

# Conformal transformations in classical gravitational theories and in cosmology

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## Abstract

In recent years, the use of conformal transformation techniques has become widespread in the literature on gravitational theories alternative to general relativity, on cosmology, and on nonminimally coupled scalar fields. Typically, the transformation to the Einstein frame is generated by a fundamental scalar field already present in the theory. In this context, the problem of which conformal frame is the physical one has to be dealt with and, in the general case, it has been clarified only recently; the formulation of a theory in the “new” conformal frame leads to departures from canonical Einstein gravity. In this article, we review the literature on conformal transformations in classical gravitational theories and in cosmology, seen both as purely mathematical tools and as maps with physically relevant aspects. It appears particularly urgent to refer the analysis of experimental tests of Brans–Dicke and scalar–tensor theories of gravity, as well as the predictions of cosmological inflationary scenarios, to the physical conformal frame, in order to have a meaningful comparison with the observations.

Keywords: *conformal transformations; scalar fields; nonminimal coupling; alternative theories of gravity; inflation.*

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## Notations and conventions

The notations and conventions used in this paper are as follows: the metric signature is  $- + + \dots +$ . To facilitate the comparison with the literature on inflationary cosmology, we use units in which the speed of light and the reduced Planck constant assume the value unity.  $G$  is Newton's constant and the Planck mass is  $m_{pl} = G^{-1/2}$  in these units. Greek indices assume the values  $0, 1, 2, \dots, n-1$ , where  $n$  is the dimension of spacetime. When  $n = 4$ , small Latin indices assume the values  $1, 2, 3$ . While we allow for  $n$  spacetime dimensions (only one of which is timelike), in most of this paper the value  $n = 4$  is assumed, except when discussing Kaluza–Klein and string theories prior to compactification.

A comma denotes ordinary differentiation, and  $\nabla_\mu$  is the covariant derivative operator. Round and square brackets around indices denote, respectively, symmetrization and antisymmetrization, which include division by the number of permutations of the indices: e.g.  $A_{(\mu\nu)} = (A_{\mu\nu} + A_{\nu\mu})/2$ . The Riemann and Ricci tensors are given in terms of the Christoffel symbols  $\Gamma_{\alpha\beta}^\delta$  by

$$R_{\alpha\beta\gamma}{}^\delta = \Gamma_{\alpha\gamma,\beta}^\delta - \Gamma_{\beta\gamma,\alpha}^\delta + \Gamma_{\alpha\gamma}^\sigma \Gamma_{\sigma\beta}^\delta - \Gamma_{\beta\gamma}^\sigma \Gamma_{\sigma\alpha}^\delta ,$$

$$R_{\mu\rho} = \Gamma_{\mu\rho,\nu}^\nu - \Gamma_{\nu\rho,\mu}^\nu + \Gamma_{\mu\rho}^\alpha \Gamma_{\alpha\nu}^\nu - \Gamma_{\nu\rho}^\alpha \Gamma_{\alpha\mu}^\nu ,$$

and  $R \equiv g^{\alpha\beta} R_{\alpha\beta}$  is the Ricci curvature.  $\square \equiv g^{\mu\nu} \nabla_\mu \nabla_\nu$  is d'Alembert's operator. A tilde denotes quantities defined in the Einstein frame, while a caret denotes quantities defined in a higher-dimensional space prior to the compactification of the extra dimensions.

## 1 Introduction

If  $(M, g_{\mu\nu})$  is a spacetime, the point-dependent rescaling of the metric tensor

$$g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} , \tag{1.1}$$

where  $\Omega = \Omega(x)$  is a nonvanishing, regular function, is called a *Weyl* or *conformal transformation*. It affects the lengths of time [space]-like intervals and the norm of time [space]-like vectors, but it leaves the light cones unchanged: the spacetimes  $(M, g_{\mu\nu})$  and  $(M, \tilde{g}_{\mu\nu})$  have the same causal structure. The converse is also true (Wald 1984). If  $v^\mu$  is a null, timelike, or spacelike vector with respect to the metric  $g_{\mu\nu}$ , it is also a null, timelike, or spacelike vector, respectively, in the rescaled metric  $\tilde{g}_{\mu\nu}$ .

Denoting by  $g$  the determinant  $\det(g_{\mu\nu})$  one has, under the action of (1.1),  $\tilde{g}^{\mu\nu} = \Omega^{-2}g^{\mu\nu}$  and  $\tilde{g} \equiv \det(\tilde{g}_{\mu\nu}) = \Omega^{2n}g$ . It will be useful to remember the transformation properties of the Christoffel symbols, Riemann and Ricci tensor, and of the Ricci curvature  $R$  under the rescaling (1.1) (Synge 1955; Birrell and Davies 1982; Wald 1984):

$$\tilde{\Gamma}_{\beta\gamma}^{\alpha} = \Gamma_{\beta\gamma}^{\alpha} + \Omega^{-1} \left( \delta_{\beta}^{\alpha} \nabla_{\gamma} \Omega + \delta_{\gamma}^{\alpha} \nabla_{\beta} \Omega - g_{\beta\gamma} \nabla^{\alpha} \Omega \right), \quad (1.2)$$

$$\begin{aligned} \widetilde{R_{\alpha\beta\gamma}{}^{\delta}} &= R_{\alpha\beta\gamma}{}^{\delta} + 2\delta_{[\alpha}^{\delta} \nabla_{\beta]} \nabla_{\gamma} (\ln \Omega) - 2g^{\delta\sigma} g_{\gamma[\alpha} \nabla_{\beta]} \nabla_{\sigma} (\ln \Omega) + 2\nabla_{[\alpha} (\ln \Omega) \delta_{\beta]}^{\delta} \nabla_{\gamma} (\ln \Omega) \\ &\quad - 2\nabla_{[\alpha} (\ln \Omega) g_{\beta]\gamma} g^{\delta\sigma} \nabla_{\sigma} (\ln \Omega) - 2g_{\gamma[\alpha} \delta_{\beta]}^{\delta} g^{\sigma\rho} \nabla_{\sigma} (\ln \Omega) \nabla_{\rho} (\ln \Omega), \end{aligned} \quad (1.3)$$

$$\begin{aligned} \tilde{R}_{\alpha\beta} &= R_{\alpha\beta} - (n-2) \nabla_{\alpha} \nabla_{\beta} (\ln \Omega) - g_{\alpha\beta} g^{\rho\sigma} \nabla_{\rho} \nabla_{\sigma} (\ln \Omega) + (n-2) \nabla_{\alpha} (\ln \Omega) \nabla_{\beta} (\ln \Omega) \\ &\quad - (n-2) g_{\alpha\beta} g^{\rho\sigma} \nabla_{\rho} (\ln \Omega) \nabla_{\sigma} (\ln \Omega), \end{aligned} \quad (1.4)$$

$$\tilde{R} \equiv \tilde{g}^{\alpha\beta} \tilde{R}_{\alpha\beta} = \Omega^{-2} \left[ R - 2(n-1) \square (\ln \Omega) - (n-1)(n-2) \frac{g^{\alpha\beta} \nabla_{\alpha} \Omega \nabla_{\beta} \Omega}{\Omega^2} \right], \quad (1.5)$$

where  $n$  ( $n \geq 2$ ) is the dimension of the spacetime manifold  $M$ . In the case  $n = 4$ , the scalar curvature has the expressions

$$\tilde{R} = \Omega^{-2} \left[ R - \frac{6\square\Omega}{\Omega} \right] = \Omega^{-2} \left[ R - \frac{12\square(\sqrt{\Omega})}{\sqrt{\Omega}} - \frac{3g^{\alpha\beta} \nabla_{\alpha} \Omega \nabla_{\beta} \Omega}{\Omega^2} \right], \quad (1.6)$$

which are useful in many applications. The Weyl tensor  $C_{\alpha\beta\gamma}{}^{\delta}$  (beware of the position of the indices!) is conformally invariant:

$$\widetilde{C_{\alpha\beta\gamma}{}^{\delta}} = C_{\alpha\beta\gamma}{}^{\delta}, \quad (1.7)$$

and the null geodesics are also conformally invariant (Lorentz 1937). The conservation equation  $\nabla^{\nu} T_{\mu\nu} = 0$  for a symmetric stress-energy tensor  $T_{\mu\nu}$  is not conformally invariant unless the trace  $T \equiv T^{\mu}{}_{\mu}$  vanishes (Wald 1984). The Klein-Gordon equation  $\square\phi = 0$  for a scalar field  $\phi$  is not conformally invariant, but its generalization

$$\square\phi - \frac{n-2}{4(n-1)} R\phi = 0 \quad (1.8)$$

( $n \geq 2$ ) is conformally invariant (note that the introduction of a nonzero cosmological constant in the Einstein action for gravity creates an effective mass, and a length scale,

in the Klein–Gordon equation, which spoils the conformal invariance (Madsen 1993)). Maxwell’s equations in four dimensions are conformally invariant (Cunningham 1909; Bateman 1910), but the equations for the electromagnetic four–potential are not (it is to be noted that, at the quantum level, the conformal invariance of the Maxwell equations may be broken by quantum corrections like the generation of mass or the conformal anomaly). The conditions for conformal invariance of fields of arbitrary spin in any spacetime dimensions were discussed in (Iorio *et al.* 1997).

In this review paper, we will limit ourselves to consider special conformal transformations, in which the dependence of the conformal factor  $\Omega(x)$  on the spacetime point  $x$  is obtained via a functional dependence (usually a power law) on a scalar field  $\phi(x)$  present in the theory:

$$\Omega(x) = \Omega[\phi(x)] . \quad (1.9)$$

A redefinition of the scalar field  $\phi$  accompanies the conformal transformation (1.1). Theories in which a fundamental scalar field appears and generates (1.1) include scalar–tensor and nonlinear theories of gravity (in which  $\phi$  is a Brans–Dicke–like field) and Kaluza–Klein theories (in which  $\phi$  is the determinant of the metric of the extra compact dimensions). Fundamental scalar fields in quantum theories include  $SO(N)$  bosons in dual models, Nambu–Goldstone bosons, Higgs fields, and dilatons in superstring theories. In addition, almost all<sup>1</sup> scenarios of cosmological inflation (Linde 1990; Kolb and Turner 1990; Liddle and Lyth 1993; Liddle 1996) are based on scalar fields, either in the context of a classical or high energy theory, or in a phenomenological approach in which a scalar field is introduced as a source of gravitation in the field equations of the theory (usually the Einstein equations of general relativity). By means of a transformation of the form (1.1), many of these scenarios are recast in the form of Einstein gravity with the scalar field(s) as a source of gravity and a power–law inflationary potential. The investigation of this mathematical equivalence has far–reaching consequences, and in many cases the mathematical equivalence provides a means to go from a physically inconsistent theory to a viable one. Unfortunately, the use of conformal transformations in gravitational theories is haunted by confusion and ambiguities, particularly in relation to the problem of identifying the conformal frame which correctly describes the physics. Despite early work on the subject, confusion still persists in the literature and considerably detracts from papers that use conformal techniques incorrectly.

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<sup>1</sup>The exception is  $R^2$  inflation (Starobinsky 1980; Starobinsky 1986; Maeda, Stein–Schabes and Futamase 1989), in which the Lagrangian term  $R^2$  itself drives inflation. However, a scalar field is sometimes added to this scenario to “help” inflation (Maeda 1989; Maeda, Stein–Schabes and Futamase 1989) and the scenario is often recast as power–law inflation by using a conformal transformation (Liddle and Lyth 1993).

It must be stressed that, in general, conformal transformations are not diffeomorphisms of the manifold  $M$ , and the rescaled metric  $\tilde{g}_{\mu\nu}$  is not simply the metric  $g_{\mu\nu}$  written in a different coordinate system: the metrics  $\tilde{g}_{\mu\nu}$  given by Eq. (1.1) and the metric  $g_{\mu\nu}$  describe different gravitational fields and different physics. Special conformal transformations originating from diffeomorphisms are called *conformal isometries* (Wald 1984). The reader should not be confused by the fact that some authors use the name “conformal transformation” for special coordinate transformations relating inertial and accelerated observers (e.g. Fulton, Rorlich and Witten 1962*a,b*; Wood, Papini and Cai 1989; Mashoon 1993). In this case the metric is left unchanged, although its coordinate representation varies. The possibility of different conformal rescalings for different metric components has also been considered (Mychelkin 1991), although it appears doubtful that this procedure can be given a covariant formulation and a physically sound motivation.

Historically, interest in conformal transformations arose after the formulation of Weyl’s (1919) theory aimed at unifying gravitation and electromagnetism, especially after its reformulation by Dirac (1973). Moreover, a conformally invariant version of special relativity was formulated (Page 1936*a,b*; Page and Adams 1936), but the conformal invariance in this case was recognized to be meaningless (Pauli 1958). Further developments of Weyl’s theory are more appealing; for example, the self-consistent, scale-invariant theory of Canuto *et al.* (1977), so far, is not in contradiction with the observations. It requires that the astronomical unit of length is related to the atomic unit by a scalar function which depends on the spacetime point. The theory contains a time-dependent cosmological “constant”  $\Lambda(t) = \Lambda_0(t_0/t)^2$ , which is sought after by many authors in modern cosmology and astroparticle physics.

