Quantum effects, soft singularities and the fate of the universe in a braneworld cosmology

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Abstract. We examine a class of braneworld models in which the expanding universe encounters a "quiescent" future singularity. At a quiescent singularity, the energy density and pressure of the cosmic fluid as well as the Hubble parameter remain finite while all derivatives of the Hubble parameter diverge (i.e., $\dot{H}$, $\ddot{H}$, etc. $\to \infty$). Since the Kretschmann invariant diverges ($R_{iklm}R^{iklm} \to \infty$) at the singularity, one expects quantum effects to play an important role as the quiescent singularity is approached. We explore the effects of vacuum polarization due to massless conformally coupled fields near the singularity and show that these can either cause the universe to recollapse or, else, lead to a softer singularity at which H, $\dot{H}$, and $\ddot{H}$ remain finite while $\dddot{H}$ and higher derivatives of the Hubble parameter diverge. An important aspect of the quiescent singularity is that it is encountered in regions of low density, which has obvious implications for a universe consisting of a cosmic web of high and low density regions — superclusters and voids. In addition to vacuum polarization, the effects of quantum particle production of non-conformal fields are also likely to be important. A preliminary examination shows that intense particle production can lead to an accelerating universe whose Hubble parameter shows oscillations about a constant value.

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1. Introduction

Braneworld models give rise to interesting new physical effects [1, 2]. The well-known Dvali–Gabadadze–Porrati (DGP) model, for instance, can lead to an accelerating universe without the presence of either a cosmological constant or some other form of dark energy [3]. Generalizations of the DGP model can result in a phantom-like acceleration of the universe at late times [4, 5]. This class of models can also lead to new cosmological behaviour at *intermediate redshifts*: the loitering [6] and mimicry [7] scenarios provide examples of cosmologies which are close to LCDM at late times but can show significant departure from LCDM-like expansion at $z \gtrsim$ few. In both cases, the age of the high-redshift universe turns out to be larger than in LCDM, while the redshift of reionization is lower. Whether the universe has properties which are easier to explain within the braneworld context is an intriguing possibility demanding further exploration. In this paper, we examine yet another property of braneworld models which does not have parallels in general relativity: the possibility that the universe may encounter a *quiescent future singularity* as it expands [8].

It is well known that the rate of expansion of the universe and its ultimate fate depend on the system of equations governing evolution (general relativity, scalar-tensor theory, braneworld theory etc.) as well as on the form of matter and, particularly, on its equation of state. In general relativity (GR), if one assumes that matter satisfies the strong energy condition (SEC) $\rho+3p \geq 0$, then, within a Friedmann–Robertson–Walker (FRW) setting, the evolution of the universe is strongly dependent upon the spatial curvature: a spatially closed universe turns around and collapses whereas open and flat cosmologies continue to expand forever. The situation becomes more complicated (and interesting!) if one of the following assumptions is made: (i) the expansion of the universe is not governed by GR, (ii) matter can violate the SEC and even the weak energy condition (WEC) $\rho + p \geq 0$. In the latter case, if $w = p/\rho < -1$, then the expanding universe can encounter a "big rip" future singularity at which the density, pressure, and Hubble parameter diverge. In the former case, if the equations of motion have been derived from a braneworld action [4], then the expanding universe can encounter a different kind of (quiescent) future singularity, at which the density, pressure and Hubble parameter *remain finite*, but derivatives of the Hubble parameter, including $\dot{H}$, diverge as the singularity is approached [8]. The occurrence of this singularity is related to the fact that the equations of motion are no longer quasi-linear (as they are, for instance, in GR) but include terms which are non-linear in the highest derivative. Examples of such singularities can be found in models other than the braneworld model; for instance, [9] refer to singularities in which the deceleration parameter tends to infinity as the "Big Brake" while, in [10], they are called "sudden" singularities (see also [11]). The geometrical reason of a quiescent singularity in braneworld models is connected with the fact that the brane embedding in the bulk becomes singular at some point (see [8] for details). Since quiescent singularities can occur both in the past and in the future, they might provide an interesting alternative to the more conventional "big bang"/"big crunch" singularities of general relativity. An important distinction between the quiescent and the sudden singularity is the following: The existence of the quiescent singularity does not require matter with unusual properties, hence, both density and pressure remain finite near this singularity. For the sudden singularity, on the other hand, the pressure diverges as the singularity is approached, implying the presence of matter with exotic properties.

In this paper, we examine the issue of how quantum effects might influence a braneworld which encounters a quiescent singularity during expansion. It is well known that quantum effects come into play when the space-time curvature becomes large, as happens, for instance, in the vicinity of a black hole or near the Big-Bang and Big-Crunch singularities of general relativity [13]. Since $R_{iklm}R^{iklm} \to \infty$ as one approaches a quiescent (sudden) singularity, one might expect quantum effects to become important in this case too (see, for instance, [8, 14]). As we demonstrate in this paper, quantum corrections to the equations of motion at the semi-classical level result in several important changes in the evolution of the universe: due to the (local) effects of vacuum polarization, (i) the quiescent singularity changes its form and becomes a much weaker "soft" singularity, at which H and $\dot{H}$ remain finite but $\dddot{H} \to \infty$; (ii) vacuum polarization effects can also cause a spatially flat universe to turn around and collapse. Both (i) and (ii) demonstrate that the incorporation of quantum effects into the braneworld equations of motion can radically alter the future of the universe and lead to behaviour which differs significantly from general-relativistic cosmology. Since the quiescent singularity (and the associated quantum effects) arise as the density drops below a threshold value, it follows that those regions which are significantly underdense (voids) may be the first to encounter the quiescent singularity. We also briefly discuss whether particle production effects could be significant as the universe approaches a quiescent future singularity. An interesting possibility that may arise in this case is that the universe expands in a regime in which the Hubble parameter vacillates about the de Sitter value. ‖

2. Equations of motion

We consider the simplest generic braneworld model with action of the form

$$S = M^3 \left[\int_{\rm bulk} (\mathcal{R} - 2\Lambda_{\rm b}) - 2 \int_{\rm brane} K \right] + \int_{\rm brane} \left(m^2 R - 2\sigma \right) + \int_{\rm brane} L\left(h_{ab}, \phi\right) . \qquad (1)$$

Here, $\mathcal{R}$ is the scalar curvature of the metric g_{ab} in the five-dimensional bulk, and R is the scalar curvature of the induced metric $h_{ab} = g_{ab} - n_a n_b$ on the brane, where n^a is the vector field of the inner unit normal to the brane, which is assumed to be a boundary of the bulk space, and the notation and conventions of [15] are used. The quantity $K = h^{ab} K_{ab}$ is the trace of the symmetric tensor of extrinsic curvature

‖ It should be noted that braneworld models which approach the first quiescent singularity in the future (and have Friedmann-like behaviour in the past) appear to be in some tension with recent observational data [12]. However, the observational status of the oscillatory model discussed in Sec. 5 remains to be studied.

