin coefficients (3.3) easily explain that the metric (2.7) admits a null vector ℓ^a which is geodesic, shear free, expanding (3.4) as well as rotating (3.5). In this section, with the help of NP quantities given in (3.3), (3.6), (3.7), we shall give known examples of rotating solutions like Kerr-Newman, rotating Vaidya [4] and rotating Vaidya-Bonnor [10]. In the next section we shall combine these solutions with other rotating solutions in order to derive new embedded rotating solutions.

5.1 Kerr-Newman solution: $e = M = \text{constant}$, $a \neq 0$

When $e = M = \text{constant}$, $a \neq 0$, the equation (3.10) reduces to the Kerr-Newman solution

$$\begin{aligned}
\mu^* &= \omega = 0 \\
\rho^* &= p = \frac{e^2}{K R^2 R^2},
\end{aligned} \tag{5.1}$$

and the only existing Weyl scalar is

$$\psi_2 = \frac{1}{\overline{R} \overline{R} R^2} (e^2 - R M).$$

Then the total energy momentum tensor takes the form

$$T_{ab} = \rho^* (\ell_a n_b + n_a \ell_b) + p (m_a \overline{m}_b + \overline{m}_a m_b)$$

$$= (e^2/KR^2R^2)\{(\ell_a n_b + n_a \ell_b) + (m_a \bar{m}_b + \bar{m}_a m_b)\}, \quad (5.2)$$

which is the EMT for non-null electromagnetic field with Maxwell scalar

$$\phi_1 \equiv \frac{1}{2}F_{ab}(\ell^a n^b + \bar{m}^a m^b) = \frac{e}{\sqrt{(2K)\bar{R}\bar{R}}} \quad (5.3)$$

for Kerr-Newman solution. Here is the birth place of Kerr-Newman solution, originally applied the Newman-Janis algorithm by Newman *et. al.* [12] to generate this well known rotating solution from the non-rotating Reissner-Nordstrom ‘seed’ solution. The line element is

$$\begin{aligned} ds^2 = & \{1 - (2rM - e^2)R^{-2}\} du^2 + 2du dr \\ & + 2aR^{-2}(2rM - e^2)\sin^2\theta du d\phi - 2a\sin^2\theta dr d\phi \\ & - R^2 d\theta^2 - \{(r^2 + a^2)^2 - \Delta^* a^2 \sin^2\theta\} R^{-2} \sin^2\theta d\phi^2, \end{aligned} \quad (5.4)$$

where $\Delta^* = r^2 - 2rM + a^2 + e^2$. The charged Kerr-Newman black hole has an *external event horizon* at $r_+ = M + \sqrt{(M^2 - a^2 - e^2)}$ and an *internal Cauchy horizon* at $r_- = M - \sqrt{(M^2 - a^2 - e^2)}$. The *stationary limit surface* $g_{uu} > 0$ of the rotating black hole i.e. $r = r_e(\theta) = M + \sqrt{(M^2 - a^2 \cos^2\theta - e^2)}$ does not coincide with the event horizon at r_+ thereby producing the *ergosphere*. This stationary limit coincides with the event horizon at the poles $\theta = 0$ and $\theta = \pi$ [20]. Naturally, this solution includes Kerr ($e = 0$), Reissner-Nordstrom ($a = 0, e \neq 0$) as well as Schwarzschild ($a = e = 0$) solutions.

5.2 Rotating Vaidya solution: $M = M(u), a \neq 0, e = 0$

In this case the energy momentum tensor (4.2) takes

$$T_{ab} = \mu^* \ell_a \ell_b + \omega \ell_{(a} \bar{m}_{b)} + \bar{\omega} \ell_{(a} m_{b)} \quad (5.5)$$

where the null density μ^* and the rotation function ω in (3.15) become

$$\begin{aligned} K \mu^* &= -\frac{1}{R^2 R^2} \{2r^2 M_{,u} + a^2 r \sin^2\theta M_{,uu}\}, \\ K \omega &= -\frac{1}{\sqrt{2}\bar{R}R^2} i a \sin\theta M_{,u}, \end{aligned} \quad (5.6)$$

and the Weyl scalars are

$$\begin{aligned} \psi_2 &= -\frac{M}{\bar{R}\bar{R}\bar{R}}, \\ \psi_3 &= -\frac{i a \sin\theta}{2\sqrt{2}\bar{R}\bar{R}R^2} \{4r M_{,u} + \bar{R} M_{,u}\}, \end{aligned} \quad (5.7)$$

$$\psi_4 = \frac{a^2 r \sin^2 \theta}{2\bar{R} R R^2 R^2} \{R^2 M_{,uu} - 2r M_{,u}\}. \quad (5.8)$$

From the above we observe that ω , ψ_3 , ψ_4 will vanish when $a = 0$, and the EMT will be that of the original non-rotational radiating Vaidya metric [28] with $\mu^* = -2 M_{,u}/K r^2$. The line element of this rotating metric is

$$\begin{aligned} ds^2 = & \{1 - 2rM(u)R^{-2}\} du^2 + 2du dr \\ & + 4arM(u)\sin^2\theta R^{-2} du d\phi - 2a\sin^2\theta dr d\phi \\ & - R^2 d\theta^2 - \{(r^2 + a^2)^2 - \Delta^* a^2 \sin^2\theta\} R^{-2} \sin^2\theta d\phi^2, \end{aligned} \quad (5.9)$$

where $\Delta^* = r^2 - 2rM(u) + a^2$. This metric represents a non-stationary rotating solution of Einstein's equations possessing an energy-momentum tensor (5.5) for a rotating null radiating fluid, and can describe a non-stationary *rotating* black hole if $M(u) > a$. The involvement of ω in the energy-momentum tensor (5.5) indicates that the null fluid is a rotating Vaidya null fluid. When $M(u) = \text{constant}$ initially, this metric would reduce to rotating vacuum Kerr solution with vanishing μ^* and ω in (5.6).

Carmeli and Kaye [4] studied the metric (5.9) after considering the mass M of the Kerr solution as a function of coordinate u . That is why, they referred to the metric (5.9) as the variable-mass Kerr solution (see also in [29,30]) and discussed the properties of the metric using the NP quantities. Carmeli [29] referred to these ω terms as residues of the black hole. However, we refer to the metric (5.9) as rotating Vaidya solution. In the next section this metric will be combining smoothly with the usual Kerr-Newman solution as rotating Kerr-Newman-Vaidya black hole.

5.3 Rotating Vaidya-Bonnor solution: $M = M(u)$, $a \neq 0$. $e = e(u)$

In this case the energy momentum tensor takes

$$T_{ab} = \mu^* l_a l_b + 2\rho^* \{\ell_{(a} n_{b)} + m_{(a} \bar{m}_{b)}\} + 2\omega \ell_{(a} \bar{m}_{b)} + 2\bar{\omega} \ell_{(a} m_{b)} \quad (5.10)$$

where

$$\begin{aligned} \mu^* = & -\frac{1}{K R^2 R^2} \{2r(r M_{,u} - e e_{,u}) + a^2 \sin^2 \theta (r M_{,u} - e e_{,u})_{,u}\}, \\ \rho^* = p = & \frac{e^2(u)}{K R^2 R^2}, \end{aligned} \quad (5.11)$$

$$\omega = \frac{-i a \sin \theta}{\sqrt{2} K R^2 R^2} \{R M_{,u} - 2e e_{,u}\},$$

and the Weyl scalars are

$$\begin{aligned}\psi_2 &= \frac{1}{\bar{R} R R^2} (e^2 - R M) \\ \psi_3 &= \frac{-i a \sin \theta}{2 \sqrt{2} \bar{R} R R^2} \left\{ 4 (r M_{,u} - e e_{,u}) + \bar{R} M_{,u} \right\}, \\ \psi_4 &= \frac{a^2 \sin^2 \theta}{2 \bar{R} R R^2 R^2} \left\{ R^2 (r M_{,u} - e e_{,u})_{,u} - 2r (r M_{,u} - e e_{,u}) \right\}.\end{aligned}\tag{5.12}$$