## 2 Conformal transformations as a mathematical tool

Conformal rescalings and conformal techniques have been widely used in general relativity for a long time, especially in the theory of asymptotic flatness and in the initial value formulation (Wald 1984 and references therein), and also in studies of the propagation of massless fields, including Fermat’s principle (Perlick 1990; Schneider, Ehlers and Falco 1992), gravitational lensing in the (conformally flat) Friedmann–Lemaître–Robertson–Walker universe (Perlick 1990; Schneider, Ehlers and Falco 1992), wave equations (Sonogo and Faraoni 1992; Noonan 1995), studies of the optical geometry near black hole horizons (Abramowicz, Carter and Lasota 1988; Sonogo and Massar 1996; Abramowicz *et al.* 1997*a,b*), exact solutions (Van den Bergh 1986*a,b,c,d,e*, 1988)

and in other contexts. Conformal techniques and conformal invariance are important also for quantum field theory in curved spaces (Birrell and Davies 1982), for statistical mechanics and for string theories (e.g. Dita and Georgescu 1989). A conformal transformation is often used as a mathematical tool to map the equations of motion of physical systems into mathematically equivalent sets of equations that are more easily solved and computationally more convenient to study. This situation arises mainly in three different areas of gravitational physics: alternative (including nonlinear) theories of gravity, unified theories in multidimensional spaces, and studies of scalar fields nonminimally coupled to gravity.

**Brans–Dicke theory:** The conformal rescaling to the minimally coupled case for the Brans–Dicke field in Brans–Dicke theory was found by Dicke (1962). One starts with the Brans–Dicke action in the so-called “Jordan frame”

$$S_{BD} = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left[ \phi R - \frac{\omega}{\phi} \nabla^\mu \phi \nabla_\mu \phi \right] + S_{matter} , \quad (2.1)$$

which corresponds to the field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi}{\phi} T_{\mu\nu} + \frac{\omega}{\phi^2} \left( \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \nabla^\alpha \phi \nabla_\alpha \phi \right) + \frac{1}{\phi} (\nabla_\mu \nabla_\nu \phi - g_{\mu\nu} \square \phi) , \quad (2.2)$$

$$\square \phi = \frac{8\pi T}{3 + 2\omega} . \quad (2.3)$$

The conformal transformation (1.1) with

$$\Omega = \sqrt{G\phi} \quad (2.4)$$

and the redefinition of the scalar field given in differential form by

$$d\tilde{\phi} = \sqrt{\frac{2\omega + 3}{16\pi G}} \frac{d\phi}{\phi} \quad (2.5)$$

( $\omega > -3/2$ ) transform the action (2.1) into the “Einstein frame” action

$$S = \int d^4x \left\{ \sqrt{-\tilde{g}} \left[ \frac{\tilde{R}}{16\pi G} - \frac{1}{2} \tilde{\nabla}^\mu \tilde{\phi} \tilde{\nabla}_\mu \tilde{\phi} \right] + \exp \left( -8\sqrt{\frac{\pi G}{2\omega + 3}} \tilde{\phi} \right) \mathcal{L}_{matter}(\tilde{g}) \right\} , \quad (2.6)$$

where  $\tilde{\nabla}_\mu$  is the covariant derivative operator of the rescaled metric  $\tilde{g}_{\mu\nu}$ . The gravitational part of the action now contains only Einstein gravity, but a free scalar field acting

as a source of gravitation *always* appears. It permeates spacetime in a way that cannot be eliminated, i.e. one cannot contemplate solutions of the vacuum Einstein equations  $R_{\mu\nu} = 0$  in the Einstein frame. In the Jordan frame, the gravitational field is described by the metric tensor  $g_{\mu\nu}$  and by the Brans–Dicke field  $\phi$ . In the Einstein frame, the gravitational field is described only by the metric tensor  $\tilde{g}_{\mu\nu}$ , but the scalar field  $\tilde{\phi}$ , which is now a form of matter, is always present, a reminiscence of its fundamental role in the “old” frame. In addition, the rest of the matter part of the Lagrangian is multiplied by an exponential factor, thus displaying an anomalous coupling to the scalar  $\tilde{\phi}$ . This anomalous coupling will be discussed in Sec. 6.

**Nonminimally coupled scalar field:** By means of a conformal rescaling, the study of a nonminimally coupled scalar field can also be reduced to that of a minimally coupled scalar. The transformation relating a massless conformally coupled and a minimally coupled scalar field was found by Bekenstein (1974) and later rediscovered and generalized to massive fields and arbitrary values of the coupling constant (Deser 1984; Schmidt 1988; Maeda 1989; Futamase and Maeda 1989; Xanthopoulos and Dialynas 1992; Klimcik 1993; Accioly *et al.* 1993). In this case, the starting point is the action for canonical gravity plus a scalar field in the Jordan frame:

$$S = \int d^4x \sqrt{-g} \left[ \left( \frac{1}{16\pi G} - \frac{\xi\phi^2}{2} \right) R - \frac{1}{2} \nabla^\mu \phi \nabla_\mu \phi - V(\phi) \right], \quad (2.7)$$

where  $V(\phi)$  is the scalar field potential (possibly including a mass term and the cosmological constant) and  $\xi$  is a dimensionless coupling constant. Note that the dimensions of the scalar field are  $[\phi] = [G^{-1/2}] = [m_{pl}]$ . The equation satisfied by the scalar  $\phi$  is

$$\square\phi - \xi R\phi - \frac{dV}{d\phi} = 0. \quad (2.8)$$

Two cases occur most frequently in the literature: “minimal coupling” ( $\xi = 0$ ) and “conformal coupling” ( $\xi = 1/6$ ); the latter makes the wave equation (2.8) conformally invariant in four dimensions if  $V = 0$  or  $V = \lambda\phi^4$  (the latter potential being used in the chaotic inflationary scenario). The conformal transformation (1.1) with

$$\Omega^2 = 1 - 8\pi G\xi\phi^2 \quad (2.9)$$

and the redefinition of the scalar field, given in differential form by

$$d\tilde{\phi} = \frac{[1 - 8\pi G\xi(1 - 6\xi)\phi^2]^{1/2}}{1 - 8\pi G\xi\phi^2} d\phi, \quad (2.10)$$

reduce (2.7) to the Einstein frame action

$$S = \int d^4x \sqrt{-\tilde{g}} \left[ \frac{\tilde{R}}{16\pi G} - \frac{1}{2} \tilde{\nabla}^\mu \tilde{\phi} \tilde{\nabla}_\mu \tilde{\phi} - \tilde{V}(\tilde{\phi}) \right], \quad (2.11)$$

where the scalar field  $\tilde{\phi}$  is now minimally coupled and satisfies the equation

$$\tilde{g}^{\mu\nu} \nabla_\mu \nabla_\nu \tilde{\phi} - \frac{d\tilde{V}}{d\tilde{\phi}} = 0. \quad (2.12)$$

The new scalar field potential is given by

$$\tilde{V}(\tilde{\phi}) = \frac{V(\phi)}{(1 - 8\pi G \xi \phi^2)^2}, \quad (2.13)$$

where  $\phi = \phi(\tilde{\phi})$  is obtained by integrating and inverting Eq. (2.10). The field equations of a gravitational theory in the case of a minimally coupled scalar field as a source of gravity are computationally much easier to solve than the corresponding equations for nonminimal coupling, and the transformation (1.1), (2.9), (2.10) is widely used for this purpose. The stress-energy tensor of a scalar field can be put in the form corresponding to a fluid, but the  $T_{\mu\nu}$  for a nonminimally coupled field is considerably more complicated than the minimal coupling case, for which the form of the  $T_{\mu\nu}$  reduces to that of a perfect fluid (Madsen 1988). It is generally assumed that the scalar field  $\phi$  assumes values in a range that makes the right hand side of Eq. (2.9) positive. For  $\xi > 0$ , this range is limited by the critical values  $\phi_{1,2} = \pm (8\pi G \xi)^{-1/2}$ .

Nonminimal couplings of the electromagnetic field to gravity have also been considered (Novello and Salim 1979; Novello and Heintzmann 1984; Turner and Widrow 1988; Novello and Elbaz 1994; Novello, Pereira and Pinto-Neto 1995; Lafrance and Myers 1995), but conformal techniques analogous to those developed for scalar fields are presently unknown. A formal method alternative to conformal transformations is sometimes useful for nonminimally coupled scalar fields, which are equivalent to an effective flat space field theory with a scalar mass that is  $\xi$ -dependent (Hochberg and Kephart 1995).

**Nonlinear theories of gravity:** The mathematical equivalence between a theory described by the gravitational Lagrangian density  $\mathcal{L}_g = \sqrt{-g}f(R)$  ("higher order theory") and Einstein gravity was found in (Teysandier and Tournenc 1983; Schmidt 1987;

Starobinsky 1987; Barrow and Cotsakis 1988; Maeda 1989; Gott, Schmidt and Starobinsky 1990; Schmidt 1990; Cotsakis and Saich 1994; Wands 1994). The field equations for this theory are of fourth order:

$$\left(\frac{df}{dR}\right) R_{\mu\nu} - \frac{1}{2} f(R) g_{\mu\nu} - \nabla_\mu \nabla_\nu \left(\frac{df}{dR}\right) + g_{\mu\nu} \square \left(\frac{df}{dR}\right) = 0, \quad (2.14)$$

and are reduced to the Einstein equations by the conformal transformation.

Quadratic Lagrangian densities with  $R^2$  terms arising from quantum corrections are the most frequently studied cases of nonlinear gravitational theories; they can be reduced to the Einstein Lagrangian density (Higgs 1959; Teyssandier and Tourenc 1983; Whitt 1984; Ferraris 1986; Berkin and Maeda 1991). These results were generalized to supergravity, Lagrangians with terms  $\square^k R$  ( $k \geq 1$ ) and polynomial Lagrangians in  $R$  (Cecotti 1987); the two-dimensional case was studied in (Mignemi and Schmidt 1995). This class of theories includes Weyl's theory (Weyl 1919; Dirac 1973) described by the Lagrangian density  $\mathcal{L} = \sqrt{-g}(R^2 + \beta F_{\mu\nu} F^{\mu\nu})$ , and theories of the form  $\mathcal{L} = R^k$  ( $k \geq 1$ ). For nonlinear theories of gravity, the conformal transformation that maps the theory into Einstein gravity becomes a Legendre transformation (Ferraris, Francaviglia and Magnano 1988; Jakubiec and Kijowski 1988; Ferraris, Francaviglia and Magnano 1990; Magnano, Ferraris and Francaviglia 1990; Magnano and Sokolowski 1994).

There are obvious advantages in performing this transformation because the higher order field equations of the nonlinear theory are reduced to the second order Einstein equations with matter. One starts with a purely gravitational nonlinear theory described by the action

$$S = \int d^m x \sqrt{-g} \left[ F(\phi, R) - \frac{\epsilon}{2} \nabla^\mu \phi \nabla_\mu \phi \right], \quad (2.15)$$

in  $m$  spacetime dimensions, where  $F(\phi, R)$  is an arbitrary (but sufficiently regular) function of  $\phi$  and  $R$ , and  $\epsilon$  is a free parameter (normally 0 or 1).

The corresponding field equations (Maeda 1989) are

$$\begin{aligned} \left(\frac{\partial F}{\partial R}\right) \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\right) &= \frac{\epsilon}{2} \left(\nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \nabla^\alpha \phi \nabla_\alpha \phi\right) + \frac{1}{2} g_{\mu\nu} \left(F - \frac{\partial F}{\partial R} R\right) \\ &+ \nabla_\mu \nabla_\nu \left(\frac{\partial F}{\partial R}\right) - g_{\mu\nu} \square \left(\frac{\partial F}{\partial R}\right), \end{aligned} \quad (2.16)$$

$$\epsilon \square \phi = - \frac{\partial F}{\partial \phi}. \quad (2.17)$$

The conformal rescaling (1.1), where

$$\Omega^2 = \left[ 16\pi G \left| \frac{\partial F}{\partial R} \right| + \text{constant} \right]^{2/(m-2)}, \quad (2.18)$$

and the redefinition of the scalar field

$$\tilde{\phi} = \frac{1}{\sqrt{8\pi G}} \sqrt{\frac{m-1}{m-2}} \ln \left[ \sqrt{32\pi G} \left| \frac{\partial F}{\partial R} \right| \right] \quad (2.19)$$

(Maeda 1989) reduce the action (2.15) to

$$S = \alpha \int d^m x \sqrt{-\tilde{g}} \left[ \frac{\tilde{R}}{16\pi G} - \frac{1}{2} \tilde{\nabla}^\mu \tilde{\phi} \tilde{\nabla}_\mu \tilde{\phi} - \frac{\epsilon\alpha}{2} \exp \left[ -\sqrt{8\pi G} \frac{m-2}{m-1} \tilde{\phi} \right] - U(\phi, \tilde{\phi}) \right] \quad (2.20)$$

where the two scalar fields  $\phi$  and  $\tilde{\phi}$  appear and

$$\alpha = \frac{\partial F / \partial R}{|\partial F / \partial R|}, \quad (2.21)$$

$$U(\phi, \tilde{\phi}) = \alpha \exp \left( -\frac{m\sqrt{8\pi G}\tilde{\phi}}{\sqrt{(m-1)(m-2)}} \right) \left[ \frac{\alpha}{16\pi G} R(\phi, \tilde{\phi}) \exp \left( \sqrt{\frac{m-2}{m-1}} \sqrt{8\pi G} \tilde{\phi} \right) - F(\phi, \tilde{\phi}) \right], \quad (2.22)$$

and  $F(\phi, \tilde{\phi}) = F(\phi, R(\phi, \tilde{\phi}))$ . The resulting system is of nonlinear  $\sigma$ -model type, canonical gravity with two scalar fields  $\phi, \tilde{\phi}$ .