$K_{ab} = h^c{}_a \nabla_c n_b$ of the brane. The symbol $L(h_{ab}, \phi)$ denotes the Lagrangian density of the four-dimensional matter fields ϕ whose dynamics is restricted to the brane so that they interact only with the induced metric h_{ab}. All integrations over the bulk and brane are taken with the corresponding natural volume elements. The symbols M and m denote the five-dimensional and four-dimensional Planck masses, respectively, $\Lambda_{\rm b}$ is the bulk cosmological constant, and σ is the brane tension.

Action (1) leads to the Einstein equation with cosmological constant in the bulk:

$$\mathcal{G}_{ab} + \Lambda_{\rm b} g_{ab} = 0 \,, \tag{2}$$

while the field equation on the brane is

$$m^2 G_{ab} + \sigma h_{ab} = T_{ab} + M^3 \left(K_{ab} - h_{ab} K \right) \,, \tag{3}$$

where T_{ab} is the stress–energy tensor of matter on the brane stemming from the last term in action (1), i.e.,

$$T_{ab} = \frac{1}{\sqrt{h}} \frac{\delta \int_{\rm brane} L\left(h_{ab}, \phi\right)}{\delta h^{ab}} \,. \tag{4}$$

The five-dimensional bulk, satisfying Eq. (2), is described by the metric

$$ds^2_{\rm bulk} = - f(r) d\tau^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_\kappa^2 \,, \quad f(r) = \kappa - \frac{\Lambda_{\rm b}}{6} r^2 - \frac{C}{r^2} \,. \tag{5}$$

Here, $\kappa = 0, \pm 1$ is the sign of the spatial curvature of the brane, $d\Omega_\kappa^2$ denotes the metric of the maximally symmetric three-dimensional Euclidean space with constant curvature corresponding to κ, and the constant C, if it is nonzero, corresponds to the presence of a black hole in the bulk. The trajectory of the brane in the bulk is given by $r = a(\tau)$, and then the brane is made the boundary by discarding either the region $r > a(\tau)$ or the region $r < a(\tau)$ from the bulk, resulting in two possible cosmological branches (see Eq. (10) below).

The cosmological evolution on the brane that follows from Eqs. (2) and (3) can be encoded in a single equation [4, 16]

$$\left(H^2 + \frac{\kappa}{a^2} - \frac{\rho + \sigma}{3m^2} \right)^2 = \frac{4}{\ell^2} \left(H^2 + \frac{\kappa}{a^2} - \frac{\Lambda_{\rm b}}{6} - \frac{C}{a^4} \right) \,, \tag{6}$$

where $H \equiv \dot{a}/a$ is the Hubble parameter, and ρ is the matter energy density on the brane. Here and below, the overdot derivative is taken with respect to the cosmological time t on the brane, which is connected with the bulk time τ through the obvious relation

$$\frac{dt}{d\tau} = \sqrt{f(a) - \frac{\left(da/d\tau\right)^2}{f(a)}} \,. \tag{7}$$

The term containing the constant C describes the so-called "dark radiation." The length scale ℓ is defined as

$$\ell = \frac{2m^2}{M^3} \,. \tag{8}$$

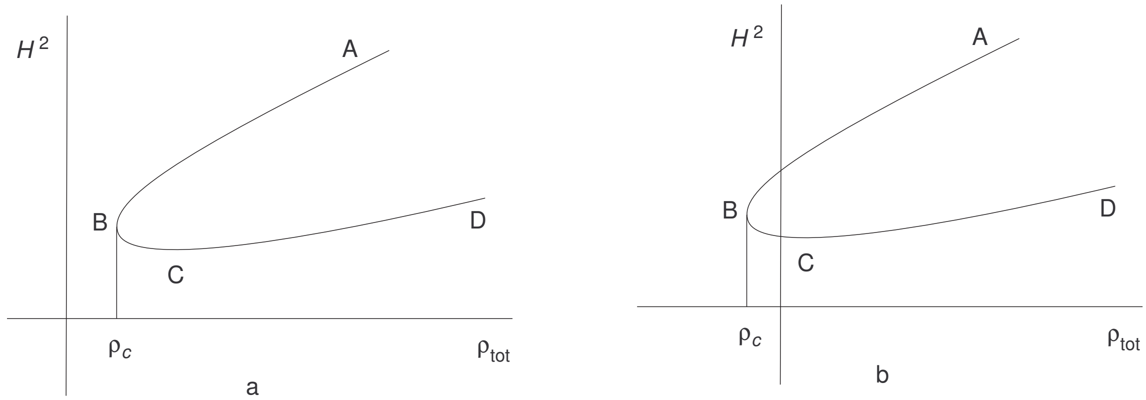

Figure 1. Plot of relation (9) in the case $\Lambda_b > 0$. Case (a) corresponds to $\rho_c > 0$, and case (b) corresponds to $\rho_c < 0$, where ρ_c is given by Eq. (12).