The line element will be in the form

$$\begin{aligned}ds^2 &= [1 - \{2rM(u) - e^2(u)\}R^{-2}] du^2 + 2du dr \\ &\quad + 2aR^{-2}\{2rM(u) - e^2(u)\}\sin^2\theta du d\phi - 2a\sin^2\theta dr d\phi \\ &\quad - R^2 d\theta^2 - \{(r^2 + a^2)^2 - \Delta^* a^2 \sin^2\theta\} R^{-2} \sin^2\theta d\phi^2,\end{aligned}\tag{5.13}$$

where $\Delta^* = r^2 - 2rM(u) + a^2 + e^2(u)$. This solution will describe a black hole when $M(u) > a^2 + e^2(u)$ and has $r_{\pm} = M(u)^* \pm \sqrt{\{M^2(u) - a^2 - e^2(u)\}}$ as the roots of the equation $\Delta^* = 0$. So the rotating Vaidya-Bonnor solution has an *external event horizon* at $r_+ = M(u) + \sqrt{\{M^2(u) - a^2 - e^2(u)\}}$ and an *internal Cauchy horizon* at $r_- = M(u) - \sqrt{\{M^2(u) - a^2 - e^2(u)\}}$. The non-stationary limit surface $g_{uu} > 0$ of the rotating black hole *i.e.* $r \equiv r_e(u, \theta) = M(u) + \sqrt{\{M^2(u) - a^2 \cos^2\theta - e^2(u)\}}$ does not coincide with the event horizon at r_+ , thereby producing the *ergosphere*. The rotating Vaidya-Bonnor metric (5.13) can be written in Kerr-Schild form on the rotating Vaidya null radiating background as

$$g_{ab}^{\text{VB}} = g_{ab}^{\text{V}} + 2Q(u, r, \theta) \ell_a \ell_b \tag{5.14}$$

where

$$Q(u, r, \theta) = \frac{e^2(u)}{2R^2}. \tag{5.15}$$

Here, g_{ab}^{V} is the rotating Vaidya metric (5.9) and ℓ_a is geodesic, shear free, expanding and rotating null vector for both g_{ab}^{V} as well as g_{ab}^{VB} and given in (2.8). The Kerr-Schild form (5.14) may be interpreted as the existence of the electromagnetic field on the rotating Vaidya null radiating background. If we set $M(u)$ and $e(u)$ are both constant, this Kerr-Schild form may be that of Kerr-Newman black hole. That is, the Kerr-Newman solution itself has the Kerr-Schild form on the Kerr background with the same null vector ℓ_a (2.8).

From this rotating Vaidya-Bonnor metric, we can clearly recover the following solutions: (i) rotating Vaidya metric (5.9) when $e(u) = 0$, (ii) rotating charged Vaidya solution when $e(u)$ becomes constant, (iii) the Kerr-Newman solution (5.4) when

$M(u) = e(u) = \text{constant}$ and (iv) well-known non-rotating Vaidya-Bonnor metric [31] when $a = 0$. It is also noted that when $e = a = 0$, the null density of Vaidya radiating fluid takes the form $\mu^* = -2 M_{,u}/K r^2$. The non-rotating Vaidya null radiating metric is of type D in the Petrov classification of spacetime whose one of the repeated principal null vectors, ℓ_a is a geodesic, shear free, non-rotating with non-zero expansion [29], while the rotating one is of algebraically special with a null vector ℓ_a which is geodesic, shear free, rotating as well as expanding. It is also noted that when $e = a = 0$, the energy-momentum tensor becomes that of the original non-rotational null-radiating Vaidya fluid with $\mu^* = -2 M_{,u}/K r^2$. It is noted that the metric (5.13) can be seen in [10]. From (5.4) and (5.13) it is observed that the rotating Vaidya-Bonnor solution is the non-stationary version of Kerr-Newman black holes. That is, the parameters M and e of Kerr-Newman solution are functions of retarded time coordinate u in rotating Vaidya-Bonnor metric.

6 Rotating solutions with $M = M(u, r)$, $e(u, r, \theta) = 0$

In this section we shall discuss the rotating solutions by considering the function $M(u, r)$ of two variables u, r only and $e(u, r, \theta) = 0$, and derive new embedded rotating solutions. We shall express all the rotating embedded solutions in Kerr-Schild types of metrics in order to show them as solutions of Einstein's field equations. In this case the energy momentum tensor takes the form

$$T_{ab} = \mu^* \ell_a \ell_b + 2 \rho^* \ell_{(a} n_{b)} + 2 p m_{(a} \bar{m}_{b)} + 2 \omega \ell_{(a} \bar{m}_{b)} + 2 \bar{\omega} \ell_{(a} m_{b)} \quad (6.1)$$

or in terms of unit vectors

$$T_{ab} = \mu^* \ell_a \ell_b + \rho^* (u_a u_b - v_a v_b) + p (w_a w_b + z_a z_b) + (\omega + \bar{\omega}) \{u_{(a} w_{b)} + v_{(a} w_{b)}\} - i(\omega - \bar{\omega}) \{u_{(a} z_{b)} + v_{(a} z_{b)}\}, \quad (6.2)$$

where

$$\begin{aligned} \mu^* &= -\frac{1}{K R^2 R^2} \{2r^2 M_{,u} + a^2 r \sin^2 \theta M_{,uu}\}, \\ \rho^* &= \frac{1}{K R^2 R^2} M_{,r}, \\ p &= -\frac{1}{K} \left\{ \frac{2a^2 \cos^2 \theta}{R^2 R^2} M_{,r} + \frac{r}{R^2} M_{,rr} \right\}, \\ \omega &= -\frac{i a \sin \theta}{\sqrt{2} K R^2 R^2} (R M_{,u} - r \bar{R} M_{,ur}). \end{aligned} \quad (6.3)$$

The line element will take the form

$$ds^2 = \left\{ 1 - 2rM(u, r)R^{-2} \right\} du^2 + 2du dr$$

$$\begin{aligned}
& +4arM(u, r)R^{-2} \sin^2\theta du d\phi - 2a \sin^2\theta dr d\phi \\
& -R^2 d\theta^2 - \left\{ (r^2 + a^2)^2 - \Delta^* a^2 \sin^2\theta \right\} R^{-2} \sin^2\theta d\phi^2,
\end{aligned} \tag{6.4}$$

where $R^2 = r^2 + a^2 \cos^2\theta$, $\Delta^* = r^2 - 2rM(u, r) + a^2$ and the Weyl scalars are

$$\begin{aligned}
\psi_2 &= \frac{1}{\overline{R}\overline{R}R^2} \left\{ -RM + \frac{\overline{R}}{6} M_{,r}(4r + 2ia \cos\theta) - \frac{r}{6} \overline{R}\overline{R} M_{,rr} \right\}, \\
\psi_3 &= -\frac{ia \sin\theta}{2\sqrt{2}\overline{R}\overline{R}R^2} \left\{ (4r + \overline{R}) M_{,u} + r \overline{R} M_{,ur} \right\}, \\
\psi_4 &= \frac{a^2 r \sin^2\theta}{2\overline{R}\overline{R}R^2 R^2} \left\{ R^2 M_{,uu} - 2r M_{,u} \right\}.
\end{aligned} \tag{6.5}$$

One can consider this rotating metric (6.4) along with the stress-energy momentum tensor (6.1) or (6.2) and the Weyl scalars as the extension of the non-rotating solutions discussed by Glass and Krisch [23] and Husain [25].

6.1 Rotating Husain's solution: $M = M(u, r)$, $a \neq 0$

Husain [25] has imposed one condition in the equation of state of non-rotating null fluid that $p = k\rho^{*b}$ and obtain the solution of the equation of state with $k \geq 1/2$ $b = 1$. However, due the present of the rotating factor a in equation (6.3), one cannot be able to get the solution. So we put $k = 1$ and $b = 1$. Then, the equation to be solved takes a simple form

$$\frac{M_{,r}}{r} = -\frac{M_{,rr}}{2}, \tag{6.6}$$

which gives the function $M(u, r)$

$$M(u, r) = f(u) - \frac{1}{r} g(u). \tag{6.7}$$

It can be treated as rotating Husain's solution for $p = k\rho^*$ with $k = 1$. This rotating Husain's solution may be degenerated to the rotating Vaidya-Bonnor solution presented above if one puts $g(u) = e^2(u)/2$ in (6.7).

6.2 Rotating Wang-Wu solutions

Wang and Wu [32] have expanded the function $M(u, r)$ of (6.3) of the non-rotating space in the power of r

$$M(u, r) = \sum_{n=-\infty}^{+\infty} q_n(u) r^n, \tag{6.8}$$

where $q_n(u)$ are arbitrary functions of u . They consider the above sum as an integral when the 'spectrum' index n is continuous. In fact Wang and Wu technique is based

on a linear superposition that a linear superposition of mass function of particular solutions is also a solution of Einstein's field equations of non-rotating spacetime. Using the expression (6.8) in equations (6.3) we can generate rotating solutions with Wang-Wu functions as

$$\begin{aligned}
\mu^* &= -\frac{r}{K R^2 R^2} \sum_{n=-\infty}^{+\infty} \left\{ 2 q_n(u)_{,u} r^{n+1} + a^2 \sin^2 \theta q_n(u)_{,uu} r^n \right\}, \\
\rho^* &= \frac{2 r^2}{K R^2 R^2} \sum_{n=-\infty}^{+\infty} n q_n(u) r^{n-1}, \\
p &= -\frac{1}{K R^2} \sum_{n=-\infty}^{+\infty} n q_n(u) r^{n-1} \left\{ \frac{2 a^2 \cos^2 \theta}{R^2} + (n-1) \right\}, \\
\omega &= -\frac{i a \sin \theta}{\sqrt{2} K R^2 R^2} \sum_{n=-\infty}^{+\infty} (R - n \bar{R}) q_n(u)_{,u} r^n.
\end{aligned} \tag{6.9}$$

Here one can observe that these rotating solutions with functions $q_n(u)$ include many known as well as unknown rotating solutions of Einstein's field equations in spherical symmetry as shown by Wang and Wu in non-rotating cases [32]. The functions $q_n(u)$ in (6.8) play a great role in generating new solutions whether rotating or non-rotating. Therefore, we will hereafter refer to these as Wang-Wu functions. Thus, rotating solutions can be derived from these solutions as follows.