In the particular case in which  $F(\phi, R)$  is a linear function of the Ricci curvature,

$$F(\phi, R) = f(\phi)R - V(\phi), \quad (2.23)$$

the redefinition of the scalar field

$$\tilde{\phi} = \frac{1}{\sqrt{8\pi G}} \int d\phi \left\{ \frac{\epsilon(m-2)f(\phi) + 2(m-1)[df(\phi)/d\phi]^2}{2(m-2)f^2(\phi)} \right\}^{1/2} \quad (2.24)$$

(where the argument of the square root is assumed to be positive) leads to the Einstein action with a single scalar field  $\phi$ :

$$S = \frac{|f|}{f} \int d^m x \sqrt{-\tilde{g}} \left[ \frac{\tilde{R}}{16\pi G} - \frac{1}{2} \tilde{\nabla}^\mu \tilde{\phi} \tilde{\nabla}_\mu \tilde{\phi} - U(\tilde{\phi}) \right]. \quad (2.25)$$

This action is equivalent to the Einstein equations

$$\tilde{R}_{\mu\nu} - \frac{1}{2}\tilde{g}_{\mu\nu}\tilde{R} = 8\pi G\tilde{T}_{\mu\nu}[\tilde{\phi}] , \quad (2.26)$$

$$\tilde{T}_{\mu\nu}[\tilde{\phi}] = \nabla_{\mu}\tilde{\phi}\nabla_{\nu}\tilde{\phi} - \frac{1}{2}\tilde{g}_{\mu\nu}\tilde{g}^{\alpha\beta}\nabla_{\alpha}\tilde{\phi}\nabla_{\beta}\tilde{\phi} + U\tilde{g}_{\mu\nu} , \quad (2.27)$$

where

$$U(\tilde{\phi}) = \frac{|f|}{f} [16\pi G |f(\phi)|^{\frac{-m}{m-2}} V(\phi)] \quad (2.28)$$

and  $\phi = \phi(\tilde{\phi})$ . The transformations (2.4), (2.9) and (2.10) are recovered as particular cases of (2.24), (2.18). In addition, all the theories described by a four-dimensional action of the form

$$S = \int d^4x \sqrt{-g} [f(\phi)R + A(\phi)\nabla^{\mu}\phi\nabla_{\mu}\phi + V(\phi)] \quad (2.29)$$

and satisfying the relation

$$2Af - 3\left(\frac{df}{d\phi}\right)^2 = 0 , \quad V(\phi) = \lambda f^2(\phi) \quad (2.30)$$

( $\lambda = \text{constant}$ ) are conformally related (Shapiro and Takata 1995); particular cases include general relativity and the case of a conformally coupled scalar field.

The conformal transformation establishes a *mathematical* equivalence between the theories formulated in the two conformal frames; the space of solutions of the theory in one frame is isomorphic to the space of solutions in the conformally related frame (which is mathematically more convenient to study). The conformal transformation can also be used as a solution-generating technique, if solutions are known in one conformal frame but not in another (Harrison 1972; Belinskii and Kalatnikov 1973; Bekenstein 1974; Van den Bergh 1980, 1982, 1983*a, b, c, d*; Froyland 1982; Accioly, Vaidya and Som 1983; Lorentz-Petzold 1984; Barrow and Maeda 1990; Klimcik and Kolnik 1993; Abreu, Crawford and Mimoso 1994). It is to be stressed that the mathematical equivalence between the two systems *a priori* implies nothing about their physical equivalence (Brans 1988; Cotsakis 1993; Magnano and Sokolowski 1994). Moreover, only the gravitational (vacuum) part of the action is conformally equivalent to Einstein gravity: if ordinary matter (i.e. matter different from the scalar field used in the conformal transformation) is added to the theory, the coupling of this matter to gravity and the conservation equations that it satisfies are different in the two conformally related frames. The advantage of the

conformal transformation in a non-purely vacuum theory is questionable: it has been argued that, because the Einstein frame scalar field is coupled to matter, a simplification of the equations of motion in this case does not occur (Barrow and Maeda 1990).

Not only it is possible to map the classes of theories considered above into canonical Einstein gravity, but it is also possible to find conformal transformations between each two of these theories (see Magnano and Sokolowski 1994 for a table of possible transformations). Indeed, one expects to be able to do that by taking appropriate compositions of different maps from gravitational theories to general relativity, and their inverse maps.

We conclude this section with a remark on the terminology: it has become common to use the word “frame” to denote a set of dynamical variables of the theory considered; the term “gauge” instead of “frame” has been (rather improperly) used (Gibbons and Maeda 1988; Brans 1988). In some papers (Cho 1987, 1990, 1992, 1994, 1997; Cho and Yoon 1993) the metric  $\tilde{g}_{\mu\nu}$  in the Einstein frame is called “Pauli metric”, as opposed to the “Jordan” or “atomic unit” metric  $g_{\mu\nu}$  of the Jordan frame.

### 3 Is the Einstein frame physical ?

Many high energy theories and many classical gravity theories are formulated by using a conformal transformation mapping the Jordan frame to the Einstein frame. Typically, the conformal factor of the transformation is a function of a dilaton, or Brans–Dicke-like field already present in the theory. The classical theories of gravity for which a conformal transformation maps the system into a new conformal frame, in which the gravitational sector of the theory reduces to the canonical Einstein form, include Brans–Dicke theory and its scalar–tensor generalizations, non-linear gravity theories, classical Kaluza–Klein theories and in general, all theories which have an extended gravitational sector or which involve a dimensional reduction and compactification of extra spacetime dimensions. Quantum theories incorporating the conformal transformation include superstring and supergravity theories and  $\sigma$ -models. The transformation to the Einstein frame seems to be universally accepted for supergravity and superstring theories (although field redefinitions may be an issue for debate (Tseytlin 1993)). It is unknown whether physics is conformally invariant at a sufficiently high energy scale, but there are indications in this sense from string theories (Green, Schwarz and Witten 1987) and from  $SU(N)$  induced gravity models in which, in the high energy limit, the scalar fields of the theory approach conformal coupling (Buchbinder, Odintsov and Shapiro 1992; Geyer and Odintsov 1996). We have no experiments capable of probing the energy scale of string theories, and conformal invariance at this energy scale cannot be directly

tested. While the low-energy Einstein gravity contains a dynamical degree of freedom connected with the “length” of the metric tensor (the determinant  $g$ ), this is absent in conformally invariant gravity (e.g. induced gravity described by the action (4.17)). The conformal invariance of a theory implies that the latter contains no intrinsic mass; a nonzero mass would introduce a preferred length scale in the theory, thus breaking the scale-invariance. The physical inequivalence of conformal frames at low energies reflects the fact that the non-negligible masses of the fields present in the theory break the conformal symmetry which is present at higher energies. In classical gravity theories, there is disagreement and confusion on the long-standing (Pauli 1955; Fierz 1956) problem of which conformal frame is the physical one. Is the Jordan frame physical and the Einstein frame unphysical? Is the conformal transformation necessary, and the Einstein frame physical? Does any other choice of the conformal factor in Eq. (1.1) map the theory into a physically significant frame, and how many of these theories are possible? Here the term “physical” theory denotes one that is theoretically consistent and predicts the values of some observables that can, at least in principle, be measured in experiments performed in four macroscopic spacetime dimensions (definitions that differ from ours are sometimes adopted in the literature, see e.g. (Garay and Garcia-Bellido 1993; Overduin and Wesson 1997)). The ambiguity in the choice of the physical conformal frame raises also problems of an almost philosophical character (Weinstein 1996).

Before attempting to answer any of these questions, it is important to recognize that, in general, the reformulation of the theory in a new conformal frame leads to a different, physically inequivalent theory. If one restricts oneself to consider the metric tensor and physics that does not involve only conformally invariant fields (e.g. a stress-energy tensor  $T_{\mu\nu}$  with nonvanishing trace), or experiments involving massive particles and timelike observers, it is obvious that metrics conformally related by a nontrivial transformation of the kind (1.1) on a manifold describe different gravitational fields and different physical situations. For example, one could consider a Friedmann-Lemaitre-Robertson-Walker metric with flat spatial sections, given by the line element

$$ds^2 = a^2(\eta) \left( -d\eta^2 + dx^2 + dy^2 + dz^2 \right) , \quad (3.1)$$

where  $\eta$  is the conformal time and  $(x, y, z)$  are spatial comoving coordinates. The metric (3.1) is conformally flat, but certainly it is not physically equivalent to the Minkowski metric  $\eta_{\mu\nu}$ , since it exhibits a nontrivial dynamics and significant (observed) cosmological effects.

The authors working in classical gravitational physics can be grouped into five categories according to their attitude towards the issue of the conformal frame (we partially follow a previous classification by Magnano and Sokolowski (1994)):

- authors that neglect the issue (Deruelle and Spindel 1990; Garcia-Bellido and Quirós 1990; Hwang 1990; Gottlöber, Müller and Starobinsky 1991; Suzuki and Yoshimura 1991; Rothman and Anninos 1991; Guendelman 1992; Guth and Jain 1992; Liddle and Wands 1992; Capozziello, Occhionero and Amendola 1993; Capozziello, de Ritis and Rubano 1993; McDonald 1993*a,b*; Barrow, Mimoso and de Garcia Maia 1993; Garcia-Bellido and Wands 1995; Laycock and Liddle 1994; Alvarez and Belén Gavela 1983; Sadhev 1984; Deruelle and Madore 1987; Van den Bergh and Tavakol 1993; Fabris and Sakellariadou 1997; Kubyshev and Martin 1995; Fabris and Martin 1993; Chatterjee and Banerjee 1993; Biesiada 1994; Liddle and Lyth 1993; Hwang 1996);
- authors that explicitly support the view that a theory formulated in one conformal frame is physically equivalent to the reformulation of the same theory in a different conformal frame (Buchmüller and Dragon 1989; Holman, Kolb and Wang 1990; Campbell, Linde and Olive 1991; Casas, Garcia-Bellido and Quirós 1991; Garay and Garcia-Bellido 1993; Levin 1995*a,b*; Shapiro and Takata 1995; Kaloper and Olive 1998);
- authors that are aware of the physical non-equivalence of conformally related frames but do not present conclusive arguments in favour of one or the other of the two versions of the theory (and/or perform computations both in the Jordan and the Einstein frame) (Brans 1988; Jakubiec and Kijowski 1988; Kasper and Schmidt 1989; Deruelle and Spindel 1990; Hwang 1990; Kolb, Salopek and Turner 1990; Gottlöber, Müller and Starobinsky 1991; Suzuki and Yoshimura 1991; Rothman and Anninos 1991; Guendelman 1992; Guth and Jain 1992; Liddle and Wands 1992; Piccinelli, Lucchin and Matarrese 1992; Damour and Nordvedt 1993*a,b*; Cotsakis and Saich 1994; Hu, Turner and Weinberg 1994; Turner 1993; Mimoso and Wands 1995*b*; Faraoni 1996*a*; Weinstein 1996; Turner and Weinberg 1997; Majumdar 1997; Capozziello, de Ritis and Marino 1997; Dick 1998);
- authors that identify the Jordan frame as physical (possibly allowing the use of the conformal transformation as a purely mathematical tool) (Gross and Perry 1983; Barrow and Maeda 1992; Berkin, Maeda and Yokoyama 1990; Damour, Gibbons and Gundlach 1990; Kalara, Kaloper and Olive 1990; Berkin and Maeda 1991; Damour and Gundlach 1991; Holman, Kolb and Wang 1991; Mollerach and Matarrese 1992; Tao and Xue 1992; Wu 1992; del Campo 1992; Tkacev 1992; Mignemi and Whiltshire 1992; Barrow 1993; Bruckman and Velazquez 1993; Cotsakis and Flessas 1993; Will and Steinhardt 1995; Scheel, Shapiro and Teukolsky

1995; Barros and Romero 1997);

- authors that identify the Einstein frame as the physical one (Van den Bergh 1981, 1983*e*; Kunstatter, Lee and Leivo 1986; Gibbons and Maeda 1988; Sokolowski 1989*a,b*; Pimentel and Stein–Schabes 1989; Kubyshev, Rubakov and Tkachev 1989; Salopek, Bond and Bardeen 1989; Kolb, Salopek and Turner 1990; Cho 1990; Deruelle, Garriga and Verdaguer 1991; Cho 1992; Amendola *et al.* 1992; Amendola, Bellisai and Occhionero 1993; Cotsakis 1993; Cho and Yoon 1993; Alonso *et al.* 1994; Magnano and Sokolowski 1994; Cho 1994; Occhionero and Amendola 1994; Lu and Cheng 1996; Fujii 1997; Cho 1997; Cho and Keum 1998).