In what follows, we consider a spatially flat universe ($\kappa = 0$) without dark radiation ($C = 0$). Then Eq. (6) takes the form

$$\left(H^2 - \frac{\rho + \sigma}{3m^2} \right)^2 = \frac{4}{\ell^2} \left(H^2 - \frac{\Lambda_b}{6} \right) . \tag{9}$$

This equation can be solved with respect to the total energy density on the brane $\rho_{tot} \equiv \rho + \sigma$:

$$\frac{\rho + \sigma}{3m^2} = H^2 \pm \frac{2}{\ell} \sqrt{H^2 - \frac{\Lambda_b}{6}} . \tag{10}$$

The "$\pm$" signs in the solution correspond to two branches defined by the two possible ways of bounding the Schwarzschild–(anti)-de Sitter bulk space by the brane, as described above [17, 18]. Discarding the region $r > a(\tau)$ (the region $r < a(\tau)$) from the bulk corresponds to the "+" ("−") sign in (10).

Alternatively, Eq. (9) can be solved with respect to H^2 with the result [4, 17, 19]

$$H^2 = \frac{\rho + \sigma}{3m^2} + \frac{2}{\ell^2} \left[1 \pm \sqrt{1 + \ell^2 \left(\frac{\rho + \sigma}{3m^2} - \frac{\Lambda_b}{6} \right)} \right] . \tag{11}$$

Models with the lower ("−") sign in this equation were called Brane 1, and models with the upper ("+") sign were called Brane 2 in [4], and we refer to them in this way throughout this paper.

3. Classical dynamics of the Brane

Before we study quantum corrections to brane equations of motion, we describe possible classical dynamical regimes. Classical dynamics depends significantly on the sign of the bulk cosmological constant Λ_b. In the ensuing discussion, we shall examine separately all three cases, namely $\Lambda_b > 0$, $\Lambda_b = 0$, and $\Lambda_b < 0$.

- The case $\Lambda_{\rm b} > 0$ is shown in Fig. 1. The graph of (9) in the $(H^2, \rho_{\rm tot})$ plane in Fig. 1 illustrates that in an expanding universe the matter density ρ decreases (except for a "phantom matter" which we do not consider in the present paper), and the point in the plane $(H^2, \rho_{\rm tot})$ moves from right to left in Fig. 1.

  A striking feature of Fig. 1 is that the value of the Hubble parameter in the braneworld can *never drop to zero*. In other words, the Friedmann asymptote $H \to 0$ is absent in our case. The upper and lower branches in Fig. 1 describe the two complementary braneworld models: branches AB and DB are associated with Brane 2 and Brane 1 of [4], respectively, while branches AC and DC correspond to the lower and upper signs in (10), respectively, and describe the two branches with different embedding in the bulk. It should be noted that, in many important cases, the behaviour of the braneworld does not have any parallel in conventional Friedmannian dynamics (by this we mean standard GR in a FRW universe). For instance, the BC part of the evolutionary track corresponds to "phantom-like" cosmology with $\dot{H} > 0$, even though matter on the brane never violates the weak energy condition.

  We would like to draw the reader's attention to the fact that, for a given $\rho_{\rm tot}$, a solution $H^2(\rho_{\rm tot})$ exists if and only if ¶

  $$\rho_{\rm tot} \geq \rho_c \equiv 3m^2\left(\Lambda_{\rm b}/6 - \ell^{-2}\right) \tag{12}$$

  (see Fig. 1). This leads to two distinct possibilities for the late-time cosmological evolution of the braneworld. (i) If $\sigma > \rho_c$, then nothing prevents the matter density from diluting to $\rho \to 0$ at late times. In this case, the braneworld approaches a De Sitter-like future attractor at which $H \to$ constant. (ii) In the opposite case, when $\sigma < \rho_c$, the braneworld dynamics is very different. In this case, Eq. (12) can be rewritten as $\rho \geq \rho_c - \sigma$, which implies that the density of matter can only drop to $\rho_c - \sigma$ and no further! Indeed, at $\rho \to \rho_c - \sigma$, the universe experiences a "quiescent" singularity, at which the density ρ and the Hubble parameter H remain finite, while derivatives of the H, including $\dot{H}$, $\ddot{H}$ etc., diverge [8]. (For $\Lambda_{\rm b} < 6\ell^{-2}$, we have $\rho_c < 0$, and, in order for the quiescent future singularity to exist, the brane tension σ must be negative (see Fig. 1b).)

- The case $\Lambda_{\rm b} = 0$ is shown in Fig 2. The Minkowski bulk ($\Lambda_{\rm b} = 0$) leads to two generic future attractors, namely, the De Sitter-like Brane 2 [3] and the Friedmann-like braneworld evolving to the point C (with $H \to 0$ and $\rho \to 0$). However, a sufficiently large negative brane tension ($\sigma < -3m^2/\ell^2$) can also trigger the formation of a quiescent future singularity at the point B.

- The case $\Lambda_{\rm b} < 0$ is shown in Fig. 3. A very interesting situation arises in this case since the Hubble parameter can reach zero within a finite interval of time with $\dot{H} < 0$ subsequently. This situation describes the recollapse of the universe.

  It is worth stressing that the recollapse of the braneworld is a consequence of the modified expansion law (11) and is not due to the presence of a spatial curvature

¶ This corresponds to the quantity under the square root in (11) being non-negative.

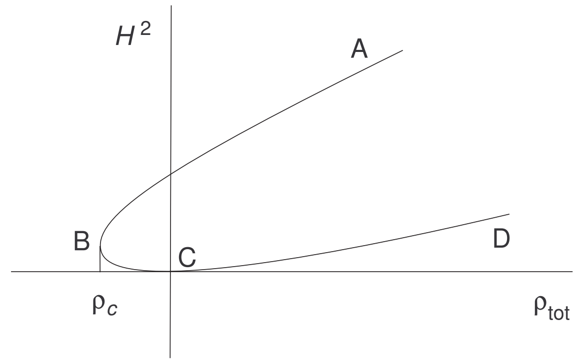

Figure 2. Plot of relation (9) in the case $\Lambda_b = 0$. The point C dividing the branches with different embedding corresponding to different signs in (10) is at the origin of coordinates.