6.2.1 Rotating monopole solution

If one chooses the functions $q_n(u)$ such that

$$q_n(u) = \begin{cases} (b/2), & \text{when } n = 1 \\ 0, & \text{when } n \neq 1 \end{cases} \tag{6.10}$$

where b is constant, then from (6.8) one can obtain

$$\begin{aligned}
M(u, r) &= b r / 2, \quad \mu^* = \omega = 0, \\
\rho^* &= \frac{r^2 b}{K R^2 R^2}, \quad p = -\frac{b a^2 \cos^2 \theta}{K R^2 R^2},
\end{aligned}$$

with the energy momentum tensor

$$T_{ab} = 2 \rho^* \ell_{(a} n_{b)} + 2 p m_{(a} \bar{m}_{b)}.$$

The Weyl scalar takes the form

$$\psi_2 = -\frac{b r}{2 \bar{R} \bar{R} \bar{R}}.$$

The rotating monopole line element will be of the following form

$$\begin{aligned}
ds^2 = & \left\{ 1 - br^2 R^{-2} \right\} du^2 + 2du dr \\
& + 2a b r^2 R^{-2} \sin^2 \theta du d\phi - 2a \sin^2 \theta dr d\phi \\
& - R^2 d\theta^2 - \left\{ (r^2 + a^2)^2 - \Delta^* a^2 \sin^2 \theta \right\} R^{-2} \sin^2 \theta d\phi^2,
\end{aligned} \tag{6.11}$$

where $R^2 = r^2 + a^2 \cos^2 \theta$, and $\Delta^* = r^2 - br^2 + a^2$.

Because of the non-vanishing Weyl scalar ψ_2 , the rotating monopole solution is stationary Petrov type D whose one of the repeated principal directions is geodesic, shear free, expanding and rotating null vector ℓ_a . The rotating monopole solution has the non-zero pressure p , which leads the difference between the rotating and the non-rotating monopole solutions. That is, when $a = 0$, one can obtain the non-rotating metric [32] with $p = 0$. To study this rotating monopole solution (6.11) will be of interest. For example, one can easily embed Kerr-Newman black hole in this rotating monopole space to study a different physical nature of the black holes.

6.2.2 Kerr-Newman solution

We can choose the Wang-Wu functions $q_n(u)$ such that

$$q_n(u) = \begin{cases} m, & \text{when } n = 0 \\ -e^2/2, & \text{when } n = -1 \\ 0, & \text{when } n \neq 0, -1 \end{cases} \tag{6.12}$$

where m and e are constants. Then we obtain the function from (6.8)

$$\begin{aligned}
M(u, r) &= m - e^2/2r, \quad \mu^* = \omega = 0, \\
\rho^* &= p = \frac{e^2}{K R^2 R^2},
\end{aligned} \tag{6.13}$$

and the Weyl scalar is

$$\psi_2 = \frac{1}{R R R^2} (e^2 - m R),$$

which are the same as given (5.1).

6.2.3 Rotating Vaidya-Bonnor solution

The rotating Vaidya-Bonnor solution presented above, can also be obtained from these rotating Wang-Wu solutions if we choose the functions as

$$q_n(u) = \begin{cases} f(u), & \text{when } n = 0 \\ -h(u)^2/2, & \text{when } n = -1 \\ 0, & \text{when } n \neq 0, -1. \end{cases} \tag{6.14}$$

Then the corresponding quantities are

$$\begin{aligned} M(u, r) &= f(u) - h^2(u)/2r \\ \rho^* = p &= \frac{h^2(u)}{K R^2 R^2}, \end{aligned} \quad (6.15)$$

$$\begin{aligned} \mu^* &= -\frac{1}{K R^2 R^2} \left\{ 2r (r f(u)_{,u} - h h_{,u}) + a^2 \sin^2 \theta (r f(u)_{,u} - h h_{,u})_{,u} \right\}, \\ \omega &= \frac{-i a \sin \theta}{\sqrt{2} K R^2 R^2} \left\{ R f(u)_{,u} - 2h h_{,u} \right\}. \end{aligned} \quad (6.16)$$

This solution includes the rotating Vaidya solution ($h(u) = 0$) obtained above in (5.9). These subsections (6.2.2) and (6.2.3), which are the repetition of the section (5.1) and (5.3) above, show that the general rotating solutions with Wang-Wu functions (6.8) can employ to derive the rotating Kerr-Newman as well as rotating Vaidya-Bonnor solutions in a much simpler way.

6.2.4 *Kerr-Newman-Vaidya solution*

Wang and Wu [32] could combine the three non-rotating solutions, namely monopole, de-Sitter and charged Vaidya solution to obtain a new solution representing the non-rotating monopole-de Sitter-Vaidya charged solutions. In the same way, we shall combine the Kerr-Newman solution with the rotating Vaidya solution obtained above in (5.9), if the Wang-Wu functions $q_n(u)$ are chosen such that

$$q_n(u) = \begin{cases} m + f(u), & \text{when } n = 0 \\ -e^2/2, & \text{when } n = -1 \\ 0, & \text{when } n \neq 0, -1, \end{cases} \quad (6.17)$$

where m and e are constants and $f(u)$ is the mass function of rotating Vaidya solution (5.9). Thus, we obtain the function from (6.8)

$$M(u, r) = m + f(u) - e^2/2r$$

and using this in (6.9) we obtain other quantities

$$\begin{aligned} \rho^* = p &= \frac{e^2}{K R^2 R^2}, \\ \mu^* &= -\frac{r}{K R^2 R^2} \left\{ 2r f(u)_{,u} + a^2 \sin^2 \theta f(u)_{,uu} \right\}, \\ \omega &= \frac{-i a \sin \theta}{\sqrt{2} K R^2 R^2} f(u)_{,u}. \end{aligned} \quad (6.18)$$

The Weyl scalars for this rotating solution are

$$\begin{aligned}\psi_2 &= \frac{1}{\overline{R} \overline{R} R^2} [e^2 - R \{m + f(u)\}] \\ \psi_3 &= \frac{-i a \sin\theta}{2\sqrt{2\overline{R} \overline{R} R^2}} \{4r f(u)_{,u} + \overline{R} f(u)_{,u}\}, \\ \psi_4 &= \frac{a^2 r \sin^2\theta}{2\overline{R} \overline{R} R^2 R^2} \{R^2 f(u)_{,uu} - 2r f(u)_{,u}\}.\end{aligned}\tag{6.20}$$

This represents a rotating non-stationary Kerr-Newman-Vaidya solution with the line element

$$\begin{aligned}ds^2 &= [1 - R^{-2}\{2r(m + f(u)) - e^2\}] du^2 + 2du dr \\ &\quad + 2aR^{-2}\{2r(m + f(u)) - e^2\} \sin^2\theta du d\phi - 2a \sin^2\theta dr d\phi \\ &\quad - R^2 d\theta^2 - \{(r^2 + a^2)^2 - \Delta^* a^2 \sin^2\theta\} R^{-2} \sin^2\theta d\phi^2,\end{aligned}\tag{6.21}$$

where $\Delta^* = r^2 - 2r\{m + f(u)\} + a^2 + e^2$. Here m and e are the mass and the charge of Kerr-Newman solution, a is the rotation per unit mass and $f(u)$ represents the mass function of rotating Vaidya null radiating fluid. The solution (6.21) will describe a black hole if $m + f(u) > a^2 + e^2$ with external event horizon at $r_+ = \{m + f(u)\} + \sqrt{[\{m + f(u)\}^2 - a^2 - e^2]}$, an internal Cauchy horizon at $r_- = \{m + f(u)\} - \sqrt{[\{m + f(u)\}^2 - a^2 - e^2]}$ and the non-stationary limit surface $r \equiv r_e(u, \theta) = \{m + f(u)\} + \sqrt{[\{m + f(u)\}^2 - a^2 \cos^2\theta - e^2]}$. When we set $f(u) = 0$, the metric (6.21) recovers the standard Kerr-Newman black hole, and if $m = 0$, then it is the rotating charged Vaidya null radiating black hole (5.9).

In this rotating solution, the Vaidya null fluid is interacting with the non-null electromagnetic field whose Maxwell scalar ϕ_1 can be obtained from (6.18). Thus, we can write the total energy momentum tensor (EMT) for the rotating solution (6.21) as follows:

$$T_{ab} = T_{ab}^{(n)} + T_{ab}^{(E)},\tag{6.22}$$

where the EMTs for the rotating null fluid as well as that of the electromagnetic field are given respectively

$$T_{ab}^{(n)} = \mu^* \ell_a \ell_b + 2\omega \ell_{(a} \overline{m}_{b)} + 2\overline{\omega} \ell_{(a} m_{b)},\tag{6.23}$$

$$T_{ab}^{(E)} = 2\rho^* \{\ell_{(a} n_{b)} + m_{(a} \overline{m}_{b)}\}.\tag{6.24}$$

The appearance of non-vanishing ω shows the null fluid is rotating as the expression (6.19) of ω involves the rotating constant a coupling with $\partial f(u)/\partial u$ - both are non-zero quantities for a rotating Vaidya null radiating universe (5.9).