Sometimes, works by the same author(s) belong to two different groups; this illustrates the confusion on the issue that is present in the literature.

The two conformal frames, however, are substantially different. Furthermore, if a preferred conformal frame does not exist, it is possible to generate an infinite number of alternative theories and of cosmological inflationary scenarios by arbitrarily choosing the conformal factor (1.9) of the transformation (1.1). Only when a physical frame is uniquely determined the theory and its observable predictions are meaningful.

Earlier attempts to solve the problem in Brans–Dicke theory advocated the equivalence principle: to this end it is essential to consider not only the gravitational, but also the matter part of the Lagrangian. The use of the equivalence principle requires a careful analysis (Brans 1988; Magnano and Sokolowski 1994); by including the Lagrangian for ordinary matter in the Jordan frame action, one finds that, after the conformal transformation has been performed, the scalar field in the Einstein frame couples minimally to gravity, but nonminimally to matter (“non–universal coupling”). Historically, the Jordan frame was selected as physical because the dilaton couples minimally to ordinary matter in this frame (Brans and Dicke 1961). Attempts were also made to derive conclusive results from the conservation laws for the stress–energy tensor of matter, favouring the Jordan frame (Brans 1988) or the Einstein frame (Cotsakis 1993; Cotsakis 1995 – see (Teyssandier 1995; Schmidt 1995; Magnano and Sokolowski 1994) for the correction of a flaw in the proof of (Cotsakis 1993; Cotsakis 1995)). Indeed, the conservation laws do not allow one to draw definite conclusions (Magnano and Sokolowski 1994).

However, the point of view that selects the Jordan frame as physical is untenable because it leads to a negative definite, or indefinite kinetic energy for the scalar field; on the contrary, the energy density is positive definite in the Einstein frame. This result was initially proved for Brans–Dicke and for Kaluza–Klein theories, and later generalized to gravitational theories with Lagrangian density  $\mathcal{L} = f(R)\sqrt{-g} + \mathcal{L}_{matter}$  (Magnano and Sokolowski 1994). This implies that the theory does not have a stable ground state, and

that the system decays into a lower energy state *ad infinitum* (Gross and Perry 1983; Appelquist and Chodos 1983; Maeda 1986*b*; Maeda 1987). While a stable ground state may not be required for certain particular solutions of the theory (e.g. cosmological solutions (Padmanabhan 1988)), or for Liouville's theory (D'Hoker and Jackiw 1982), it is certainly necessary for a viable theory of classical gravity. The ground state of the system must be stable against small fluctuations and not fine-tuned, i.e. nearby solutions of the theory must have similar properties (Strater and Wightman 1964; Epstein, Glaser and Jaffe 1965; Abbott and Deser 1982). The fact that the energy is not positive definite is usually associated with the formulation of the theory in unphysical variables. On the contrary, the energy conditions (Wald 1984) are believed to be satisfied by all classical matter and fields (not so in quantum theories – see Sec. 8). This decisive argument was first used to select the Einstein frame in Kaluza–Klein and Brans–Dicke theories (Bombelli *et al.* 1987; Sokolowski and Carr 1986; Sokolowski 1989*a,b*; Sokolowski and Golda 1987; Cho 1990; Cho 1994), and later generalized to scalar–tensor theories (Cho 1997) and nonlinear gravity theories (Magnano and Sokolowski 1994). Also, the uniqueness of a physical conformal frame was proved (Sokolowski 1989*a,b*; Magnano and Sokolowski 1994).

For completeness, we mention other arguments supporting the Einstein frame as physical that have appeared in the literature: however, they are either highly questionable (sometimes to the point of not being valid), or not as compelling as the one based on the positivity of energy. The Hilbert and the Palatini actions for scalar–tensor theories are equivalent in the Einstein but not in the Jordan frame (Van den Bergh 1981, 1983*e*). Some authors choose the Einstein frame on the basis of the resemblance of its action with that of general relativity (Gibbons and Maeda 1988; Pimentel and Stein–Schabes 1989; Alonso *et al.* 1994; Amendola, Bellisai and Occhionero 1993); others (Salopek, Bond and Bardeen 1989; Kolb, Salopek and Turner 1990) find difficulties in quantizing the scalar field fluctuations in the linear approximation in the Jordan frame, but not in the Einstein frame; quantization and the conformal transformation do not commute (Fujii and Nishioka 1990; Nishioka and Fujii 1992; Fakir and Habib 1993). Other authors claim that the Einstein frame is forced upon us by the compactification of the extra dimensions in higher dimensional theories (Kubyshev, Rubakov and Tkachev 1989; Deruelle, Garriga and Verdaguer 1991).

A possible alternative to the Einstein frame formulation of the complete theory (gravity plus matter) has been supported (Magnano and Sokolowski 1994), and consists in starting with the introduction of matter non-minimally coupled to the Brans–Dicke scalar in the Jordan frame, with the coupling tuned in such a way that the Einstein frame action exhibits matter minimally coupled to the Einstein frame scalar field, after

the conformal transformation has been performed. This procedure arises from the observation (Magnano and Sokolowski 1994) that the traditional way of prescribing matter minimally coupled in the Jordan frame relies on the implicit assumptions that

- i)* the equivalence principle holds;
- ii)* the Jordan frame is the physical one.

While these assumptions are not justified *a priori*, as noted by Magnano and Sokolowski (1994), the possibility of adding matter in the Jordan frame with a coupling that exactly balances the exponential factor appearing in the Einstein frame appears to be completely *ad hoc* and is not physically motivated; by proceeding along these lines, one could arbitrarily change the theory without theoretical justification.

As a summary, the Einstein frame is the physical one (and the Jordan frame and all the other conformal frames are unphysical) for the following classes of theories:

- scalar–tensor theories of gravity described by the Lagrangian density

$$\mathcal{L} = \sqrt{-g} \left[ f(\phi)R - \frac{\omega(\phi)}{\phi} \nabla^\mu \phi \nabla_\mu \phi + \Lambda(\phi) \right] + \mathcal{L}_{matter} , \quad (3.2)$$

which includes Brans–Dicke theory as a special case (see Sec. 4 for the corresponding field equations);

- classical Kaluza–Klein theories;
- nonlinear theories of gravity whose gravitational part is described by the Lagrangian density  $\mathcal{L} = \sqrt{-g}f(R)$  (see Sec. 4 for the corresponding field equations).

Since the Jordan frame formulation of alternative theories of gravity is unphysical, one reaches the conclusion that the Einstein frame formulation is the only possible one for a classical theory. In other words, this amounts to say that Einstein gravity is essentially the only viable classical theory of gravity (Bicknell 1974; Magnano and Sokolowski 1994; Magnano 1995; Sokolowski 1997). We remark that this statement is strictly correct only if the purely gravitational part of the action (without matter) is considered: in fact, when matter is included into the action, in general it exhibits an anomalous coupling to the scalar field which does not occur in general relativity.

Finally, we comment on the case of a nonminimally coupled scalar field described by the action (2.7). From the above discussion, one may be induced to believe that the Einstein frame description is necessary also in this case: this conclusion would be incorrect because the kinetic term in the action (2.7) is canonical and positive definite, and the problem discussed above for other theories of gravity of the form  $\mathcal{L} = \sqrt{-g}f(R)$

does not arise. It is, however, still true that the Einstein and the Jordan frame are physically inequivalent: the conformal transformation (1.1), (2.9), (2.10) implies only a mathematical, not a physical equivalence, despite strong statements on this regard that point to the contrary (Accioly *et al.* 1993).

## 4 Conformal transformations in gravitational theories

In this section, we review in greater detail the arguments that led to the conclusions of the previous section, devoting more attention to specific classical theories of gravity.

**Brans–Dicke theory:** The Jordan–Fierz–Brans–Dicke theory (Jordan 1949; Jordan 1955; Fierz 1956; Brans and Dicke 1961) described by the action (2.1) (where  $\phi$  has the dimensions of the inverse gravitational constant,  $[\phi] = [G^{-1}]$ ) has been the subject of renewed interest, especially in cosmology in the extended inflationary scenario (La and Steinhardt 1989; Kolb, Salopek and Turner 1990; Laycock and Liddle 1994). The recent surge of interest appears to be motivated by a restricted conformal invariance of the gravitational part of the Lagrangian that mimics the conformal invariance of string theories before conformal symmetry is broken (Cho 1992; Cho 1994; Turner 1993; Kolitch and Eardley 1995; Brans 1997; Cho and Keum 1998). The conformal transformation (1.1) with  $\Omega = (G\phi)^\alpha$ , together with the redefinition of the scalar field  $\tilde{\phi} = G^{-2\alpha}\phi^{1-2\alpha}$  ( $\alpha \neq 1/2$ ) maps the Brans–Dicke action (2.1) into an action of the same form, but with parameter

$$\tilde{\omega} = \frac{\omega - 6\alpha(\alpha - 1)}{(1 - 2\alpha)^2}. \quad (4.1)$$

If  $\omega = -3/2$ , the action (2.1) is invariant under the conformal transformation; this case corresponds to the singularity  $\alpha \rightarrow -1/2$  in the expression (4.1), but the field equations (2.2), (2.3) are not defined in this case. This conformal invariance is broken when a term describing matter with  $T \equiv T^\mu{}_\mu \neq 0$  is added to the purely gravitational part of the Brans–Dicke Lagrangian. This property of conformal invariance of the gravitational sector of the theory is enjoyed also by a subclass of more general tensor–multiscalar theories of gravity (Damour and Esposito–Farèse 1992) and has not yet been investigated in depth in the general case. The study of the conformal invariance property of Brans–Dicke theory helps to solve the problems arising in the  $\omega \rightarrow \infty$  limit of Brans–Dicke theory (Faraoni 1998). This limit is supposed to give back general relativity, but it

fails to do so when  $T = 0$ . The differences between the Jordan and the Einstein frame formulations of Brans–Dicke theory have been pointed out clearly in (Guth and Jain 1992). It has been noted in studies of gravitational collapse to a black hole in Brans–Dicke theory that the noncanonical form of the Brans–Dicke scalar energy–momentum tensor in the Jordan frame violates the null energy condition ( $R_{\alpha\beta}l^\alpha l^\beta \geq 0$  for all null vectors  $l^\alpha$ ). This fact is responsible for a *decrease* in time of the horizon area during the dynamical phase of the collapse, contrarily to the case of general relativity (Scheel, Shapiro and Teukolsky 1995). The violation of the weak energy condition in the Jordan frame has also been pointed out (Weinstein 1996; Faraoni and Gunzig 1998a).

Brans–Dicke theory must necessarily be reformulated in the Einstein frame; the strongest argument supporting this conclusion is obtained by observing that the kinetic energy term for the Brans–Dicke field in the Jordan frame Brans–Dicke Lagrangian does not have the canonical form for a scalar field, and it is negative definite (Gross and Perry 1983; Appelquist and Chodos 1983; Maeda 1986*b*; Maeda 1987; Sokolowski and Carr 1986; Maeda and Pang 1986; Sokolowski 1989*a,b*; Cho 1992, 1993; Magnano and Sokolowski 1994). The fact that this energy argument was originally developed for Brans–Dicke and for Kaluza–Klein theories is not surprising, owing to the fact that Brans–Dicke theory can be derived from a Kaluza–Klein theory with  $n$  extra dimensions and Brans–Dicke parameter  $\omega = -(n - 1)/n$  (Jordan 1959; Brans and Dicke 1961; Bergmann 1968; Wagoner 1970; Harrison 1972; Belinskii and Kalatnikov 1973; Freund 1982; Gross and Perry 1983; Cho 1992); this derivation is seen as a motivation for Brans–Dicke theory, and provides a useful way of generating exact solutions in one theory from known solutions in the other (Billyard and Coley 1997). Despite this derivation from Kaluza–Klein theory, the Jordan frame Brans–Dicke theory is sometimes considered in  $D > 4$  spacetime dimensions (e.g. Majumdar 1997).

An independent argument supporting the choice of the Einstein frame as the physical one is obtained by considering (Cho 1992; Damour and Nordvedt 1993*a,b*) the linearized version of the theory. In the Jordan frame the metric is  $\gamma_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$  (where  $\eta_{\mu\nu}$  is the Minkowski metric), while in the Einstein frame the conformally transformed metric is

$$\tilde{\gamma}_{\mu\nu} = \gamma_{\mu\nu} \exp \left( \sqrt{\frac{16\pi G}{2\omega + 3}} \tilde{\phi} \right) \simeq \eta_{\mu\nu} + \rho_{\mu\nu}, \quad (4.2)$$

where

$$\rho_{\mu\nu} = h_{\mu\nu} + \left( \sqrt{\frac{16\pi G}{2\omega + 3}} \tilde{\phi} \right) \eta_{\mu\nu}. \quad (4.3)$$

The canonical action for a spin 2 field is not obtained from the metric  $h_{\mu\nu}$ , but it is

instead given by  $\rho_{\mu\nu}$ , and the spin 2 gravitational field is described by the Einstein frame corrections  $\rho_{\mu\nu}$  to the flat metric. The Jordan frame corrections  $h_{\mu\nu}$  to  $\eta_{\mu\nu}$  describe a mixture of spin 0 and spin 2 fields (the fact that spin 0 and spin 2 modes are mixed together can also be seen from the full equations of motion of the theory).