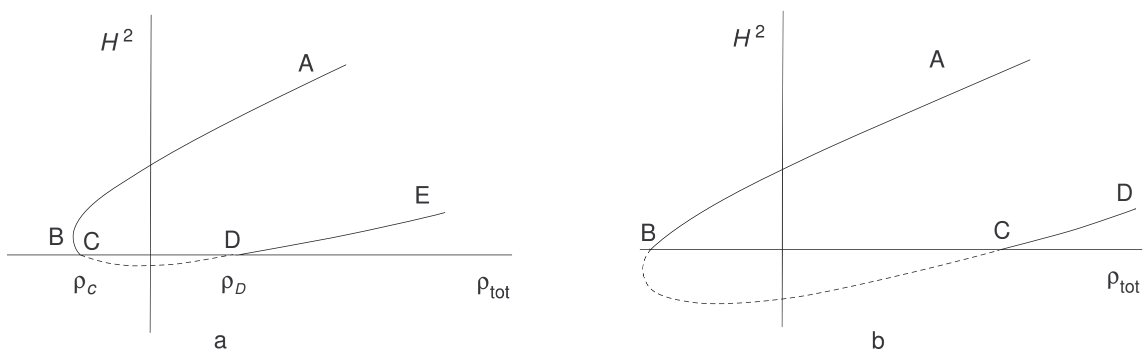

Figure 3. Plot of relation (9) in the case $\Lambda_b < 0$. Case (a) corresponds to the condition $|\Lambda_b| < 6\ell^{-2}$, and case (b) corresponds to the condition $|\Lambda_b| > 6\ell^{-2}$. The dashed part corresponds to the forbidden nonphysical branch.

term — the recollapsing braneworld is spatially flat!
It is easy to show that Brane 1 (lower branch) invariably recollapses provided the brane tension is less then $\rho_D = M^3\sqrt{-3\Lambda_b/2}$, which is the energy density at the point D in Fig. 3a. The critical case $\sigma = \rho_D$ leads to a Friedmann late-time dynamics [20]. The Brane 2 case (the upper branch AB), however, leads to two different possibilities. Furthermore, this difference is important if we allow for the existence of negative brane tensions. For instance, in Fig. 3a, a Brane 2 evolves along the AB part and encounters a quiescent singularity at the point B. This requires negative tension $\sigma < 3m^2(\Lambda_b/6 - \ell^{-2})$ [8]. The part BC corresponds to a hypothetical universe which is born at the singularity B, expands to the point C, then contracts and ends up in the quiescent singularity at B. The BC track is characterized by the negative derivative of $\rho_{\rm tot}$ as a function of H^2. Simple calculations lead to the condition $|\Lambda_b| < 6\ell^{-2}$ for the existence of this branch. For $|\Lambda_b| > 6\ell^{-2}$, the situation is presented in Fig. 3b. In this case, the Brane 2 universe

also recollapses (provided the brane tension is negative and sufficiently large by absolute value).

4. Quantum corrections to the equations of motion

It is well known that quantum effects, including vacuum polarization and particle production, generically occur within regions of strong space-time curvature such as in the vicinity of black holes and near the cosmological Big-Bang singularity. As we have seen earlier, the braneworld — in addition to having the usual Big-Bang and Big-Crunch singularities — also possesses quiescent singularities, which an observer can encounter while the universe *is still expanding*. This singularity is specific to the braneworld, since the density of matter, its pressure, and the Hubble parameter all freeze to constant values, whereas $\dot{H}$, $\ddot{H}$ etc. diverge as the singularity is approached. Since this particular singularity can also develop in an accelerating universe akin to ours [8], it is of considerable interest to ask whether the nature of the quiescent singularity will in any way be affected once quantum effects are incorporated into the treatment.

In general, quantum effects in curved space-time can arise on account of the vacuum polarization as well as particle production. It is well known that the latter is absent for conformally invariant fields (which we shall consider in this section) and that, in this case, quantum corrections to the equations of motion are fully described by the renormalized vacuum energy–momentum tensor which has the form [13]

$$\langle T_{ab} \rangle = k_1 H^{(4)}_{ab} + k_2 H^{(3)}_{ab} + k_3 H^{(1)}_{ab} \, , \tag{13}$$

where

$$H^{(1)}_{ab} \equiv 2 D_a D_b R - 2 h_{ab} D^c D_c R + \frac{1}{2} h_{ab} R - 2 R R_{ab} \, , \tag{14}$$

$$H^{(3)}_{ab} \equiv \, - R^c{}_a R_{cb} + \frac{2}{3} R R_{ab} + \frac{1}{2} h_{ab} R^{cd} R_{cd} - \frac{1}{4} h_{ab} R^2 \, , \tag{15}$$

and $H^{(4)}_{ab}$ is a local non-geometric tensor which depends upon the choice of the vacuum state and has vanishing trace $\left(H^{(4)a}_{a} = 0 \right)$; k_1, k_2, k_3 depend upon the spin weights of the different fields contributing to the vacuum polarization.

Equations (14) and (15) lead to the following vacuum expectation value for the energy density:

$$\rho_q \equiv \langle T_{00} \rangle = k_2 H^4 + k_3 \left(2 \ddot{H} H + 6 \dot{H} H^2 - \dot{H}^2 \right) \, . \tag{16}$$

In order to assess the effects of the vacuum polarization on the dynamics of the braneworld, one must add ρ_q to the matter density in (9), (11) or (10) so that $\rho \to \rho + \rho_q$ in those equations. An important consequence of this operation is that the form of the equation of motion changes dramatically — the original algebraic equation changes to a differential equation! The dynamical equation (10) now takes the form

$$\ddot{H} H = \frac{1}{2} \dot{H}^2 - 3 \dot{H} H^2 + (2 k_3)^{-1} \left(- k_2 H^4 + 3 m^2 H^2 - \rho_{\rm tot} \pm 3 M^3 \sqrt{H^2 - \Lambda_{\rm b}/6} \right) \, . \tag{17}$$

The two signs in (17) correspond to two *different* dynamical equations. As explained in Sec. 2, the sign is fixed by specifying one of the two possible ways of embedding the brane in the bulk.

The goal of the present paper is to study the stability of the classical solutions when vacuum polarization terms are taken into account. The k_2-term in (16), which does not contain time derivatives of H, can only change the position of the future stable points. On the contrary, due to the k_3-term in (16), some classical solutions can lose stability. Therefore, for simplicity (and without loss of generality), we set $k_2 = 0$ in our calculations. This assumption simplifies the situation significantly since, in this case, the stationary points of Eq. (17) are precisely the solutions of the classical equation (9) with $\rho = 0$.