This Kerr-Newman-Vaidya metric (6.21) can be written in Kerr-Schild ansatz on the Kerr-Newman background as

$$g_{ab}^{\text{KNV}} = g_{ab}^{\text{KN}} + 2Q(u, r, \theta)\ell_a\ell_b \quad (6.25)$$

where

$$Q(u, r, \theta) = -rf(u)R^{-2}, \quad (6.26)$$

and the vector ℓ_a is a geodesic, shear free, expanding as well as rotating null vector of both g_{ab}^{KN} as well as g_{ab}^{KNV} and given in (2.8) and g_{ab}^{KN} is the Kerr-Newman metric (5.4) with $m = e = \text{constant}$. This null vector ℓ_a is one of the double repeated principal null vectors of the Weyl tensor of g_{ab}^{KN} . This completes the proof of theorem 1 stated above.

It appears from (6.25) that the Kerr-Newman geometry can be thought of joining smoothly to the rotating Vaidya geometry at its null radiative boundary, as shown by Glass and Krisch [23] in the case of Schwarzschild geometry joining to the non-rotating Vaidya space-time. The Kerr-Schild form (6.25) will recover that of Xanthopoulos [33] $g'_{ab} = g_{ab} + \ell_a\ell_b$, when $Q(u, r, \theta) \rightarrow 1/2$ and that of Glass and Krisch [23] $g'_{ab} = g_{ab}^{\text{Sch}} - \{2f(u)/r\}\ell_a\ell_b$ when $e = a = 0$ for non-rotating Schwarzschild background space. Thus, it can be regarded that the Kerr-Schild form presented in (6.25) above will be the extension of those of Xanthopoulos as well as Glass and Krisch. When we set $a = 0$, this Kerr-Newman-Vaidya solution (6.21) will recover to non-rotating Reissner-Nordstrom-Vaidya solution with the Kerr-Schild form $g_{ab}^{\text{RNV}} = g_{ab}^{\text{RN}} - \{2f(u)/r\}\ell_a\ell_b$, which is still a generalization of Xanthopoulos and Glass and Krisch in the charged Reissner-Nordstrom solution. It is worth to mention that the new solution (6.21) cannot be considered as a bimetric theory as $g_{ab}^{\text{KNV}} \neq \frac{1}{2}(g_{ab}^{\text{KN}} + g_{ab}^{\text{V}})$.

To interpret the Kerr-Newman-Vaidya solution as a black hole during the early inflationary phase of rotating Vaidya null radiating universe *i.e.*, the Kerr-Newman black hole embedded in rotating Vaidya null radiating background space, we can also write the Kerr-Schild form (6.25) as

$$g_{ab}^{\text{KNV}} = g_{ab}^{\text{V}} + 2Q(r, \theta)\ell_a\ell_b \quad (6.27)$$

where

$$Q(r, \theta) = -(rm - e^2/2)R^{-2}, \quad (6.28)$$

Here, the constants m and e are the mass and the charge of Kerr-Newman black hole, g_{ab}^{V} is the rotating Vaidya null radiating black hole (5.9) and ℓ_a is the geodesic null vector given in (2.8) for both g_{ab}^{KNV} and g_{ab}^{V} . When we set $f(u) = a = 0$, g_{ab}^{V} will recover the flat metric, then g_{ab}^{KNV} becomes the original Kerr-Schild form written in spherical symmetric flat background.

These two Kerr-Schild forms (6.25) and (6.27) certainly confirm that the metric g_{ab}^{KNV} is a solution of Einstein's field equations since the background rotating metrics g_{ab}^{KN} and g_{ab}^{V} are solutions of Einstein's equations. They both have different stress-energy tensors $T_{ab}^{(\text{E})}$ and $T_{ab}^{(\text{n})}$ given in (6.24) and (6.23) respectively. Looking at the Kerr-Schild form (6.27), the Kerr-Newman-Vaidya black hole can be treated as a generalization of Kerr-Newman black hole by incorporating Visser's suggestion [34] that *Kerr-Newman black hole embedded in an axisymmetric cloud of matter would be of interest*. Hawking et al [35] have also mentioned the possibility to embed the rotating black hole solutions with a theory for which they know the corresponding conformal field theory.

6.2.5 Kerr-Newman-Vaidya-Bonnor solution

Similarly, one can combine the rotating Vaidya-Bonnor solution obtained above in (5.13) with the Kerr-Newman solution (5.4) to generate another rotating solution with the mass function

$$M(u, r) = m + f(u) - (e^2 + h^2(u))/2r, \quad (6.29)$$

representing a Kerr-Newman-Vaidya-Bonnor solution:

$$\begin{aligned} ds^2 = & [1 - R^{-2}\{2r(m + f(u)) - e^2 - h^2(u)\}] du^2 + 2du dr \\ & + 2aR^{-2}\{2r(m + f(u)) - e^2 - h^2(u)\} \sin^2\theta du d\phi - 2a \sin^2\theta dr d\phi \\ & - R^2 d\theta^2 - \{(r^2 + a^2)^2 - \Delta^* a^2 \sin^2\theta\} R^{-2} \sin^2\theta d\phi^2, \end{aligned} \quad (6.30)$$

where $\Delta^* = r^2 - 2r\{m + f(u)\} + a^2 + e^2 + h^2(u)$. This rotating solution can also be written in Kerr-Schild form (6.25) with the function:

$$Q(u, r, \theta) = -\{r f(u) - h^2(u)/2\} R^{-2},$$

When the charge e of the Kerr-Newman solution vanishes, this *rotating* solution (6.30) will reduce to a rotating Kerr-Vaidya-Bonnor solution with the mass function:

$$M(u, r) = m + f(u) - h^2(u)/2r. \quad (6.31)$$

It suggests that by choosing the Wang-Wu functions $q_n(u)$ properly one can generate as many rotating solutions as required. However, the generation of these types of rotating solutions will be restricted that the energy-momentum tensor of the fluids must be of the form given in (4.1).

6.2.6 Kerr-Newman-de Sitter metrics

Here we shall present the rotating de Sitter as well as Kerr-Newman-de Sitter metrics in NP formalism.

A. *Rotating de Sitter solution:*

First we will derive a rotating de Sitter solution of Einstein's equations. For this we choose the Wang-Wu functions as

$$q_n(u) = \begin{cases} \Lambda^*/6, & \text{when } n = 3 \\ 0, & \text{when } n \neq 3 \end{cases} \quad (6.32)$$

to obtain the mass function

$$M(u, r) = \frac{\Lambda^* r^3}{6}, \quad (6.33)$$

The line element for the rotating de Sitter metric is

$$\begin{aligned} ds^2 = & \left\{ 1 - \frac{\Lambda^* r^4}{3 R^2} \right\} du^2 + 2 du dr \\ & + 2a \frac{\Lambda^* r^4}{3} R^{-2} \sin^2 \theta du d\phi - 2a \sin^2 \theta dr d\phi \\ & - R^2 d\theta^2 - \left\{ (r^2 + a^2)^2 - \Delta^* a^2 \sin^2 \theta \right\} R^{-2} \sin^2 \theta d\phi^2, \end{aligned} \quad (6.34)$$

where $R^2 = r^2 + a^2 \cos^2 \theta$, $\Delta^* = r^2 - \Lambda^* r^4/3 + a^2$. This corresponds to the rotating de Sitter solution for $\Lambda^* > 0$, and to the anti-de Sitter solution for $\Lambda^* < 0$. In general Λ^* denotes the cosmological constant of the de Sitter space. Then the changed NP quantities are

$$\gamma = -\frac{1}{2\overline{R} R^2} \left\{ \left(1 - \frac{1}{3} \Lambda^* r^2 \right) r \overline{R} + \Delta^* \right\},$$

$$\phi_{11} = -\frac{1}{2 R^2 \overline{R}^2} \Lambda^* r^2 a^2 \cos^2 \theta, \quad (6.35)$$

$$\psi_2 = \frac{1}{3\overline{R} \overline{R} R^2} \Lambda^* r^2 a^2 \cos^2 \theta, \quad (6.36)$$

$$\Lambda = \frac{\Lambda^* r^2}{6 R^2}. \quad (6.37)$$

This means that in rotating de Sitter cosmological universe, the Λ^* is coupling with the rotational parameter a . From these NP quantities we can clearly observe that the rotating de Sitter cosmological metric is a Petrov type D gravitational field whose one of the repeated principal null vectors, ℓ_a is geodesic, shear free, expanding as well

as non-zero twist. The rotating cosmological space possesses an energy-momentum tensor

$$T_{ab} = 2\rho^* \ell_{(a} n_{b)} + 2p m_{(a} \bar{m}_{b)}, \quad (6.38)$$

where $K\rho^* = 2\phi_{11} + 6\Lambda$ and $Kp = 2\phi_{11} - 6\Lambda$ are related to the density and the pressure of the cosmological matter which is, however not a perfect fluid. If we set the rotational parameter $a = 0$, we will recover the non-rotating de Sitter metric [36], which is a solution of the Einstein's equations for an empty space with $\Lambda \equiv g_{ab}R^{ab} = \Lambda^*/6$ or constant curvature. However, it is observed that the rotating de Sitter metric (6.34) is neither *empty* nor *constant curvature*. It certainly describes a stationary rotating spherical symmetric solution representing Petrov type D space-time. So it is noted that to the best of the present author's knowledge, this rotating de Sitter metric has not been seen discussed before.