A third argument has been proposed against the choice of the Jordan frame as the physical one: when quantum corrections are taken into account, one cannot maintain the minimal coupling of ordinary (i.e. other than the dilaton) matter to the Jordan metric (Cho 1997). This nullifies the traditional statement that the Jordan frame is to be preferred because the scalar couples minimally to all forms of matter in this frame.

These results are of the outmost importance for the experiments aimed at testing Einstein's theory: the Jordan frame versions of alternative classical theories of gravity are simply nonviable. However, despite the necessity of formulating Brans–Dicke theory in the Einstein frame, the classical tests of gravity for this theory are studied only for the Jordan frame formulation. In general, the authors working on the experimental tests of general relativity and alternative gravity theories do not seem to be aware of this paradoxical situation (e.g. Reasenberg *et al.* 1979; Will 1993).

The conformal rescaling has been used as a mathematical technique to generate exact solutions of Brans–Dicke theory from known solutions of the Einstein equations (Harrison 1972; Belinskii and Kalatnikov 1973; Lorentz–Petzold 1984) and approximate solutions in the linearized theory (Barros and Romero 1997).

**(Generalized) scalar–tensor theories:** This class of theories (Bergmann 1968; Wagoner 1970; Nordvedt 1970; Will 1993) is described by the Lagrangian density

$$S = \int d^4x \sqrt{-g} \left[ f(\phi)R - \frac{\omega}{2} \nabla^\alpha \phi \nabla_\alpha \phi - V(\phi) \right] + S_{matter} , \quad (4.4)$$

where  $\omega = \omega(\phi)$  and  $V = V(\phi)$  (or by the more general action (2.15)). The corresponding field equations are

$$f(\phi) \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) = \frac{1}{2} T_{\mu\nu} + \frac{\omega}{2} \left( \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \nabla^\alpha \phi \nabla_\alpha \phi \right) + \frac{1}{2} g_{\mu\nu} [Rf(\phi) - 2V] + \nabla_\mu \nabla_\nu f - g_{\mu\nu} \square f , \quad (4.5)$$

$$\square \phi + \frac{1}{\omega} \left( \frac{1}{2} \frac{d\omega}{d\phi} \nabla^\alpha \phi \nabla_\alpha \phi + \frac{df}{d\phi} - \frac{dV}{d\phi} \right) = 0 , \quad (4.6)$$

where  $T_{\mu\nu} = 2(-g)^{-1/2} \delta \mathcal{L}_{matter} / \delta g^{\mu\nu}$ . The action (4.4) contains Brans–Dicke theory (2.1) and the nonminimally coupled scalar field theory (2.7) as particular cases. Theories with

more than one scalar field have also been investigated (Damour and Esposito-Farèse 1992; Berezin *et al.* 1989; Rainer and Zuhk 1996). A revival of interest in scalar-tensor theories was generated by the fact that in supergravity and superstring theories, scalar fields are associated to the metric tensor field, and that a coupling between a scalar field and gravity seems unavoidable in string theories (Green, Schwarz and Witten 1987). Indeed, scalar fields have been present in relativistic gravitational theories even before general relativity was formulated (see Brans 1997 for an historical perspective).

The necessity of the conformal transformation to the Einstein frame has been advocated in (Cho 1992, 1997) by investigating which linearized metric describes the physical spin 2 gravitons. A similar argument was presented in (Damour and Nordvedt 1993*a,b*), although these authors did not see it as a compelling reason to select the Einstein frame as the physical one. It has also been pointed out (Teyssandier and Tournenc 1983; Damour and Esposito-Farèse 1992; Damour and Nordvedt 1993*a,b*) that the mixing of  $g_{\mu\nu}$  and  $\phi$  in the Jordan frame equations of motion makes the Jordan frame variables an inconvenient set for formulating the Cauchy problem. Moreover, the generalization to the case of tensor-multi scalar theories of gravitation, where several scalar fields instead of a single one appear, is straightforward in the Einstein frame but not so in the Jordan frame (Damour and Esposito-Farèse 1992). In the Einstein frame, ordinary matter does not obey the conservation law  $\tilde{\nabla}^\nu \tilde{T}_{\mu\nu} = 0$  (with the exception of a radiative fluid with  $\tilde{T} = 0$ , which is conformally invariant) because of the coupling to the dilaton  $\phi$ . Instead, the equation

$$\tilde{\nabla}_\nu \tilde{T}^{\mu\nu} = -\frac{1}{\Omega} \frac{\partial \Omega}{\partial \phi} \tilde{T} \tilde{\nabla}^\mu \phi \quad (4.7)$$

is satisfied. The total energy-momentum tensor of matter plus the scalar field is conserved (see Magnano and Sokolowski 1994 for a detailed discussion of conservation laws in both conformal frames).

The phenomenon of the propagation of light through scalar-tensor gravitational waves and the resulting time-dependent amplification of the light source provide an example of the physical difference between the Jordan and the Einstein frame. In the Jordan frame the amplification effect is of first order in the gravitational wave amplitude (Faraoni 1996*a*), while it is only of second order in the Einstein frame (Faraoni and Gunzig 1998*a*).

It is interesting to note that, while the observational constraints on the Brans-Dicke parameter  $\omega$  is  $\omega > 500$  (Reasenberg *et al.* 1979), Brans-Dicke theory in the Einstein frame is subject to the much more stringent constraint  $\omega > 10^8$  (Cho 1997). However, since the Einstein frame is the physical one, it is not very meaningful to present constraints on the Jordan frame parameter  $\omega$ . Other formal and physical differences occur

in the Jordan and the Einstein frame: the singular points  $\omega \rightarrow \infty$  in the  $\omega$ -parameter space of the Jordan frame correspond to a minimum of the coupling factor  $\ln \Omega(\phi)$  in the Einstein frame (Damour and Nordvedt 1993*a,b*). Singularities of the scalar-tensor theory may be smoothed out in the Jordan frame, but they reappear in the Einstein frame and plague the theory again due to the fact that the kinetic terms are canonical and the energy conditions (which are crucial in the singularity theorems) are satisfied in the Einstein frame (Kaloper and Olive 1998).

In (Bose and Lohiya 1997), the quasi-local mass defined in general relativity by the recent Hawking-Horowitz prescription (Hawking and Horowitz 1996) was generalized to  $n$ -dimensional scalar-tensor theories. It was shown that this quasi-local mass is invariant under the conformal transformation that reduces the gravitational part of the scalar-tensor theory to canonical Einstein gravity. The result holds under the assumptions that the conformal factor  $\Omega(\phi)$  is a monotonic function of the scalar field  $\phi$ , and that a global foliation of the spacetime manifold with spacelike hypersurfaces exists, but it does not require asymptotic flatness. Conformal invariance of the quasi-local mass was previously found in another generalization to scalar-tensor theories of the Hawking-Horowitz quasi-local mass (Chan, Creighton and Mann 1996).

The conformal transformation technique has been used to derive new solutions to scalar-tensor theories from known solutions of Einstein's theory (Van den Bergh 1980, 1982, 1983*a,b,c,d*; Barrow and Maeda 1990).

**Nonlinear gravitational theories:** a yet more general class of theories than the previous one is described by the Lagrangian density

$$\mathcal{L} = f(R)\sqrt{-g} + \mathcal{L}_{matter} , \quad (4.8)$$

which generates the field equations

$$\left(\frac{df}{dR}\right) R_{\mu\nu} - \frac{1}{2} f(R) g_{\mu\nu} - \nabla_{\mu} \nabla_{\nu} \left(\frac{df}{dR}\right) + g_{\mu\nu} \square \left(\frac{df}{dR}\right) = T_{\mu\nu} , \quad (4.9)$$

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta \mathcal{L}_{matter}}{\delta g^{\mu\nu}} . \quad (4.10)$$

It is claimed in (Magnano and Sokolowski 1994) that the Einstein frame is the only physical one for this class of theories, using the energy argument of Sec. 3. The idea underlying the proof is to expand the function  $f(R)$  as

$$f(R) = R + aR^2 + \dots , \quad a > 0 , \quad (4.11)$$

and then prove a positive energy theorem in the Einstein frame and the indefiniteness of the energy sign in the Jordan frame. The occurrence of singularities in higher order theories of gravity of the form (4.8) has been studied in (Barrow and Cotsakis 1988; Miritzis and Cotsakis 1996; Kaloper and Olive 1998), both in the Jordan and in the Einstein frame.

**Kaluza–Klein theories:** In classical Kaluza–Klein theories (Appelquist, Chodos and Freund 1987; Bailin and Love 1987; Overduin and Wesson 1997), the scalar field (dilatation) has a geometrical origin and corresponds to the scale factor of the extra spatial dimensions. The extra dimensions manifest themselves as matter (scalar fields) in the 4–dimensional spacetime. In the simplest version of the theory with a single scalar field<sup>2</sup>, one starts with the  $(4 + d)$ –dimensional action of vacuum general relativity

$$\hat{S} = \frac{1}{16\pi G} \int d^{(4+d)}x (\hat{R} + \hat{\Lambda}) \sqrt{-\hat{g}}, \quad (4.12)$$

where a caret denotes higher–dimensional quantities, the  $(4 + d)$ –dimensional metric has the form

$$(\hat{g}_{AB}) = \begin{pmatrix} \hat{g}_{\mu\nu} & 0 \\ 0 & \hat{\phi}_{ab} \end{pmatrix}, \quad (4.13)$$

and  $\hat{\Lambda}$  is the cosmological constant of the  $(4 + d)$ –dimensional spacetime manifold. The latter is assumed to have the structure  $M \otimes K$ , where  $M$  is 4–dimensional and  $K$  is  $d$ –dimensional. Here the notations depart from those introduced at the beginning of this paper: the indices  $A, B, \dots = 0, 1, 2, 3, \dots, (4 + d)$ ;  $\mu, \nu, \dots = 0, 1, 2, 3$ , and  $a, b, \dots = 4, 5, \dots, (4 + d)$ . Dimensional reduction and the conformal transformation (1.1) with  $\Omega = \sqrt{\phi}$ ,  $\phi = |\hat{\phi}_{ab}|$ , together with the redefinition of the scalar field

$$d\sigma = \frac{1}{2} \left( \frac{d+2}{16\pi G d} \right)^{1/2} \frac{d\phi}{\phi}, \quad (4.14)$$

leads to the Einstein frame action

$$S = \int d^4x \left[ \frac{R}{16\pi G} - \frac{1}{2} \nabla_\mu \sigma \nabla^\mu \sigma - V(\sigma) \right] \sqrt{-g}, \quad (4.15)$$

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<sup>2</sup>See e.g. (Berezin *et al.* 1989; Rainer and Zuhk 1996) for Kaluza–Klein theories with multiple dilatons.

$$V(\sigma) = \frac{R_K}{16\pi G} \exp\left(-\sqrt{\frac{16\pi G(d+2)}{d}}\sigma\right) + \frac{\hat{\Lambda}}{16\pi G} \exp\left(-\sqrt{\frac{16\pi Gd}{d+2}}\sigma\right), \quad (4.16)$$

were  $R_K$  is the Ricci curvature of the metric on the submanifold  $K$ . Note that  $\phi$  is dimensionless. However the redefined scalar field  $\sigma$  has the dimensions  $[\sigma] = [G^{-1/2}]$ , and is usually measured in Planck masses.

Unfortunately, the omission of a factor  $1/\sqrt{16\pi G}$  in the right hand side of Eq. (4.14) seems to be common in the literature on Kaluza–Klein cosmology (cf. (Faraoni, Cooperstock and Overduin 1995) and footnote 11 of (Kolb, Salopek and Turner 1990)) and it leads to a non-canonical kinetic term  $(16\pi G)^{-1}\nabla_\mu\sigma\nabla^\mu\sigma$  instead of  $\nabla_\mu\sigma\nabla^\mu\sigma/2$  in the final action, and to a dimensionless field  $\sigma$  instead of one with the correct dimensions  $[\sigma] = [G^{-1/2}]$ . The error is perhaps due to the different notations used by particle physicists and by relativists; however insignificant it may appear to be, it profoundly affects the viability of the Kaluza–Klein cosmological model considered, since the spectral index of density perturbations is affected through the arguments of the exponentials in the scalar field potential (4.16) (Faraoni, Cooperstock and Overduin 1995). In the Jordan frame, the scalar field originating from the presence of the extra dimensions has kinetic energy that is negative definite or indefinite and an energy spectrum which is unbounded from below, implying that the ground state is unstable (Maeda 1986*a*; Maeda and Pang 1986; Sokolowski and Carr 1986; Sokolowski and Golda 1987). These defects are removed by the conformal rescaling (1.1) of the 4-dimensional metric. The requirement that the conformally rescaled system in 4 dimensions has positive definite energy (a purely classical argument) singles out a *unique* conformal factor. A proof of the uniqueness in 5-dimensional Kaluza–Klein theory was given in (Bombelli *et al.* 1987) and later generalized to an arbitrary number of extra spatial dimensions (Sokolowski 1989*a, b*).