For $m = 0$ (the Randall–Sundrum brane), Eq. (17) becomes

$$\ddot{H}H = \frac{1}{2}\dot{H}^2 - 3\dot{H}H^2 + \left(-k_2H^4 - \rho_{\rm tot} \pm 3M^3\sqrt{H^2 - \Lambda_{\rm b}/6}\right)\Big/\, 2k_3 \qquad (18)$$

We can see immediately that the stationary points can exist only for the "+" sign in (18). This statement is valid also for $k_2 > 0$. A negative value of k_2 leads to a De Sitter solution, which has no analogs in the classical case, so we do not consider it here. By linearizing the brane equation of motion near the classical stationary point, it is easy to find that the condition for this stationary point to be stable is $k_3 < 0$, which is the same condition as in the standard cosmology.

We now turn to the case $m \neq 0$. If the brane has nonzero tension σ, the stationary points of (17) in the case $k_2 = 0$ can be found by substituting σ into (6) and setting $\rho = 0$. After that, we linearize Eq. (17) at these stationary points and find the eigenvalues of the corresponding linearized system. The condition of stability of the stationary point is that its eigenvalues are negative.

The eigenvalues at the stationary points where $\dot{H} = 0$ are given by

$$\mu_{1,2} = \frac{1}{2}\left(f_1 \pm \sqrt{f_1^2 + 4f_2}\right), \qquad (19)$$

where we have made the notation

$$f_1 = -3H\,, \qquad (20)$$

$$f_2 = \frac{1}{2k_3}\left(1 + \frac{\lambda}{H^2} \pm \frac{2\ell^{-1}\Lambda_{\rm b}/6}{H^2\sqrt{H^2 - \Lambda_{\rm b}/6}}\right), \qquad (21)$$

$$\lambda = \frac{\sigma}{3m^2}\,. \qquad (22)$$

Two different signs in Eq. (21) correspond to two different equations of motion, while, in Eq. (19), we have two different eigenvalues of a single equation.

Since f_1 is negative, the eigenvalue μ_2 corresponding to the "−" sign in Eq. (19) is also always negative. Moreover, μ_1 is positive if and only if f_2 is positive. As a result, the stability of a fixed point is equivalent to the condition $f_2 < 0$.

From now on, we consider only the second eigenvalue, μ_2, so the two signs in the expressions given below always corresponds to two different equations of motion.

The equations for the fixed points are

$$H - \frac{\lambda}{H} \pm \frac{2\ell^{-1}}{H}\sqrt{H^2 - \Lambda_{\rm b}/6} = 0\,. \tag{23}$$

Substituting the value of λ obtained from this equation into (21), we can rewrite it in the form

$$f_2 = \frac{1}{2k_3}\left(2 \pm \frac{2\ell^{-1}}{\sqrt{H^2 - \Lambda_{\rm b}/6}}\right)\,. \tag{24}$$

We classify solutions corresponding to the upper sign ("+") and the lower sign ("−") as the "+" branch and "−" branch, respectively. Then, for the "+" branch, the expression in the brackets is positive, so we make our first conclusion:

- Fixed points of the "+" branch are stable with respect to quantum corrections if $k_3 < 0$.

Proceeding further, we solve Eq. (23) with respect to H^2. The solutions are given by

$$H^2 = \lambda + 2\ell^{-2} \pm 2\ell^{-1}\sqrt{\lambda - \Lambda_{\rm b}/6 + \ell^{-2}}, \tag{25}$$

however, it is necessary to be careful about the two different branches. There are two possible cases:

(i) $\lambda > \Lambda_{\rm b}/6$. In this case, we have one solution for the "+" branch (with the upper sign in (25)), and one solution for the "−" branch (with the lower sign in (25)).

(ii) $\Lambda_{\rm b}/6 - \ell^{-2} < \lambda < \Lambda_{\rm b}/6$. In this case, two solutions (25) belong to the "−" branch, while the "+" branch does not have fixed points.

For $\lambda < \Lambda_{\rm b}/6 - \ell^{-2}$, there are no fixed points for both equations of motion.

Consider the first situation. We already know that stability of a fixed point for the "+" branch requires $k_3 < 0$. For the "−" branch, we have

$$f_2 = \frac{1}{2k_3}\left(2 - \frac{2\ell^{-1}}{\sqrt{\lambda - \Lambda_{\rm b}/6 + 2\ell^{-2} + 2\ell^{-1}\sqrt{\lambda - \Lambda_{\rm b}/6 + \ell^{-2}}}}\right) \tag{26}$$

at a fixed point. As we have now $\lambda - \Lambda_{\rm b}/6 > 0$, the second term in the brackets is always smaller than $\sqrt{2}$, so the whole expression in the brackets is positive. This implies the following conclusion:

- If the "−" branch has only one stable point, its stability requires $k_3 < 0$.

For the second case, both solutions (25) belong to the "−" branch. The upper sign gives the same expression for f_2 as in (26). The condition $\lambda - \Lambda_{\rm b}/6 + \ell^{-2} > 0$ makes the second term in the brackets of (24) smaller then 2, and again the total number in the brackets is positive. On the other hand, for the lower sign in (25), we have

$$f_2 = \frac{1}{2k_3}\left(2 - \frac{2\ell^{-1}}{\sqrt{\lambda - \Lambda_{\rm b}/6 + 2\ell^{-2} - 2\ell^{-1}\sqrt{\lambda - \Lambda_{\rm b}/6 + \ell^{-2}}}}\right)\,. \tag{27}$$

One can see that the number in the brackets is negative. As a result, we arrive at the following conclusion:

- If two stationary points (25) belong to the "$-$" branch, then one of them is stable and the other one is unstable.

Having in mind the description of the fixed points given in Sec. 3, we obtain the following picture:

The simplest case corresponds to the classical points described by Fig. 3b. Note that now the horizontal axis represents σ, because we are dealing with the stationary points only. Each equation, with "+" and "$-$" sign, has only one possible stationary point, and these are stable for $k_3 < 0$.