B. Kerr-Newman-de Sitter solution:

By choosing the Wang-Wu function as

$$q_n(u) = \begin{cases} m, & \text{when } n = 0 \\ -e^2/2, & \text{when } n = -1 \\ \Lambda^*/6, & \text{when } n = 3 \\ 0, & \text{when } n \neq 0, -1, 3 \end{cases} \quad (6.39)$$

we can obtain the function from (6.8)

$$M(u, r) = m - \frac{e^2}{2r} + \frac{\Lambda^* r^3}{6}, \quad (6.40)$$

where m and e are constants and are the mass and the charge of the Kerr-Newman solution. The line element with the function (6.39) is

$$\begin{aligned} ds^2 = & \left\{ 1 - R^{-2} \left(2mr - e^2 + \frac{\Lambda^* r^4}{3} \right) \right\} du^2 + 2du dr \\ & + 2aR^{-2} \left(2mr - e^2 + \frac{\Lambda^* r^4}{3} \right) \sin^2\theta du d\phi - 2a \sin^2\theta dr d\phi \\ & - R^2 d\theta^2 - \left\{ (r^2 + a^2)^2 - \Delta^* a^2 \sin^2\theta \right\} R^{-2} \sin^2\theta d\phi^2, \end{aligned} \quad (6.41)$$

where $R^2 = r^2 + a^2 \cos^2\theta$, $\Delta^* = r^2 - 2mr - \Lambda^* r^4/3 + a^2 + e^2$. Then the changed NP quantities are

$$\begin{aligned} \gamma &= \frac{1}{2\bar{R}R^2} \left[\left(r - m - \frac{2}{3}\Lambda^* r^3 \right) \bar{R} - \Delta^* \right], \\ \phi_{11} &= \frac{1}{2R^2R^2} \left(e^2 - \Lambda^* r^2 a^2 \cos^2\theta \right), \end{aligned}$$

$$\begin{aligned}\psi_2 &= \frac{1}{R \bar{R} R^2} \left\{ e^2 - m R + \frac{\Lambda^* r^2}{3} a^2 \cos^2 \theta \right\} \\ \Lambda &= \frac{\Lambda^* r^2}{6R^2}.\end{aligned}\tag{6.42}$$

We have seen from the above that in each expression of ϕ_{11} and ψ_2 , the cosmological constant Λ^* is coupling with the rotational parameter a . This means that the cosmological parameter Λ^* has the effect of its presence in the curvature of the embedded Kerr-Newman black hole. The metric (6.41) admits the following energy momentum tensor

$$T_{ab} = 2 \rho^* \ell_{(a} n_{b)} + 2 p m_{(a} \bar{m}_{b)},$$

with the density and the pressure of the matter field

$$\begin{aligned}\rho^* &= \frac{1}{K R^2 R^2} (e^2 + \Lambda^* r^4), \\ p &= \frac{1}{K R^2 R^2} \left\{ e^2 - \Lambda^* r^2 (r^2 + 2a^2 \cos^2 \theta) \right\}\end{aligned}$$

respectively. Without loss of generality, we can write this T_{ab} with the decomposition of $\rho^* = \rho^{*(E)} + \rho^{*(C)}$ and $p = p^{(E)} + p^{(C)}$ that

$$T_{ab} = 4 \rho^{*(E)} \{ \ell_{(a} n_{b)} + m_{(a} \bar{m}_{b)} \} + 2 \{ \rho^{*(C)} \ell_{(a} n_{b)} + 2 p^{(C)} m_{(a} \bar{m}_{b)} \},\tag{6.43}$$

where

$$\begin{aligned}\rho^{*(E)} &= p^{(E)} = \frac{e^2}{K R^2 R^2}, \\ \rho^{*(C)} &= \frac{\Lambda^* r^4}{K R^2 R^2}, \quad p^{(C)} = \frac{-\Lambda^* r^2}{K R^2 R^2} (r^2 + 2a^2 \cos^2 \theta).\end{aligned}$$

The advantage of writing T_{ab} in the form (6.43) is that, for Reissner-Nordstrom-de Sitter metric ($a = 0$), the energy-momentum tensor can be written in the form of Guth's modification of T_{ab} for inflationary universe [37] as

$$T_{ab} = T_{ab}^{(E)} + \Lambda^* g_{ab}\tag{6.44}$$

where $T_{ab}^{(E)}$ is the energy-momentum tensor for non-null electromagnetic field and g_{ab} is the Reissner-Nordstrom-de Sitter metric. This indicates that Guth's modification of T_{ab} is acceptable only in the case of non-rotating metrics, and its extension in the case of the rotating solutions ($a \neq 0$) will take the form given in (6.43) above where $\rho^{*(C)} \neq p^{(C)}$.

The metric (6.41) describes a rotating stationary solution and is Petrov type D with $(\psi_2 \neq 0)$, whose one of the repeated principal null directions is ℓ_a . That is, the metric can be written in Kerr-Schild form on the de Sitter background as

$$g_{ab}^{\text{KNdS}} = g_{ab}^{\text{dS}} + 2Q(r, \theta)\ell_a\ell_b \quad (6.45)$$

where $Q(r, \theta) = -(rm - e^2/2)R^{-2}$, and the vector ℓ_a is a geodesic, shear free, expanding as well as rotating null vector of both g_{ab}^{KNdS} as well as g_{ab}^{dS} and given in (2.8) and g_{ab}^{KN} is the Kerr-Newman metric (5.4) with $m = e = \text{constant}$. This null vector ℓ_a is one of the double repeated principal null vectors of the Weyl tensor of g_{ab}^{KNdS} and g_{ab}^{dS} . This completes the proof of theorem 2 stated above. We can also write the Kerr-Schild form (6.45) on the Kerr-Newman background as

$$g_{ab}^{\text{KNdS}} = g_{ab}^{\text{KN}} + 2Q(r, \theta)\ell_a\ell_b, \quad (6.46)$$

where $Q(r, \theta) = -(\Lambda^*r^4/6)R^{-2}$, and Λ^* is the cosmological constant and g_{ab}^{KN} is the Kerr-Newman metric (5.4) with $m = e = \text{constant}$.

It is quite natural to recover rotating de Sitter ($m = 0, a \neq 0, e = 0$) and Reissner-Nordstrom-de Sitter ($m \neq 0, a = 0, e \neq 0$) metrics from the Kerr-Newman-de Sitter solution ($m \neq 0, a \neq 0, e \neq 0$). It is also worth mentioning that from the Kerr-Newman-de Sitter metric, we can recover a rotating charged de Sitter cosmological universe ($m = 0, a \neq 0, e \neq 0$). It is noted that one may find the difference between this Kerr-Newman-de Sitter metric (6.41) and that of Mallett [8] used by Koberlin [38]. Mallett's derivation of Kerr-Newman-de Sitter metric is based on the direct application of Newman-Janis algorithm to the Reissner-Nordstrom-de Sitter 'seed' solution. It is also found that the Kerr-Newman-de Sitter solution (6.41) is different, in the terms involving the cosmological constant Λ^* , from the one derived by Carter [39] and used by Gibbon and Hawking [40], Khanal [41], Hawking, et al [35], and others.