From a quantum point of view, arguments in favour of the conformal rescaling have been pointed out (Maeda 1986*b*) and, in the context of 10- and 11-dimensional supergravity, the need for a conformal transformation in order to identify the physical fields was recognized (Scherk and Schwarz 1979; Chamseddine 1981; Dine *et al.* 1985). The requirement that the supersymmetry transformation of 11-dimensional supergravity take an  $SU(8)$  covariant form leads to the same conformal factor (de Witt and Nicolai 1986). The conformal transformation which works as a cure for the dimensionally reduced  $(4+d)$ -dimensional Einstein gravity does not work for the dimensionally reduced Gauss–Bonnet theory (Sokolowski *et al.* 1991).

It is unfortunate that in the literature on Kaluza–Klein theories many authors neglected the conformal rescaling and only performed computations in the Jordan frame. Many results of classical Kaluza–Klein theories should be reanalysed in the Einstein frame (e.g. Alvarez and Belén Gavela 1983; Sadhev 1984; Deruelle and Madore 1987; Van den Bergh and Tavakol 1993; Fabris and Sakellariadou 1997; Kubyshev and Martin 1995; Fabris and Martin 1993; Chatterjee and Banerjee 1993; Biesiada 1994).

**Torsion gravity:** Theories of gravity with torsion have been studied in order to incorporate the quantum mechanical spin of elementary particles, or in attempts to formulate gauge theories of gravity (Hehl *et al.* 1976). An example is given by a theory of gravity with torsion, related to string theories, recently formulated both in the Jordan and in the Einstein frame (Hammond 1990, 1996). Torsion acts as a source of the scalar field; ordinary (i.e. different from the scalar field appearing in (1.9)) matter is added to the theory formulated in the Jordan or in the Einstein frame. This possibility differs from a conformal transformation of the total (gravity plus matter) system to the Einstein frame, and it does not appear to be legitimate since ordinary matter cannot be created as an effect of a conformal transformation. Although mathematically possible, this procedure appears to be very artificial, and it has been considered also in (Magnano and Sokolowski 1994) by including a nonminimal coupling of the scalar field to matter in the Jordan frame. The coupling was tuned in such a way that the Einstein frame matter is minimally coupled to the corresponding scalar field. The Jordan frame formulation of this theory is unviable because the large effects of the dilaton contradict the observations (Hammond 1996), and the Einstein frame version of this theory is the only possible option.

Induced gravity, which is described by the action

$$S = \int d^4x \sqrt{-g} \left[ -\frac{\xi}{2} R \phi^2 - \frac{1}{2} \nabla^\mu \phi \nabla_\mu \phi - V(\phi) \right], \quad (4.17)$$

is conformally invariant if  $\xi = 1/6$ . The field equations are

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{1}{\xi \phi^2} \left[ (1 - 4\xi) \nabla_\mu \nabla_\nu \phi + g_{\mu\nu} \left( 2\xi - \frac{1}{2} \right) \nabla^\alpha \phi \nabla_\alpha \phi - V g_{\mu\nu} + 2\xi g_{\mu\nu} \phi \square \phi \right], \quad (4.18)$$

$$\square \phi - \xi R \phi - \frac{dV}{d\phi} = 0. \quad (4.19)$$

Induced gravity with torsion in Riemann–Cartan spacetimes has been studied in (Park and Yoon 1997), and a generalization of the concept of conformal invariance has been

formulated.

**Superstring theories:** Although superstring theories are not classical theories of gravity, the effective action in the low energy limit is used to make predictions in the classical domain, and we comment upon this. The low-energy effective action for the bosonic string theory is given by (Callan *et al.* 1985)

$$S = \int d^{10}x \sqrt{-g} \left[ e^{-2\Phi} R + 4\nabla^\mu \Phi \nabla_\mu \Phi \right] + S_{matter} , \quad (4.20)$$

where  $\Phi$  is the dimensionless string dilaton and the totally antisymmetric 3-form  $H_{\mu\nu\lambda}$  appearing in the theory has been set equal to zero together with the cosmological constant (however, this is not always the case in the literature). By means of dimensional reduction and a conformal transformation, this model is reduced to 4-dimensional canonical gravity with two scalar fields:

$$\psi_1 = \frac{1}{\sqrt{16\pi G}} \left( 6 \ln b - \frac{\Phi}{2} \right) , \quad (4.21)$$

$$\psi_2 = \sqrt{\frac{3}{8\pi G}} \left( 2 \ln b + \frac{\Phi}{2} \right) , \quad (4.22)$$

where  $b$  is the radius of the manifold of the compactified extra dimensions. The action (4.20) has provided theoreticians with several cosmological models (Gasperini, Maharana and Veneziano 1991; Garcia-Bellido and Quiròs 1992; Gasperini and Veneziano 1992; Gasperini, Ricci and Veneziano 1993; Gasperini and Ricci 1993; Copeland, Lahiri and Wands 1994, 1995; Batakis 1995; Batakis and Kehagias 1995; Barrow and Kunze 1997). The issue of which conformal frame is physical in the low energy limit of string theories was raised in (Dick 1998).

## 5 Conformal transformations in cosmology

The standard big-bang cosmology based on general relativity is a very successful description of the universe that we observe, although cosmological solutions have been studied also in alternative theories of gravity. However, the need to solve the horizon, flatness and monopole problem, and to find a viable mechanism for the generation of density fluctuations evolving into the structures that we see today (galaxies, clusters, superclusters and voids) motivated research beyond the big-bang model and led to the

idea of cosmological inflation (see Linde 1990; Kolb and Turner 1990; Liddle and Lyth 1993; Liddle 1996 for reviews). There is no universally accepted model of inflation, and several scenarios based either on general relativity or on alternative theories of gravity have been proposed. Since many of the alternative theories used involve a conformal transformation to a new conformal frame, it is natural that the problem of whether the Jordan or the Einstein frame is the physical one resurfaces in cosmology, together with the use of conformal rescalings to simplify the study of the equations of motion. It is possible that general relativity behaves as an attractor for scalar-tensor theories of gravity, and that a theory which departs from general relativity at early times in the history of the universe approaches general relativity during the matter-dominated era (Garcia-Bellido and Quirós 1990; Damour and Nordvedt 1993*a,b*; Mimoso and Wands 1995*a*; Oukouiss 1997) or even during inflation (Bekenstein and Meisels 1978; Garcia-Bellido and Quirós 1990; Barrow and Maeda 1990; Steinhardt and Accetta 1990; Damour and Vilenkin 1996) (unfortunately only the Jordan frame was considered in (Garcia-Bellido and Quirós 1990; Mimoso and Wands 1995*a*)). The convergence to general relativity cannot occur during the radiation-dominated era (Faraoni 1998).

One of the most important predictions of an inflationary scenario is the spectral index of density perturbations, which can already be compared with the observations of cosmic microwave background anisotropies and of large scale structures (Liddle and Lyth 1993). The spectral index is, in general, different in versions of the same scalar-tensor theory formulated in different conformal frames. For example, it is known that most classical Kaluza-Klein inflationary models based on the Jordan frame are allowed by the observations but are theoretically unviable (Sokolowski 1989*a,b*; Cho 1992) because of the energy argument discussed in Sec. 3; on the contrary, their Einstein frame counterparts are theoretically consistent but they are severely restricted or even forbidden by the observations of cosmic microwave background anisotropies (Faraoni, Cooperstock and Overduin 1995). In extended (La and Steinhardt 1989; Kolb, Salopek and Turner 1990; Laycock and Liddle 1994) and hyperextended (Steinhardt and Accetta 1990; Liddle and Wands 1992; Crittenden and Steinhardt 1992) inflation, differences between the density perturbations in the two frames have been pointed out (Kolb, Salopek and Turner 1990). The existing confusion on the problem of whether the Jordan or the Einstein frame is the physical one is particularly evident in the literature on inflation, and deeply affects the viability of the inflationary scenarios based on a theory of gravity which has a conformal transformation as an ingredient. Among these are extended (La and Steinhardt 1989; Laycock and Liddle 1994) and hyperextended (Kolb, Salopek and Turner 1990; Steinhardt and Accetta 1990; Liddle and Wands 1992; Crittenden and Steinhardt 1992) inflation, Kaluza-Klein (Yoon and Brill 1990; Cho and Yoon 1993; Cho 1994),

$R^2$ -inflation (Starobinski 1980; Starobinski 1986; Maeda, Stein-Schabes and Futamase 1989; Liddle and Lyth 1993), soft and induced gravity inflation (Accetta, Zoller and Turner 1985; Accetta and Trester 1989; Salopek, Bond and Bardeen 1989). While several authors completely neglect the problem of which frame is physical, other authors present calculations in only one frame, and others again perform calculations in both frames, without deciding whether one of the two is physically preferred. Sometimes, the two frames are implicitly treated as if they both were simultaneously physical, and part of the results are presented in the Jordan frame, part in the Einstein frame. It is often remarked that all models of inflation based on a first order phase transition can be recast as slow-roll inflation using a conformal transformation (Kolb, Salopek and Turner 1990; Kalara, Kaloper and Olive 1990; Turner 1993; Liddle 1996), but the conformal rescaling is often performed without physical motivation. The justification for studying the original (i.e. prior to the conformal transformation) theory of gravity or inflationary scenario, which often relies on a specific theory of high energy physics, is then completely lost in this way. For example, one can start with a perturbatively renormalizable potential in the Jordan frame and most likely one ends up with a non-renormalizable potential in the Einstein frame. The conformal rescaling has even been used to vary the Jordan frame gravitational theory in order to obtain a pre-determined scalar field potential in the Einstein frame (Cotsakis and Saich 1994).

It is to be noted that the conformal transformation to a new conformal frame is sometimes used as a purely mathematical device to compute cosmological solutions by reducing the problem to a familiar (and computationally more convenient) scenario. The conformal transformation technique has been used to study also cosmological perturbations in generalized gravity theories (Hwang 1990; Mukhanov, Feldman and Brandenberger 1992; Hwang 1997*a*). This technique is certainly legitimate and convenient at the classical level, but it leads to problems when quantum fluctuations of the inflaton field are computed in the new conformal frame, and the result is mapped back into the "old" frame. Problems arise already at the semiclassical level (Duff 1981). This difficulty does not come as a surprise, since the conformal transformation introduces a mixing of the degrees of freedom corresponding to the scalar and the tensor modes. In general, the fluctuations in the two frames are physically inequivalent (Fujii and Nishioka 1990; Makino and Sasaki 1991; Nishioka and Fujii 1992; Fakir and Habib 1993; Fabris and Tossa 1997). There is ambiguity in the choice of vacuum states for the quantum fields: if a vacuum is chosen in one frame, it is unclear into what state the field is mapped in the other conformal frame, and one will end up, in general, with two different quantum states. The use of gauge-invariant quantities does not fix this problem (Fakir, Habib and Unruh 1992). The problem that plagues quantum fluctuations becomes relevant for

present-day observations because the quantum perturbations eventually become classical (Kolb and Turner 1990; Liddle and Lyth 1993; Tanaka and Sakagami 1997) and seed galaxies, clusters and superclusters.

Although the problem is not solved in general, the situation is not so bad in certain specific inflationary scenarios. In (Sasaki 1986; Makino and Sasaki 1991; Fakir, Habib and Unruh 1992), chaotic inflation with the quartic potential  $V = \lambda\phi^4$  and nonminimal coupling of the scalar field was studied, and it was found that the amplitude of the density perturbations does not change under the conformal transformation. This result, however, relies on the assumption that one can split the inflaton field into a classical background plus quantum fluctuations (preliminary results when the decomposition is not possible have been obtained in (Nambu and Sasaki 1990)). Under slow-roll conditions in induced gravity inflation, the spectral index of density perturbations is frame-independent to first order in the slow-roll parameters (Kaiser 1995a). When the expansion of the universe is de Sitter-like,  $a(t) \propto e^{Ht}$ ,  $\dot{H} \approx 0$ , it was found that the magnitude of the two-point correlation function is affected by the conformal transformation, but its dependence on the wavenumber, and consequently also the spectral index, is not affected (Kaiser 1995b). The spectral indices differ in the two conformal frames when the expansion of the scale factor is close to a power law<sup>3</sup> (Kaiser 1995b); often, workers in the field have not been sufficiently careful in this regard. Certain gauge-invariant quantities related to the cosmological perturbations turn out to be also conformally invariant under a mathematical condition satisfied by power law inflation and by the pole-like inflation encountered in the pre-big bang scenario of low energy string theory (Hwang 1997b).

At the level of the classical, *unperturbed* cosmological model, the occurrence of slow-roll inflation in the Einstein frame does not necessarily imply that inflation occurs also in the Jordan frame, or that it is of the slow-roll type, and the expansion law of the scale factor is in general different in the two conformal frames<sup>4</sup> (see Abreu, Crawford and Mimoso 1994 for an example).