For the range of the parameter $\Lambda_{\rm b}$ corresponding to Fig. 3a, stationary points of the "$-$" sign of Eq. (17) are located in the DE part. Their stability also requires $k_3 < 0$. However, stationary points of the equations of motion with "+" sign are located in the whole admissible region AC. For $\sigma < \rho_c$ (see Fig. 3a), we have two possible stationary points, one on the AB branch, and the other on the BC branch. Their stability properties are opposite to each other. In particular, the unusual condition $k_3 > 0$ is required for the points on BC branch to be stable. This conclusion is, however, not so important because the branch BC cannot be reached as a result of conventional cosmological evolution in the classical picture.

More interesting situation is realized for $\Lambda_{\rm b} > 0$ (see Fig. 1). Equation (17) with "$-$" sign again has stationary points on the CD branch, and that with "+" sign has it on the AC branch. If $k_3 < 0$, points on the AB branch are stable, and those on the BC branch are unstable; for $k_3 > 0$, the situation is opposite. However, now classical and quantum dynamics differs significantly. In the classical picture, the BC branch is part of the evolution track of the Brane 2, and it can be reached during a cosmological evolution from some high-energy initial phase. The effects of vacuum polarization make this branch unreachable for a braneworld with $k_3 < 0$.

These results can be interpreted from a different point of view. It is reasonable to assume that the evolution of a brane should be close to the classical picture in regions which are far removed from singularities. That is why we are interested only in the case $k_3 < 0$. What is the fate of a braneworld if we take quantum corrections into account? In general, this problem requires numerical integration of Eq. (17); however, we can make several qualitative statements. Fig. 3b contains only stable branches, so the dynamics with quantum corrections is qualitatively the same, with recollapse somewhere in the neighbourhood of the points B and D. In the case shown in Fig. 3a, the Brane 1 evolution DE does not change much and results in a recollapse. The Brane 2 branch AB in the classical picture meets a quiescent future singularity with H finite and $\dot{H} \to -\infty$ at the point B. Such a singularity is absent in the quantum picture because the expression under the square root in (17) is positive if $\Lambda_{\rm b} < 0$. Numerical simulations show that the ultimate fate of the Brane 2 universe in the case with $\Lambda_{\rm b} < 0$ and with a large negative σ is also a recollapse.

On the other hand, in the case $\Lambda_b > 0$, recollapse becomes impossible since the existence of a square root in (17) requires $H^2 > \Lambda_b/6$. This restriction is valid also for the classical equation (6). However, during the classical evolution, a braneworld reaches the point C, where $H^2 = \Lambda_b/6$, with $\dot{H} = 0$, and then enters the CB branch, which corresponds to super-exponential expansion ($\dot{H} > 0$) but which is impossible in the quantum case. Our numerical results show that, instead, the universe reaches some point with $H^2 = \Lambda_b/6$ and $\dot{H} < 0$. This causes the square root in (17) to vanish, which, in turn, leads to the divergence of $\dddot{H}$, while $\ddot{H}$ remains finite.[+] As a result, instead of a quiescent future singularity with $|\dot{H}| \to \infty$ at the point B, the universe encounters a much weaker singularity with $\dddot{H} \to \infty$ in the neighborhood of the point C. Similarly, the Brane 2 universe misses the point B and falls into a singularity with $\dddot{H} \to \infty$.

5. Quiescent singularities in an inhomogeneous universe: a preliminary analysis

The preceding discussion focussed on a homogeneous and isotropic universe whose expansion was governed by the brane equations of motion. Since the real universe is quite inhomogeneous on spatial scales $\lesssim 100$ Mpc, it is worthwhile to ask whether any of our previous results may be generalized to this case.

Although we are not yet able to provide a self-consistent treatment of the brane equations for this important case, still, some aspects of the problem can be discussed at the phenomenological level. Consider, for instance, the expansion law (11)

$$H^2 = \frac{\rho+\sigma}{3m^2} + \frac{2}{\ell^2}\left[1 \pm \sqrt{1 + \ell^2\left(\frac{\rho+\sigma}{3m^2} - \frac{\Lambda_b}{6}\right)}\right]$$
$$= \frac{\Lambda_b}{6} + \frac{1}{\ell^2}\left[\sqrt{1 + \ell^2\left(\frac{\rho+\sigma}{3m^2} - \frac{\Lambda_b}{6}\right)} \pm 1\right]^2 , \qquad (28)$$

A necessary condition for the existence of a quiescent singularity is that the matter density ρ drop to a value which is small enough for the square root on the right-hand side of (28) to vanish. When this happens, the universe encounters the quiescent singularity at which ρ and H remain finite, but $\ddot{a}$ and higher derivatives of the scale factor diverge. Note, however, that, according to (28), the universe encounters the quiescent singularity *homogeneously*, i.e., every part of the (spatially infinite) universe encounters the singularity at *one and the same* instant of time. This follows from the fact that the density in (28) depends only upon the cosmic time and upon nothing else. In practice, however, the universe is anything but homogeneous, its density varying from place to place. For instance, it is well known that the density of matter in galaxies is $\gtrsim 10^6$ times the average value while, in voids, it drops to only a small fraction of the average value. This immediately suggests that the brane should encounter the quiescent

[+] A similar result was obtained in [14] for the case of the Randall–Sundrum braneworld model containing matter with unusual properties resulting in sudden singularity.

singularity in a very inhomogeneous fashion. Underdense regions (voids) will be the first to encounter the singularity. Even in this case, since the density in individual voids is inhomogeneously distributed, more underdense regions lying closer to the void center will be the first to experience the singularity. It therefore follows that the quiescent singularity will first form near the centers of very underdense regions. As the void expands, its density at larger radii will drop below ρ_s, where

$$\sqrt{1 + \ell^2 \left(\frac{\rho_s + \sigma}{3m^2} - \frac{\Lambda_{\rm b}}{6} \right)} = 0\,; \tag{29}$$

consequently, the singularity will propagate outward from the void center in the form of a quasi-spherical singular front. (For simplicity, we have assumed that all voids have a spherical density profile; this assumption may need to be modified for more realistic cases; see [21, 22] and references therein.)