6.2.7 Kerr-Newman-Vaidya-de Sitter solution

Here we shall combine the Kerr-Newman-de Sitter solution (6.41) with the rotating Vaidya solution given in (5.9). For this purpose we choose the Wang-Wu functions $q_n(u)$ as follows

$$q_n(u) = \begin{cases} m + f(u), & \text{when } n = 0 \\ -e^2/2, & \text{when } n = -1 \\ \Lambda^*/6, & \text{when } n = 3 \\ 0, & \text{when } n \neq 0, -1, 3, \end{cases} \quad (6.47)$$

where m and e are constants and $f(u)$ is related with the mass of rotating Vaidya solution (5.9). Thus, using (6.47) in (6.8) we have the mass function as

$$M(u, r) = m + f(u) - \frac{e^2}{2r} + \frac{\Lambda^* r^3}{6}$$

and other quantities are

$$\begin{aligned} \rho^* &= \frac{1}{K R^2 R^2} (e^2 + \Lambda^* r^4), \\ p &= \frac{1}{K R^2 R^2} \left\{ e^2 - \Lambda^* r^2 (r^2 + 2a^2 \cos \theta) \right\}, \end{aligned} \quad (6.48)$$

$$\begin{aligned} \mu^* &= -\frac{r}{K R^2 R^2} \left\{ 2r f(u)_{,u} + a^2 \sin^2 \theta f(u)_{,uu} \right\}, \\ \omega &= \frac{-i a \sin \theta}{\sqrt{2} K \bar{R} R^2} f(u)_{,u}, \\ \Lambda &\equiv g^{ab} R_{ab} = \frac{\Lambda^* r^2}{6 R^2}, \end{aligned} \quad (6.49)$$

and ϕ_{11} , ϕ_{12} , ϕ_{22} can be obtained from equations (6.48) with (3.10). The Weyl scalars are given below

$$\begin{aligned} \psi_2 &= \frac{1}{\bar{R} R R^2} \left[e^2 - R \{m + f(u)\} + \frac{\Lambda^* r^2}{3} a^2 \cos^2 \theta \right], \\ \psi_3 &= \frac{-i a \sin \theta}{2\sqrt{2} \bar{R} R R^2} \left\{ (4r + \bar{R}) f(u)_{,u} \right\}, \\ \psi_4 &= \frac{a^2 r \sin^2 \theta}{2\bar{R} R R^2 R^2} \left\{ R^2 f(u)_{,uu} - 2r f(u)_{,u} \right\}. \end{aligned} \quad (6.50)$$

This represents a Kerr-Newman-Vaidya-de Sitter solution with the line element

$$\begin{aligned} ds^2 &= \left[1 - R^{-2} \left\{ 2r(m + f(u)) + \frac{\Lambda^* r^4}{3} - e^2 \right\} \right] du^2 + 2du dr \\ &+ 2aR^{-2} \left\{ 2r(m + f(u)) + \frac{\Lambda^* r^4}{3} - e^2 \right\} \sin^2 \theta du d\phi - 2a \sin^2 \theta dr d\phi \\ &- R^2 d\theta^2 - \left\{ (r^2 + a^2)^2 - \Delta^* a^2 \sin^2 \theta \right\} R^{-2} \sin^2 \theta d\phi^2, \end{aligned} \quad (6.51)$$

where $\Delta^* = r^2 - 2r\{m + f(u)\} - \Lambda^* r^4/3 + a^2 + e^2$. Here m and e are the mass and the charge of Kerr-Newman solution, a is the non-zero rotation parameter per unit mass and $f(u)$ represents the mass function of rotating Vaidya null radiating fluid. When we set $f(u) = 0$, the metric (6.51) recovers the Kerr-Newman-de Sitter black hole (6.41), and if $m = 0$, then it is the rotating charged Vaidya null radiating

black hole (5.13). When one sets $\Lambda^* = 0$, this metric will recover the Kerr-Newman-Vaidya metric ((6.21). In this rotating solution, the Vaidya null fluid is interacting with the non-null electromagnetic field whose Maxwell scalar ϕ_1 can be obtained from (6.48). Thus, we can write the total energy momentum tensor (EMT) for the rotating solution (6.51) as follows:

$$T_{ab} = T_{ab}^{(n)} + T_{ab}^{(E)} + T_{ab}^{(C)}, \quad (6.52)$$

where the EMTs for the rotating null fluid, the electromagnetic field and cosmological matter field are given respectively

$$\begin{aligned} T_{ab}^{(n)} &= \mu^* \ell_a \ell_b + 2\omega \ell_{(a} \bar{m}_{b)} + 2\bar{\omega} \ell_{(a} m_{b)}, \\ T_{ab}^{(E)} &= 4\rho^{*(E)} \{\ell_{(a} n_{b)} + m_{(a} \bar{m}_{b)}\}, \\ T_{ab}^{(C)} &= 2\{\rho^{*(C)} \ell_{(a} n_{b)} + 2p^{(C)} m_{(a} \bar{m}_{b)}\}, \end{aligned} \quad (6.53)$$

where μ^* and ω are given in (6.49) and

$$\begin{aligned} \rho^{*(E)} &= p^{(E)} = \frac{e^2}{K R^2 R^2}, \\ \rho^{*(C)} &= \frac{\Lambda^* r^4}{K R^2 R^2}, \quad p^{(C)} = -\frac{\Lambda^* r^2}{K R^2 R^2} (r^2 + 2a^2 \cos^2 \theta). \end{aligned} \quad (6.54)$$

The appearance of non-vanishing ω shows that the null fluid is rotating as the expression of ω (6.49) involves the rotating parameter a coupling with $\partial f(u)/\partial u$ – both are non-zero quantities for a rotating Vaidya null radiating universe.

This Kerr-Newman-Vaidya-de Sitter metric (6.51) can be written in Kerr-Schild form on the de Sitter background as

$$g_{ab}^{\text{KNVdS}} = g_{ab}^{\text{dS}} + 2Q(u, r, \theta) \ell_a \ell_b \quad (6.55)$$

where

$$Q(u, r, \theta) = -[r\{m + f(u)\} - e^2/2]R^{-2}, \quad (6.56)$$

and the vector ℓ_a is a geodesic, shear free, expanding as well as rotating null vector of both g_{ab}^{dS} as well as g_{ab}^{KNVdS} and given in (2.8). We can also write this solution (6.51) in another Kerr-Schild form on the Kerr-Newman background as

$$g_{ab}^{\text{KNVdS}} = g_{ab}^{\text{KN}} + 2Q(u, r, \theta) \ell_a \ell_b \quad (6.57)$$

where $Q(u, r, \theta) = -\{rf(u) + \Lambda^* r^4/6\}R^{-2}$. These two Kerr-Schild forms (6.55) and (6.57) certainly confirm that the metric g_{ab}^{KNVdS} is a solution of Einstein's field equations since the background rotating metrics g_{ab}^{KN} and g_{ab}^{dS} are both solutions of Einstein's field equations. They both have different stress-energy tensors $T_{ab}^{(E)}$ and $T_{ab}^{(C)}$ given in (6.53).

6.2.8 Rotating Vaidya-Bonnor-de Sitter solution

We shall combine the rotating Vaidya-Bonnor solution (5.13) with the rotating de Sitter solution obtained above in (6.34), if the Wang-Wu functions $q_n(u)$ are chosen such that

$$q_n(u) = \begin{cases} f(u), & \text{when } n = 0 \\ -e^2(u)/2, & \text{when } n = -1 \\ \Lambda^*/6, & \text{when } n = 3 \\ 0, & \text{when } n \neq 0, -1, 3, \end{cases} \quad (6.58)$$

where $f(u)$ and $e(u)$ are related with the mass and the charge of rotating Vaidya-Bonnor solution (5.13). Thus, using this $q_n(u)$ in (6.8) we obtain the mass function

$$M(u, r) = f(u) - \frac{e^2(u)}{2r} + \frac{\Lambda^* r^3}{6}, \quad (6.59)$$

and other quantities are

$$\rho^* = \frac{1}{K R^2 R^2} \{e^2(u) + \Lambda^* r^4\},$$

$$p = \frac{1}{K R^2 R^2} \{e^2(u) - \Lambda^* r^2 (r^2 + 2a^2 \cos\theta)\}, \quad (6.60)$$

$$\mu^* = -\frac{1}{K R^2 R^2} \left[2r^2 \left\{ f(u)_{,u} - \frac{1}{r} e(u) e(u)_{,u} \right\} + a^2 \sin^2\theta \left\{ f(u)_{,u} - \frac{1}{r} e(u) e(u)_{,u} \right\}_{,u} \right],$$

$$\omega = \frac{-i a \sin\theta}{\sqrt{2} K R^2 R^2} \{R f(u)_{,u} - 2e(u) e(u)_{,u}\}.$$

and ϕ_{11} , ϕ_{12} , ϕ_{22} can be obtained from equations (6.60) with (3.10). The Weyl scalars are given below

$$\begin{aligned} \psi_2 &= \frac{1}{\overline{R} R R^2} \left[e^2(u) - R f(u) + \frac{\Lambda^* r^2}{3} a^2 \cos^2\theta \right], \\ \psi_3 &= \frac{-i a \sin\theta}{2\sqrt{2} \overline{R} R R^2} \left[(4r + \overline{R}) f(u)_{,u} \right] - 4e(u) e(u)_{,u}, \\ \psi_4 &= \frac{a^2 \sin^2\theta}{2\overline{R} R R^2 R^2} \left[R^2 \left\{ r f(u)_{,u} - e(u) e(u)_{,u} \right\}_{,u} \right. \\ &\quad \left. - 2r \left\{ r f(u)_{,u} - e(u) e(u)_{,u} \right\} \right]. \end{aligned} \quad (6.61)$$