Possible approaches to this problem are outlined in (Fakir and Habib 1993). Even if the same expansion law is achieved in the Jordan and the Einstein frame, the corresponding scalar field potentials can be quite different in the two frames. For example, power-law inflation is achieved by an exponential potential for a minimally coupled

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<sup>3</sup>If inflation occurs in the early universe, it is not necessarily of the slow-roll type. The most well studied case of inflation without slow rolling is power law inflation which occurs for exponential potentials, obtained in almost all theories formulated in the Einstein frame.

<sup>4</sup>For extended inflation in Brans-Dicke theory with  $\omega \gg 1$ , it has been proved that slow-roll inflation in the Einstein frame implies slow-roll inflation in the Jordan frame (but not viceversa) (Lidsey 1992; Green and Liddle 1996).

scalar field in the Einstein frame, and by a polynomial potential for its nonminimally coupled cousin in the Jordan frame (Abreu, Crawford and Mimoso 1994; Futamase and Maeda 1989; Faraoni 1996*b*).

Another cosmologically relevant aspect of the scalar field appearing in (1.9) is that it may contribute a significant fraction of the dark matter in the universe (Cho 1990; Cho and Yoon 1993; Delgado 1994; McDonald 1993*a,b*; Gasperini and Veneziano 1994; Gasperini 1994; Cho and Keum 1998). If one accepts the idea that the scalar field appearing in the expression for the conformal factor (1.9) is the field driving inflation (Salopek 1992; Cho 1992, 1994), then the inflationary scenario is completely determined. In fact, the conformal transformation to the Einstein frame in cosmology leads to either *a*) an exponential potential for the scalar field and to power-law inflation; *b*) a potential with more than one exponential term in Kaluza-Klein theories (Yoon and Brill 1990; Cho 1990; Cho and Yoon 1993), and to a kind of inflation that interpolates between power-law and de Sitter inflation (Easther 1994). It is also to be noted that, if a cosmological constant is present in a theory formulated in the Jordan frame, the new version of the theory in the Einstein frame has no cosmological constant (Collins, Martin and Squires 1989; Fujii 1997; Maeda 1992) but, instead, it exhibits an exponential term in the potential for the “new” scalar field.

The problem of whether a Noether symmetry is preserved by the conformal transformation has been analysed in (de Ritis *et al.* 1990; Demianski *et al.* 1991; Capozziello, de Ritis and Rubano 1993; Capozziello and de Ritis 1993; Capozziello, de Ritis and Marino 1997; Capozziello and de Ritis 1996, 1997*b*). The asymptotic evolution to an isotropic state of anisotropic Bianchi cosmologies in higher order theories with Lagrangian density of the form  $\mathcal{L} = f(R)\sqrt{-g} + \mathcal{L}_{matter}$  was studied in (Miritzis and Cotsakis 1996) using the conformal rescaling as a mathematical tool. This study is relevant to the issue of cosmic no-hair theorems in these gravitational theories. In the Einstein frame, a homogeneous universe with matter satisfying the strong and dominant energy conditions and with a scalar field with a potential  $V(\phi)$  locally convex and with zero minimum, can isotropize only if it is of Bianchi type I, V or VII. This result holds also in the Jordan frame if, in addition, the pressure of matter is positive (Miritzis and Cotsakis 1996).

## 6 Experimental consequences of the Einstein frame reformulation of gravitational theories

In most unified field theories, the conformal factor used in the conformal transformation is constructed using a physical field present in the gravitational theory (like a dilaton or a Brans–Dicke field) and therefore it is not surprising that it has certain physical effects which are, in principle, susceptible of experimental verification. The reality of the interaction with gravitational strength described by the dilaton was already stressed by Jordan (1949; 1955). The dilaton field in the Einstein frame couples differently to gravity and to matter (e.g. Horowitz 1990; Garfinkle, Horowitz and Strominger 1991), and the anomalous coupling results in a violation of the equivalence principle. Consider for example the action (4.4) plus a matter term in the Jordan frame: after the rescaling (1.1), (2.18), (2.24) has been performed, the scalar  $\tilde{\phi}$  is minimally coupled to gravity, but it couples nonminimally to the other forms of matter via a field-dependent exponential factor:

$$S = \int d^4x \left\{ \sqrt{-\tilde{g}} \left[ \frac{\tilde{R}}{16\pi G} - \frac{1}{2} \tilde{\nabla}^\mu \tilde{\phi} \tilde{\nabla}_\mu \tilde{\phi} \right] + e^{-\alpha\sqrt{G}\tilde{\phi}} \mathcal{L}_{matter} \right\} . \quad (6.1)$$

This leads to a violation of the equivalence principle which can, in principle, be tested by free fall experiments (Taylor and Veneziano 1988; Brans 1988; Cvetic 1989; Ellis *et al.* 1989; Cho and Park 1991; Cho 1992; Damour and Esposito–Farèse 1992; Cho 1994; Damour and Polyakov 1994*a, b*; Brans 1997). It is probably this anomalous coupling and the subsequent violation of the equivalence principle that explain the prejudice of many theoreticians against the use of the Einstein frame (which is not, however, a matter of taste, but is motivated by the independent energy arguments of Sec. 3). However, it is well known that although the Brans–Dicke scalar couples universally to all forms of ordinary matter in the Jordan frame, the *strong* equivalence principle is violated in this frame. This is sometimes understood as the fact that gravity determines a *local* value of the effective gravitational “constant”  $G = \phi^{-1}$  (e.g. Brans 1997). In any case, the dilaton dependence of the coupling constants is to be regarded as an important prediction of string theories in the low energy limit, and as a new motivation for improving the present precision of tests of the equivalence principle.

By describing the gravitational interaction between two point masses  $m_1$  and  $m_2$  with the force law

$$F = \frac{Gm_1m_2}{r^2} (1 + \lambda e^{-\mu r}) , \quad (6.2)$$

where  $\lambda$  and  $\mu$  are, respectively, the strength and the range of the fifth force, one obtains

constraints on the range of these parameters. Due to the smallness of the values of  $\lambda$  allowed by the theory, the null results of the experiments looking for a fifth force still leave room for a theory formulated in the Einstein frame and with anomalous coupling (Cho 1992, 1994; Damour and Polyakov 1994*a,b*; Cho and Keum 1998).

There are also post-Newtonian effects and departures from general relativity in the strong gravity regime (Damour and Esposito-Farèse 1992), as well as differences in the gravitational radiation emitted and absorbed as compared to general relativity (Eardley 1975; Will and Eardley 1977; Will 1977; Will and Zaglauer 1989; Damour and Esposito-Farèse 1992).

If  $\alpha(\phi) = \partial(\ln \Omega)/\partial\phi$ , where  $\Omega$  is the conformal factor in (1.1), then the post-Newtonian parameters  $\gamma$  and  $\beta$  (Will 1993) are given by (Damour and Esposito-Farèse 1992; Damour and Nordvedt 1993*a,b*)

$$\gamma - 1 = - \frac{2\alpha^2}{1 + \alpha^2} \Big|_{\phi_0}, \quad (6.3)$$

$$\beta = 1 + \frac{\alpha^2}{2(1 + \alpha^2)^2} \frac{\partial^2(\ln \Omega)}{\partial\phi^2} \Big|_{\phi_0}, \quad (6.4)$$

where  $\phi_0 = \phi(t_0)$  is the value of the scalar field at the present time  $t_0$ , and it is assumed that the Brans-Dicke field only depends on time. The  $1\sigma$  limits on  $\gamma$  from the Shapiro time delay experiment in the Solar System (Will 1993) are  $|\gamma - 1| < 2 \cdot 10^{-3}$  (which implies  $\alpha^2 < 10^{-3}$ ) and the combination  $\eta \equiv 4\beta - \gamma - 3$  is subject to the constraint  $|\eta| < 5 \cdot 10^{-3}$ . By contrast, in a scalar-tensor theory, one expects  $\alpha \approx 1$ . This value of  $\alpha$  could have been realistic early in the history of the universe with scalar-tensor gravity converging to general relativity at a later time during the matter-dominated epoch (Damour and Nordvedt 1993*a,b*). Accordingly, the Jordan and the Einstein frame would coincide today, the rescaling (1.1) differing from the identity only before the end of the matter-dominated era.

## 7 Nonminimal coupling of the scalar field

The material contained in this section is a summary of the state of the art on issues that have been only partially explored, results whose consequences are largely unknown, and problems that are still open. We try to point out the directions that, at present, appear most promising for future research. The reader should be aware of the fact that due to

the nature of such a discussion, the selection of topics presented here does not exhaust all the aspects involved.

The generalization to a curved spacetime of the flat space Klein–Gordon equation for a scalar field  $\phi$ ,

$$\square\phi - \xi R\phi - \frac{dV}{d\phi} = 0, \quad (7.1)$$

includes the possibility of an explicit coupling term  $\xi R\phi$  between the field  $\phi$  and the Ricci curvature of spacetime (Callan, Coleman and Jackiw 1970). There are many reasons to believe that a nonminimal (i.e.  $\xi \neq 0$ ) coupling term appears: a nonminimal coupling is generated by quantum corrections even if it is absent in the classical action, or it is required in order to renormalize the theory (Freedman, Muzinich and Weinberg 1974; Freedman and Weinberg 1974). It has also been argued in quantum field theory in curved spaces that a nonminimal coupling term is to be expected whenever the spacetime curvature is large. This leads to what we will call the “ $\xi$ -problem”, i.e. the problem of whether physics uniquely determines the value of  $\xi$ . The answer to this question is affirmative in many theories; several prescriptions for the coupling constant  $\xi$  exist and they differ according to the theory of the scalar field adopted. In general relativity and in all metric theories of gravity in which the scalar field  $\phi$  has a non-gravitational origin, the value of  $\xi$  is fixed to the value  $1/6$  by the Einstein equivalence principle (Chernikov and Tagirov 1968; Sonogo and Faraoni 1993; Grib and Poberii 1995; Grib and Rodrigues 1995; Faraoni 1996*b*). This is in contrast with a previous claim that nonminimal coupling spoils the equivalence principle (Lightman *et al.* 1975). However this claim has been shown to be based on flawed arguments; instead, it is the minimal coupling of the scalar field that leads to pathological behaviour (Grib and Poberii 1995; Grib and Rodrigues 1995). It is interesting that the derivation of the value  $\xi = 1/6$  is completely independent of conformal transformations, the conformal structure of spacetime, the spacetime metric and the field equations for the metric tensor of the theory. The fact that the conformal coupling constant  $\xi = 1/6$  emerges from these considerations is extremely unlikely to be a coincidence, but at present there is no satisfactory understanding of the reason why this happens, apart from the following naive consideration. No preferred length scale is present in the flat space massless Klein–Gordon equation and therefore no such scale must appear in the limit of the corresponding curved space massless equation when small regions of spacetime are considered, if the Einstein equivalence principle holds.

In all theories formulated in the Einstein frame, instead, the scalar field is minimally coupled ( $\xi = 0$ ) to the Ricci curvature, as is evident from the actions (2.6), (2.11), (2.20), (2.25).