The above approach provides us with a very different perspective of the quiescent singularity than that adopted in the previous sections (and in [8]). For one thing, the singularity may be present in certain regions of the universe *right now*, so it concerns us directly (as astrophysicists) and not as some abstract point to which we may (or may not) evolve in the distant future. The second issue is related to the first, since the singularity could already exist within several voids (there are as many as a million voids in the visible universe in at least some of which the condition $\rho \simeq \rho_s$ could be satisfied), a practical observational strategy needs to be adopted to search for singularities in voids. (Similar strategies combined with strenuous observational efforts have led to the discovery of dozens of black holes in the centers of galaxies [23].)

A number of important issues therefore need to be addressed:

(i) Since $R_{iklm}R^{iklm} \to \infty$ within a finite region at the very center of a void, it follows that, unless this region is contained within an event horizon, we will find ourselves staring at a naked singularity! (As shown earlier, quantum effects do soften the singularity so that $R_{iklm}R^{iklm}$ may remain finite if these effects are included.)

(ii) In the discussion in Sec. 4, the issue of particle production was ignored since it was assumed that we were dealing with conformally coupled fields which are not created quantum mechanically in the (conformally flat) homogeneous and isotropic universe which we have been considering up to now. However, the moment we drop the homogeneity assumption, the issue of particle production immediately crops up, and we must take it into account if our treatment is to be at all complete [24]. (In a related context, the quantum creation of gravitons takes place even in a homogeneous and isotropic universe, since these fields couple minimally, and not conformally, to gravity [25].)

Let us discuss the possible effect of particle production in more detail. First, we consider the model of homogeneous universe taking it as an approximation to the situation inside an underdensity region (void). Gravitational quantum particle production occurs as the singularity is approached. Since the local value of the Hubble parameter remains finite at the singularity, production of the conformally coupled particles (like photons) is expected to be negligible. However, particles that are non-conformally coupled to gravity (which could be, for example, Higgs bosons in the Standard Model) will be copiously produced as the acceleration of the universe $\ddot{a}$ rapidly increases. The rate of particle production depends not only on their coupling to gravity but also on their coupling between themselves. Gravitationally created primary particles will decay into conformally coupled secondaries (electrons, photons, neutrino, etc.), which will influence the rate of production of the primaries by causing decoherence in their quantum state. The whole process is thus not easy to calculate in detail. However, from very general arguments it can be seen that creation of matter due to quantum particle production is important for the dynamics of the universe during its later stages. *

For the sake of physical simplicity, we restrict ourselves to the case of vanishing bulk cosmological constant $\Lambda_{\rm b}$ and write Eq. (28) in the form

$$H = \frac{1}{\ell} + \sqrt{\frac{\Delta\rho}{3m^2}}\,, \tag{30}$$

where

$$\Delta\rho = \rho - \rho_s\,, \quad \rho_s = -\sigma - \frac{3m^2}{\ell^2}\,, \tag{31}$$

and where we have chosen the physically interesting "+" sign in Eq. (28). Thus, we have two free parameters in our theory, namely, ℓ and ρ_s. The value of m is assumed to be of the order of the Planck mass. In this case the early-time behaviour of the universe follows the standard Friedmann model, as can be seen from (28) or (30).

Let the average particle energy density production rate be $\dot{\rho}_{\rm prod}$. Then, differentiating Eq. (30), we obtain

$$\dot{H} = \frac{\dot{\rho}_{\rm prod} - \gamma H \rho}{2\left(3m^2\Delta\rho\right)^{1/2}}\,, \tag{32}$$

where $\gamma > 0$ corresponds to the effective equation of state of matter in the universe: if $p = w\rho$, then $\gamma = 3(1+w)$. The second term in the numerator of (32) follows from the conservation law and describes the effect of the universe expansion on the matter density. (Note that ρ includes contributions from quantum and classical matter.)

In order to qualitatively assess the effects of particle production, let us examine two fundamentally distinct possibilities.

(i) Suppose that, in the course of evolution, $\Delta\rho \to 0$ is reached in a finite interval of time. Since the Hubble parameter is a unique function of the energy density, given by (30), and since the singularity value ρ_s of the energy density is approached from

* Effects of particle production are negligible in the neighbourhood of the usual cosmological singularity of the Friedmann universe because the energy density of ordinary matter and radiation strongly diverges and thus dominates at this singularity [24, 13]. In our case, the energy density of ordinary matter remains finite during the classical approach to the quiescent singularity, hence, particle production effects are of crucial significance.

above, it follows that $\dot{H} \le 0$ in the neighbourhood of the singular point. In the purely classical case we find, after setting $\dot{\rho}_{\rm prod}$ to zero in (32), that $\dot{H} \to -\infty$ as the quiescent singularity is approached. It is well known, however, that particle production effects are sensitive to the change in the rate of expansion [13], and it is expected that $\dot{\rho}_{\rm prod}$ will go to infinity as $\dot{H} \to -\infty$. Since $\dot{\rho}_{\rm prod} \gg \gamma H \rho$, this will result in $\dot{H}$ becoming positive, which contradicts the assumption that $\dot{H} \le 0$.
Therefore, under the assumption that the critical density ρ_s is reached in a finite time, the only possibility for $\dot{H}$ is to *remain bounded*. In other words, the rate of particle production should exactly balance the decrease in the matter density due to expansion, turning the numerator in (32) to zero:

$$\dot{\rho}_{\rm prod} - \gamma H \rho \to 0 \quad \Rightarrow \quad \dot{\rho}_{\rm prod} \to \frac{\gamma \rho_s}{\ell} \quad \text{as} \quad \rho \to \rho_s \, . \tag{33}$$

In this case, the universe reaches its singular state with the energy density due to particle production exactly balancing the density decrease caused by expansion, as given by (33).