This represents a rotating Vaidya-Bonnor-de Sitter solution with the line element

$$ds^2 = \left[1 - R^{-2} \left\{ 2r f(u) + \frac{\Lambda^* r^4}{3} - e^2(u) \right\} \right] du^2 + 2du dr$$

$$\begin{aligned}
& +2aR^{-2}\left\{2rf(u) + \frac{\Lambda^* r^4}{3} - e^2(u)\right\} \sin^2\theta du d\phi - 2a \sin^2\theta dr d\phi \\
& -R^2 d\theta^2 - \left\{(r^2 + a^2)^2 - \Delta^* a^2 \sin^2\theta\right\} R^{-2} \sin^2\theta d\phi^2,
\end{aligned} \tag{6.62}$$

where $\Delta^* = r^2 - 2rf(u) - \Lambda^* r^3/3 + a^2 + e^2(u)$. Here, a is the non-zero rotational parameter per unit mass and $f(u)$ represents the mass function of rotating Vaidya null radiating fluid. We can write the total energy momentum tensor (EMT) for the rotating solution (6.62) as follows:

$$T_{ab} = T_{ab}^{(n)} + T_{ab}^{(E)} + T_{ab}^{(C)}, \tag{6.63}$$

where the EMTs for the rotating null fluid, the electromagnetic field and cosmological matter field are given respectively

$$T_{ab}^{(n)} = \mu^* \ell_a \ell_b + 2\omega \ell_{(a} \bar{m}_{b)} + 2\bar{\omega} \ell_{(a} m_{b)}, \tag{6.64}$$

$$T_{ab}^{(E)} = 4\rho^{*(E)} \{\ell_{(a} n_{b)} + m_{(a} \bar{m}_{b)}\}, \tag{6.65}$$

$$T_{ab}^{(C)} = 2\{\rho^{*(C)} \ell_{(a} n_{b)} + 2p^{(C)} m_{(a} \bar{m}_{b)}\}, \tag{6.66}$$

where

$$\begin{aligned}
\rho^{*(E)} &= p^{(E)} = \frac{e^2(u)}{K R^2 R^2}, \\
\rho^{*(C)} &= \frac{\Lambda^* r^4}{K R^2 R^2}, \quad p^{(C)} = -\frac{\Lambda^* r^2}{K R^2 R^2} (r^2 + 2a^2 \cos^2\theta).
\end{aligned}$$

If we set $a = 0$, we recover the non-rotating Vaidya-Bonnor-de Sitter solution, and then the energy-momentum tensor (6.63) will take the form of Guth's modification of T_{ab} for inflationary scenario [37] as

$$T_{ab} = T_{ab}^{(n)} + T_{ab}^{(E)} + \Lambda^* g_{ab} \tag{6.67}$$

where $T_{ab}^{(E)}$ is the energy-momentum tensor for non-null electromagnetic field and g_{ab} is the non-rotating Vaidya-Bonnor-de Sitter metric tensor. From this, without loss of generality, the EMT (6.63) can be regarded as the extension of Guth's modification of energy-momentum tensor in rotating spaces.

The Vaidya-Bonnor-de Sitter metric can be written in Kerr-Schild form

$$g_{ab}^{\text{VBdS}} = g_{ab}^{\text{dS}} + 2Q(u, r, \theta) \ell_a \ell_b \tag{6.68}$$

where $Q(u, r, \theta) = -\{rf(u) - e^2(u)/2\} R^{-2}$. Here, g_{ab}^{dS} is the rotating de Sitter metric (6.34) and ℓ_a is geodesic, shear free, expanding and non-zero twist null vector for

both g_{ab}^{dS} as well as g_{ab}^{VBdS} and given in (2.8). The above Kerr-Schild form can also be written on the rotating Vaidya-Bonnor background given in (5.13).

$$g_{ab}^{\text{VBdS}} = g_{ab}^{\text{VB}} + 2Q(r, \theta)\ell_a\ell_b \quad (6.69)$$

where $Q(r, \theta) = -(\Lambda^*r^4/6)R^{-2}$. These two Kerr-Schild forms (6.68) and (6.69) prove the non-stationary version of theorem 2 in the case of rotating Vaidya-Bonnor-de Sitter solution. If we set $f(u)$ and $e(u)$ are both constant, this Kerr-Schild form (6.68) will be that of Kerr-Newman-de Sitter black hole (6.45). The rotating Vaidya-Bonnor-de Sitter metric will describe a non-stationary spherically symmetric solution whose Weyl curvature tensor is algebraically special in Petrov classification possessing a geodesic, shear free, expanding and non-zero twist null vector ℓ_a given in (2.8). One can easily recover a rotating Vaidya-de Sitter metric from this Vaidya-Bonnor-de Sitter solution by setting the charge $e(u) = 0$. If one sets $a = 0$, $e(u) = 0$ in (6.62), one can also obtain the standard non-rotating Vaidya-de Sitter solution [42]. Ghosh and Dadhich [43] have studied the gravitational collapse problem in non-rotating Vaidya-de Sitter space by identifying the de Sitter cosmological constant Λ^* with the bag constant of the null strange quark fluid. Also if one sets $a = 0$ in (6.62) one can recover the non-rotating Vaidya-Bonnor-de Sitter black hole [44]. It certainly indicates that all embedded solutions (6.21), (6.30), (6.41), (6.62) can be derived by using Wang-Wu functions (6.8) in the rotating solutions (6.4).

7 Conclusion

In this paper, we have calculated NP quantities for a rotating spherically symmetric metric with three variables. With the help of these NP quantities, we have first given examples of rotating solutions like Kerr-Newman, rotating Vaidya and rotating Vaidya-Bonnor. Then, with the help of Wang-Wu functions, we come to the unpublished rotating metrics that we combined them with other rotating solutions in order to derive new embedded rotating solutions, and studied the gravitational structure of the solutions by observing the nature of the energy-momentum tensors of respective spacetime metrics. The embedded rotating solutions have also been expressed in terms of Kerr-Schild forms in order to show them as solutions of Einstein's field equations.

We would like to mention that Chandrasekhar [20] has established a relation of spin coefficients ρ , μ , τ , π in the case of an affinely parameterized geodesic vector, generating an integral which is constant along the geodesic in a vacuum Petrov type D space-time

$$\frac{\rho}{\bar{\rho}} = \frac{\mu}{\bar{\mu}} = \frac{\tau}{\bar{\tau}} = \frac{\pi}{\bar{\pi}}. \quad (7.1)$$

This relation is being derived on the basis of the vacuum Petrov type D space-time with $\psi_2 \neq 0$, $\psi_0 = \psi_1 = \psi_3 = \psi_4 = 0$ and $\phi_{01} = \phi_{02} = \phi_{10} = \phi_{20} = \phi_{12} = \phi_{21} = \phi_{00} = \phi_{22} = \phi_{11} = \Lambda = 0$. However, it has been shown in [22] that the *non-vacuum* Petrov type D spacetimes *i.e.* Kerr-Newman solution possessing electromagnetic field and Kantowski-Sachs metric with dust energy-momentum tensor, satisfy the Chandrasekhar's relation (7.1). We will show that the very general metric (2.7) with three variables, which is of *algebraically special* in Petrov classification having non-zero ψ_2, ψ_3, ψ_4 and stress-energy tensor (4.1), still satisfies the relation (7.1) as follows

$$\frac{\rho}{\bar{\rho}} = \frac{\mu}{\bar{\mu}} = \frac{\tau}{\bar{\tau}} = \frac{\pi}{\bar{\pi}} = \frac{R}{\bar{R}}. \quad (7.2)$$

This relation (7.2) shows that all rotating solutions, stationary and non-stationary, discussed here satisfy the relation (7.1). Thus, it seems reasonable to refer to the relation (7.1) as *Chandrasekhar's identity* as mentioned by Fernandes and Lun [45]. This certainly indicates that the NP spin coefficients (3.3) can be used to extend the known vacuum results like the relation (7.1) to the non-vacuum ones. Further, we also observe from the energy-momentum tensor (4.1) that the metric (2.7) with three variables does not include the perfect fluid. From the above discussion, it certainly indicates that the Newman-Janis algorithm can be used to generate rotating solutions as shown in section 6, if the metric function $M(u, r)$ is expressed in terms of Wang-Wu functions given in (6.8). However such generated rotating solutions from the application of Newman-Janis algorithm have limitations that these rotating solutions are not included spacetimes admitting rotating perfect fluid. To have a spacetime admitting a rotating perfect fluid one has to look for another algorithm rather than that of Newman and Janis.

From the above results presented in this paper, it suggests that rotating spacetime geometries must also have rotating matter fields, described by the stress-energy tensors T_{ab} with non zero rotation parameter a . It means that the matching of a rotating non-stationary space-time geometry with the usual non-rotating perfect fluid will not have a reasonably good sense. So one needs to look for a rotating perfect fluid to match with a rotating non-stationary spacetime geometry. Some of the rotating solutions discussed above include rotating non-stationary solutions, like Kerr-Newman-Vaidya black hole, Kerr-Newman-Vaidya-de Sitter black hole, Vaidya-Bonnor-de Sitter black hole. They possess rotating non-perfect fluids as shown above by the respective T_{ab} . To study the nature of these rotating black holes will certainly be a new area of interest in classical General Relativity, since all known black hole theorems, like 'no hair theorem', Penrose's theorems are based on stationary black holes, rotating or non-rotating.