(ii) It is not clear whether the above regime will be realized or whether, if realized, it will be stable, since it requires the exact balancing of rates (33) at the singularity. A second distinct possibility is that, due to the presence of particle production, the value of $\Delta\rho = \rho - \rho_s$ always remains bounded from below by a nonzero density. In this scenario, $\dot{H}$ initially decreases ($|\dot{H}|$ increases) under the influence of the increasing factor $1/\sqrt{\Delta\rho}$ in (32). However, a large value of $|\dot{H}|$ induces active particle production from the vacuum which leads to an increase in the value of $\dot{\rho}_{\rm prod}$ in (32). As the value of $\Delta\rho$ reaches its (nonzero) minimum, we have $\dot{H} = 0$ at this point, according to (30), after which the rate $\dot{H}$ becomes positive due to self-sustained particle production that continues because of the large value of the second time derivative $\ddot{H}$. After a period of extensive particle production, the universe reaches another turning point $\dot{H} = 0$ after which is continues to expand according to (30) with decreasing energy density. Thus, we arrive at a model of cyclic evolution with periods of extensive particle production alternating with periods of classical expansion during which quantum particle production is negligible. This scenario bears a formal resemblance to quasi-steady-state cosmology proposed in a very different context by Hoyle, Burbidge, and Narlikar [26]. The particle production rate in our case is estimated by the quantity $\dot{\rho}_{\rm prod}$ given in (33), which is approximately the value it takes at the turning points where $\dot{H} = 0$. The Hubble parameter in this scenario periodically varies being of the order of $H \sim \ell^{-1}$, and the energy density is of the order ρ_s, so that particle production rate is

$$\dot{\rho}_{\rm prod} \sim \frac{\gamma \rho_s}{\ell} \, . \tag{34}$$

Our discussion thus far was limited to quantum processes within a single underdense region (void) which was assumed for simplicity to be perfectly homogeneous. Let us now (qualitatively) discuss whether this scenario can be generalized to the real (inhomogeneous) universe. Clearly, the particle production rate $\dot{\rho}_{\rm prod}$ in this case should be regarded as being averaged over the entire universe, to which several significantly underdense voids are contributing. Equation (32) should therefore be treated as an ensemble average, where the *mean* particle production rate depends upon the distribution as well as dynamics of *local* underdensity regions. As a result, equation (32) is not expected to explicitly depend upon the behaviour of the Hubble parameter and, in principle, particle production can proceed even in a De Sitter-like universe, in which the Hubble parameter H remains constant in time. The rate of particle production in this case is given by equality (32) with zero left-hand side:

$$\dot{\rho}_{\rm prod} = \gamma H \rho\,. \tag{35}$$

The value of the Hubble parameter in such a steady-state universe can be related to the Ω-parameter in matter

$$\Omega_{\rm m} = \frac{\rho}{\rho_s} = \frac{\rho}{3m^2 H^2}\,, \tag{36}$$

where we have used the basic Eq. (30). For the average energy density, we obviously have $\rho - \rho_s \approx \rho$. Hence,

$$H = \frac{1}{\ell} + \sqrt{\frac{\Delta\rho}{3m^2}} \approx \frac{1}{\ell} + \sqrt{\frac{\rho}{3m^2}} = \frac{1}{\ell} + H\sqrt{\Omega_{\rm m}}\,, \tag{37}$$

or, finally,♯

$$H \approx \frac{1}{\ell\left(1 - \sqrt{\Omega_{\rm m}}\right)}\,. \tag{38}$$

In principle, one might use these preliminary results to construct a braneworld version of steady-state cosmology, in which matter is being created at a steady rate in voids rather than in overdense regions (as hypothesized in the original version [26]). This would then add one more model to the steadily growing list of dark-energy cosmologies [2]. These conclusions must, however, be substantiated by a more detailed treatment which takes into account the *joint* effect of vacuum polarization and particle production near the quiescent singularity. Clearly, whether one or the other effect dominates will depend upon the number of conformal and non-conformal fields contributing to the vacuum, their spin weights, etc. We propose to return to this issue in a future work.

6. Conclusions

Cosmological models based on braneworld gravity have the interesting property of giving rise to singularities which are not commonly encountered in general relativity. This is largely due to the possibility of different kinds of embedding of the brane in the higher dimensional (bulk) space-time. Singular embedding implies that the expansion (in time) of the brane cannot be continued indefinitely. The singularities which we have examined in this paper arise because of this reason. They have the property that, while the density, pressure, and Hubble parameter on the brane remain finite, higher derivatives of the

♯ For comparison, the late-time value of the Hubble parameter in LCDM is [27] $H = H_0\sqrt{1-\Omega_{\rm m}}$.

Hubble parameter blow up as the singularity is approached. For this reason, we refer to these singularities as being "quiescent." Despite its deceptively mild nature, the quiescent singularity is a real curvature singularity at which the Kretschmann invariant diverges ($R_{iklm}R^{iklm} \to \infty$). The importance of quantum effects in regions of large space-time curvature has been demonstrated in a number of papers [13], and it should therefore come as no surprise that these effects can significantly alter the classical behaviour near the quiescent singularity, as demonstrated by us in this paper. Examining the vacuum polarization caused by massless conformally coupled fields, we find that these weaken the quiescent singularity resulting in an exceedingly mild (soft) singularity at which the values of H, $\dot{H}$, and $\ddot{H}$ remain finite while $\dddot{H}$ and higher derivatives of the Hubble parameter diverge.

Unlike the classical Big-Bang singularity, the quiescent singularity in braneworld models is reached in regions of *low density* and is therefore encountered during the course of the universe expansion rather than its collapse. Densities lower than the mean value are known to occupy a large filling fraction within the cosmic web [22, 28]. Therefore, if the braneworld model is a true representation of reality, one might speculate that it is likely to encounter the quiescent singularity (or its quantum-corrected counterpart, the "soft singularity") within large underdense regions, or voids. The rapidly varying space-time geometry near the quiescent singularity can, in addition to vacuum polarization, also give rise to quantum creation of fields which do not couple conformally to gravity. (Examples include massive (leptons, higgs, etc.) as well as massless (gravitons) particles.) A preliminary estimate made by us in this paper shows that, if this process is sufficiently intensive, then the universe will expand at late times in a manner which is reminiscent of quasi-steady-state cosmology, with the Hubble parameter showing oscillations about a constant value. A more detailed estimate of particle production, however, lies beyond the scope of the present paper, and we hope to return to it in a future work.

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