It is also found that the technique of Wang and Wu with the functions $q_n(u)$ in (6.8) can be used to generate rotating solutions as shown in section 6.2 above. By

choosing a suitable Wang-Wu function $q(u)$, we obtain a rotating de Sitter space-time model. We can also recover the widely used (i) Schwarzschild-de Sitter solution, (ii) Reissner-Nordstrom-de Sitter black hole solution, (iii) Kerr-de Sitter solution, (iv) Kerr-Newman-de Sitter solution for early inflation scenarios from the rotating Vaidya-Bonnor-de Sitter solution (6.62). These embedded de Sitter space-times can generate by using Wang-Wu functions in rotating solutions given in (6.9). This shows that Wang-Wu functions in rotating space-time geometry can be applied to generate Kerr-de Sitter and Kerr-Newman-de Sitter solutions in a simple way – not directly applying Newman-Janis algorithm to Schwarzschild-de Sitter as well as Reissner-Nordstrom-de Sitter ‘seed’ solutions to derive Kerr-de Sitter as well as Kerr-Newman-de Sitter solutions respectively. Thus, the rotating solutions (6.9) with Wang-Wu functions can avoid the difficulties suggested by Xu [9]. The definitions of embedded spaces used here are in agreement with the one defined by Cai et al. [46]. It is worth mentioning that the rotating embedded solution, namely Kerr-Newman-de Sitter solution (6.41) is found different from the ones discussed in [8,9,40]. Hence, to the best of the author’s knowledge, the rotating embedded solutions (6.41), (6.51), (6.62) and other reducible solutions from (6.62), and also (6.21), (6.30) have not been seen published before. Other non-rotating embedded solutions of Einstein’s equations can be found in Kramer, et al. [47] (and references there in) and Hodgkinson [48].

Looking at these overall rotating solutions derived above one can conclude that

1. all stationary rotating solutions including (a) Kerr-Newman, (b) rotating monopole, (c) rotating de Sitter, (d) Kerr-Newman de Sitter solutions which are derivable from the application of Newman-Janis algorithm, are Petrov type D and each spacetime has the repeated principal null vector ℓ_a , which is geodesic, shear free, expanding as well as rotating. This completes the proof of theorem 3.
2. rotating Vaidya (5.9), rotating Vaidya-Bonnor (5.13), Kerr-Newman-Vaidya (6.21), Kerr-Newman-Vaidya-Bonnor (6.26), rotating Vaidya-de Sitter (6.62) when $e(u) = 0$ and rotating Vaidya-Bonnor-de Sitter (6.62) are all non-stationary spherically symmetric solutions. Their Weyl curvature tensors are algebraically special in the Petrov classification with null vector ℓ_a given in (2.8). This leads the proof of the theorem 4 stated in the introduction.

The remarkable feature of the analysis of rotating solutions in this paper is that all the rotating solutions, stationary Petrov type D and non-stationary algebraically special, presented here possess the same null vector ℓ_a , which is geodesic, shear free, expanding as well as non-zero twist. From the studies of the rotating solutions we found that some solutions after making rotation have disturbed their gravitational structure. For example, the rotating monopole solution (6.11) possesses the energy-momentum tensor with the monopole pressure p , where the monopole constant b couples with

the rotating parameter a . Similarly, the rotating de Sitter solution (6.34) becomes Petrov type D spacetime metric, where the rotating parameter a is coupled with the cosmological constant and so on. We have shown that all the rotating embedded solutions presented here can be written in Kerr-Schild forms, showing the extension of those of Xanthopoulos [33] and of Glass and Krisch [23].

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References

- [1] Ibohal N 2002 *Class.Quantum Grav.* **19** 4327
- [2] Hawking S W 1974 *Nature* **248** 30
Hawking S W 1975 *Commun. Math. Phys.* **43** 199
- [3] Vaidya P C and Patel L K 1973 *Phys. Rev. D* **35** 1481.
- [4] Carmeli M and Kaye M 1977 *Ann.Phys. (N.Y.)* **103** 97
- [5] Herrera L and Martinez J 1998 *J. Math. Phys* **39** 3260
- [6] Herrera L, Hernandez H, Nunez L A and Percoco U 1998 *Class. Quantum Grav.* **15** 187
- [7] Gonzalez C, Kerrera l and Jimenez J 1979 *J. Math. Phys.* **20** 837
- [8] Mallett R L 1988 *Phys. Lett A* **126** 226
- [9] Xu D 1998 *Class. Quantum Grav.* **15**, 153
- [10] Jing J and Wang Y 1996 *Int. J. Theo. Phys* **35** 1481
- [11] Newman E T and Janis A I 1965 *J. Math. Phys.* **6** 915
- [12] Newman E T, Couch E, Chinnapared K, Exton A, Prakash A and Torrence R 1965 *J. Math. Phys.* **6** 918
- [13] Herrera L and Jimenez J 1982 *J. Math. Phys.* **23** 2339

- [14] Drake S P and Turolla R 1997 *Class Quantum. Grav.* **14** 1883
- [15] Drake S P and Szekeres P 2000 *Gen. Rel. Grav.* **32**, 445
- [16] Yazadjiev S 2000 *Gen. Rel. Grav.* **32**, 2345
- [17] Sen A 1992 *Phys. Rev. Lett.* **69**, 1006
- [18] Newman E T and Penrose R 1962 *J. Math. Phys.* **3**, 566
- [19] McIntosh C B G and Hickman M S 1985 *Gen. Rel. Grav.* **17**, 111
- [20] Chandrasekhar, S.: 1983. *The Mathematical Theory of Black Holes*, Clarendon Press, Oxford.
- [21] Wainwright J 1970 *Commun. math. Phys* **17** 42
- [22] Ibohal N 1997 *Astrophys SpaceSci* **249** 73
- [23] Glass E N and Krisch J P 1998 *Phys. Rev. D* **57** R5945;
Glass E N and Krisch J P 1999 *Class. Quantum Grav.* **16**, 1175
- [24] Joshi P S 1993 *Global Aspects in Gravitation and Cosmology* (Oxford, Clarendon)
- [25] Husain V 1996 *Phys. Rev. D* **53** R1759
- [26] Pereira R C T and Wang A 2000 *Gen. Rel. and Grav.* **32** 2189
- [27] Chan R, da Silva M F A and da Rocha J F V 2003 *Int. J Mod. Phys. D* **12** 347
- [28] Vaidya P C 1951 *Proc. Indian Acad. Sci.* **A33** 264; Reprinted 1999 *Gen. Rel. Grav* **31** 119
- [29] Cameli 1982 *Classical Fields, General Relativity and Gauge Theory* (New York, Wiley)
- [30] Wu S Q and Cai X 2001 *Gen. Rel. Grav* **33** 1181
- [31] Bonnor W and Vaidya P C 1970 *Gen. Rel. Grav.* **1** 127
- [32] Wang A and Wu Y 1999 *Gen. Rel. Grav.* **31** 107
- [33] Xanthopoulos B C 1978 *J. Math. Phys.* **19** 1607
- [34] Visser M 1992 *Phys. Rev. D* **46** 2445
- [35] Hawking S W, Hunter C J and Taylor-Robinson M 1999 *Phys. Rev. D* **59** 064005

- [36] Hawking S W and Ellis G F R 1973 *The large scale structure of space-time*, Cambridge University Press, Cambridge.
- [37] Guth A H 1981 *Phys. Rev D* **33** 347
- [38] Koberlin B D 1995 *Phys. Rev. D* **51** 6783
- [39] Carter B 1973 in *Black holes* edited by C Dewitt and B.C. Dewitt (New York, Gordon and Breach Science Publ)
- [40] Gibbons G W and Hawking S W 1977 *Phys. Rev. D* **15** 2738
- [41] Khanal U 1983 *Phys. Rev. D* **28** 1291
- [42] Mallett R L 1985 *Phys. Rev. D* **51** 416;
Koberlein B D and Mallett R L 1994 *Phys. Rev. D* **49** 5111
- [43] Ghosh S G and Dadhich N 2003 *Gen. Rel. Grav.* **35** 359
- [44] Patino A and Rago H 1987 *Phys Lett. A* **121** 329;
Zhong-heng L, You L and Li-qin M 1999 **38** 925;
Wu S Q and Cai X 2001 *Inter. J Math. Phys.* **40** 1349
- [45] Fernandes J. F. Q. and Lun A. W. C. 1997 *J. Math. Phys.* **38** 330
- [46] Cai R G, Ji J Y and Soh K S 1998 *Class Quantum Grav.* **15** 2783
- [47] Kramer D, Stephani H, Herlt E, MacCallum M and Schmutzer E 1980 *Exact solutions of Einstein's field equations* (Cambridge University Press, Cambridge)
- [48] Hodgkinson D E 2001 *J Math. Phys* **42** 863