In many quantum theories of the scalar field  $\phi$  there is a unique solution to the

$\xi$ -problem, or there is a restricted range of values of  $\xi$ . If  $\phi$  is a Goldstone boson in a theory with spontaneous symmetry breaking,  $\xi = 0$  (Voloshin and Dolgov 1982). If  $\phi$  represents a composite particle, the value of  $\xi$  should be fixed by the known dynamics of its constituents: for example, for the Nambu–Jona–Lasinio model,  $\xi = 1/6$  in the large  $N$  approximation (Hill and Salopek 1992). In the  $O(N)$ -symmetric model with  $V = \alpha\phi^4$ , in which the constituents of the  $\phi$  boson are scalars themselves,  $\xi$  depends on the coupling constants of the elementary scalars (Reuter 1994): if the coupling of the elementary scalars is  $\xi_0 = 0$ , then  $\xi \in [-1, 0]$  while, if  $\xi_0 = 1/6$ , then  $\xi = 0$ . For Higgs scalar fields in the standard model and canonical gravity, the allowed range of values of  $\xi$  is  $\xi \leq 0$ ,  $\xi \geq 1/6$  (Hosotani 1985). The back reaction of gravity on the stability of the scalar  $\phi$  in the potential  $V(\phi) = \eta\phi^3$  leads to  $\xi = 0$  (Hosotani 1985). The stability of a nonminimally coupled scalar field with the self-interaction potential

$$V(\phi) = \alpha\phi + m^2\phi^2/2 + \beta\phi^3 + \lambda\phi^4 - \Lambda \quad (7.2)$$

was shown to restrict the possible values of  $\xi$  and of the other parameters of this model (Bertolami 1987). Quantum corrections lead to a typical value of  $\xi$  of order  $10^{-1}$  (Allen 1983; Ishikawa 1983). In general, in a quantum theory  $\xi$  is renormalized together with the other coupling constants of the theory and the particles' masses (Birrell and Davies 1980; Nelson and Panangaden 1982; Parker and Toms 1985; Hosotani 1985; Reuter 1994); this makes an unambiguous solution of the  $\xi$ -problem more difficult. In the context of cosmological inflation, a significant simplification occurs due to the fact that inflation is a classical, rather than quantum, phenomenon: the energy scale involved is well below the Planck scale. The potential energy density of the inflaton field 50 e-folds before the end of inflation is subject to the constraint  $V_{50} \leq 6 \cdot 10^{-11} m_{Pl}^4$ , where  $m_{Pl}$  is the Planck mass (Kolb and Turner 1990; Turner 1993; Liddle and Lyth 1993). Moreover, the trajectory of the inflaton is peaked around classical trajectories (Mazenko, Unruh and Wald 1985; Evans and McCarthy 1985; Guth and Pi 1985; Pi 1985; Mazenko 1985*a,b*; Semenoff and Weiss 1985). Nevertheless, attempts have been made to begin the inflationary epoch in the context of string theory or quantum cosmology. A running coupling constant in inflationary cosmology was introduced in (Hill and Salopek 1992) and used to improve the chaotic inflationary scenario in (Futamase and Tanaka 1997). Asymptotically free theories in an external gravitational field described by the Lagrangian density

$$\mathcal{L} = \sqrt{-g} \left( aR^2 + bG_{GB} + cC_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta} + \xi R\phi^2 \right) + \mathcal{L}_{matter}, \quad (7.3)$$

where  $G_{GB}$  is the Gauss–Bonnet invariant, have a coupling constant  $\xi(t)$  that depends on time and tends to  $1/6$  when  $t \rightarrow \infty$  (Buchbinder 1986; Buchbinder, Odintsov and

Shapiro 1986). In the renormalization group approach to grand unification theories in curved spaces it was found that, at the one loop level,  $\xi(t) \rightarrow 1/6$  or  $\xi(t) \rightarrow \infty$  exponentially (Buchbinder and Odintsov 1983, 1985; Buchbinder, Odintsov and Lichzier 1989; Odintsov 1991; Muta and Odintsov 1991; Elizalde and Odintsov 1994). However, this result is not free of controversies (Bonanno 1995; Bonanno and Zappalà 1997).

Nonminimal couplings of the scalar field have been widely used in cosmology, and therefore the above prescriptions have important consequences for the viability of inflationary scenarios. In fact, the nonminimal coupling constant  $\xi$  becomes an extra parameter of inflation, and it is well known that it affects the viability of many scenarios (Abbott 1981; Starobinsky 1981; Yokoyama 1988; Futamase and Maeda 1989; Futamase, Rothman and Matzner 1989; Amendola, Litterio and Occhionero 1990; Accioly and Pimentel 1990; Barroso *et al.* 1992; Garcia-Bellido and Linde 1995; Faraoni 1996*b*). The occurrence of inflation in anisotropic spaces is also affected by the value of  $\xi$  (Starobinsky 1981; Futamase, Rothman and Matzner 1989; Capozziello and de Ritis 1997*b*), which is relevant for the cosmic no-hair theorems. In many papers on inflation, the nonminimal coupling was used to improve the inflationary scenario; however, the feeling is that, in general, it actually works in the opposite direction (Faraoni 1997*a*). In some cases it may be possible to compare the spectral index of density perturbations predicted by the inflationary theory with the available observations of cosmic microwave background anisotropies in order to determine the value of  $\xi$  (Kaiser 1995*a*; Faraoni 1996*b*), or to obtain other observational constraints (Fukuyama *et al.* 1996). In cosmology, for chaotic inflation with the potential  $V = \lambda\phi^4$ , a nonminimal coupling to the curvature lessens the fine-tuning on the self-coupling parameter  $\lambda$  imposed by the cosmic microwave background anisotropies (Salopek, Bond and Bardeen 1989; Fakir and Unruh 1990*a,b*; Kolb, Salopek and Turner 1990; Makino and Sasaki 1991),  $\lambda < 10^{-12}$ . A nonminimal coupling term can also enhance the growth of density perturbations (Maeda 1992; Hirai and Maeda 1994; Hirai and Maeda 1997). For scalar fields in a Friedmann universe, the long wavelengths  $\lambda$  do not scale with the usual reshift formula  $\lambda/\lambda_0 = a(t)/a(t_0)$ , but exhibit diffractive corrections if  $\xi \neq 1/6$  (Hochberg and Kephart 1991).

The value of the coupling constant  $\xi$  affects also the success of the so-called “geometric reheating” of the universe after inflation (Bassett and Liberati 1997), which is achieved via a nonminimal coupling of the inflaton with the Ricci curvature, instead of the usual coupling to a second scalar field.

The “late time mild inflationary” scenario of the universe predicts very short periods of exponential expansion of the universe interrupting the matter era (Fukuyama *et al.* 1997). The model is based on a massive nonminimally coupled scalar field acting as dark matter. The success of the scenario depends on the value of  $\xi$ , and a negative sign of  $\xi$

is necessary. However, the mechanism proposed in (Fukuyama *et al.* 1997) to achieve late time mild inflation turns out to be physically pathological from the point of view of wave propagation in curved spaces (Faraoni and Gunzig 1998*b*). At present, it is unclear whether alternative mechanisms can successfully implement the idea of late time mild inflation.

The case  $\xi \neq 0$  for a scalar field in the Jordan frame of higher dimensional models has been shown to have desirable properties in shrinking the extra dimensions (Sunahara, Kasai and Futamase 1990; Majumdar 1997), and has been used also for the Brans–Dicke field in generalized theories of gravity (Linde 1994; Laycock and Liddle 1994; Garcia–Bellido and Linde 1995). Exact solutions in cosmology have been obtained by using the conformal transformation (1.1), (2.9), (2.10) and starting from known solutions in the Einstein frame, in which the scalar field is minimally coupled (Bekenstein 1974; Froyland 1992; Accioly, Vaidya and Som 1983; Futamase and Maeda 1989; Abreu, Crawford and Mimoso 1994). From what we have already said in the previous sections, it is clear that, in general relativity with a nonminimally coupled scalar field, the Einstein and the Jordan frames are physically inequivalent but neither is physically preferred on the basis of energy arguments.

Nonminimal couplings of the scalar field in cosmology have been explored also in contexts different from inflation (Dolgov 1983; Ford 1987; Suen and Will 1988; Fujii and Nishioka 1990; Morikawa 1990; Hill, Steinhardt and Turner 1990; Morikawa 1991; Maeda 1992; Sudarsky 1992; Salgado, Sudarsky and Quevedo 1996, 1997; Faraoni 1997*b*) during the matter–dominated era (in the radiation–dominated era of a Friedmann–Lemaître–Robertson–Walker solution, or in any spacetime with Ricci curvature  $R = 0$ , the explicit coupling of the scalar field to the curvature becomes irrelevant). In particular, a nonself–interacting, massless scalar field nonminimally coupled to the curvature with negative  $\xi$  has been considered as a mechanism to damp the cosmological constant (Dolgov 1983; Ford 1987; Suen and Will 1988) and solve the cosmological constant problem.

From the mathematical point of view, the action (2.7) is the only action such that the nonminimal coupling of  $\phi$  to  $R$  involves only the scalar field but not its derivatives, and the coupling is characterized by a dimensionless constant (Birrell and Davies 1982). The Klein–Gordon equation arising from the action (2.7) is conformally invariant if  $\xi = 1/6$  and  $V(\phi) = 0$ , or  $V(\phi) = \lambda\phi^4$ . Many authors choose to reason in terms of an effective renormalization of the gravitational coupling constant

$$G_{eff} = \frac{G}{1 - 8\pi G\xi\phi^2}. \quad (7.4)$$

If  $\phi = \phi(t)$ , as in spatially homogeneous cosmologies or in homogeneous regions of space–

time, then the effective gravitational coupling  $G_{eff} = G_{eff}(t)$  varies on a cosmological time scale. The possibility of a negative  $G_{eff}$  at high energies, corresponding to an antigravity regime in the early universe has also been considered (Pollock 1982; Novello 1982), also at the semiclassical level (Gunzig and Nardone 1984).

The solution of the  $\xi$ -problem is also relevant for the problem of backscattering of waves of the scalar  $\phi$  off the background curvature of spacetime, and the creation of “tails” of radiation. If the Klein–Gordon wave equation (7.1) is conformally invariant, tails are absent in any conformally flat spacetime, including the cosmologically relevant case of Friedmann–Lemaître–Robertson–Walker metrics (Sonogo and Faraoni 1992; Noonan 1995). Other areas of gravitational physics for which the solution of the  $\xi$ -problem is relevant include the collapse of scalar fields (Frolov 1998), the theory of the structure and stability of boson stars (Van der Bij and Gleiser 1987; Liddle and Madsen 1992; Jetzer 1992), which is linked to inflation by the hypothesis that particles associated with the inflaton field may survive as dark matter in the form of boson stars. The  $\xi$ -problem is also relevant for the field of classical and quantum wormholes, in which negative energy fluxes are eliminated by restricting the allowed range of values of  $\xi$  (Ford 1987; Hiscock 1990; Coule 1992; Bleyer, Rainer and Zhuk 1994). Also the Ashtekar formulation of general relativity has been studied in the presence of nonminimally coupled scalar fields using a conformal transformation; the field equations in these variables are nonpolynomial, in contrast to the polynomial case of minimal coupling (Capovilla 1992).

## 8 Conclusions

Conformal transformations are extensively used in classical theories of gravity, higher-dimensional theories and cosmology. Sometimes, the conformal transformation is a purely mathematical tool that allows one to map complicated equations of motion into simpler equations, and constitutes an isomorphism between spaces of solutions of these equations. In this sense, the conformal transformation is a powerful solution-generating technique. More often, the conformal transformation to the Einstein frame is a map from a nonviable classical theory of gravity formulated in the Jordan frame to a viable one which, however, is not as well motivated as the starting one from the physical perspective. A key role in establishing the viability of the Einstein frame version of the theory is played by the positivity of the energy and by the existence and stability of a ground state in the Einstein frame. It is to be remarked that the energy argument of Sec. 3 selecting the Einstein frame as the physical one is not applicable to quantum theories; in fact, the positivity of energy and the energy conditions do not hold for

quantum theories. The weak energy condition is violated by quantum states (Ford and Roman 1992, 1993, 1995) and a theory can be unstable in the semiclassical regime (Witten 1982), or not have a ground state (e.g. Liouville's theory (D'Hoker and Jackiw 1982)).

Conformal transformations, nonminimal coupling, and the related aspects are important also for quantum and string theories (e.g. Stahlhofen and Schramm 1989 – see Fulton, Rorlich and Witten 1962 for an early review) and for statistical mechanics (Dita and Georgescu 1989). For example, the conformal degree of freedom of a conformally flat metric has been studied in (Padmanabhan 1988) in order to get insight into the quantization of gravity in the particularly simple case when the spacetime metric is conformally flat:  $g^{\mu\nu} = \Omega^2 \eta^{\mu\nu}$ . In the context of quantum gravity, lower-dimensional theories of gravity have been under scrutiny for several years: when the spacetime dimension is 2 or 3, the metric has only the conformal degree of freedom (Brown, Henneaux and Teitelboim 1986), because the Weyl tensor vanishes and any two- or three-dimensional metric is conformally equivalent to the Minkowski spacetime of corresponding dimensionality (Wald 1984). The properties of the quantum-corrected Vlasov equation under conformal transformations have been studied in (Fonarev 1994). A nonminimal coupling of a quantum scalar field in a curved space can induce spontaneous symmetry breaking without negative squared masses (Madsen 1988; Moniz, Crawford and Barroso 1990; Gib and Robert 1995). However, all these topics are beyond the purpose of the present work, which is limited to classical theories.

Many works that appeared and still appear in the literature are affected by confusion about the conformal transformation technique and the issue of which conformal frame is physical. Hopefully, these papers will be reanalysed in the near future in the updated perspective on the issue of conformal transformations summarized in this article. A change in the point of view is particularly urgent in the analysis of experimental tests of gravitational theories: most of the current literature refers to the Jordan frame formulation of Brans-Dicke and scalar-tensor theories, but it is the Einstein frame which has been established to be the physical one. A revision is also needed in the applications of gravitational theories to inflation; the predicted spectrum of density perturbations must be computed in the physical frame. In fact, only in this case it is meaningful to compare the theoretical predictions with the data from the high precision satellite experiments which map the anisotropies in the cosmic microwave background – those already ongoing (COBE (Smoot *et al.* 1992; Bennett *et al.* 1996), and those planned for the early 2000s (NASA's MAP (MAP 1998) and ESA's PLANCK (PLANCK 1998))), and from the observations of large scale structures.

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## References

- Abbott, L.F. (1981), *Nucl. Phys. B* **185**, 233.
- Abbott, L.F. and Deser, S. (1982), *Nucl. Phys. B* **195**, 76.
- Abramowicz, M.A., Carter, B. and Lasota, J.P. (1988), *Gen. Rel. Grav.* **20**, 1173.
- Abramowicz, M.A., Lanza, A., Miller, J.C. and Sonego, S. (1997a), *Gen. Rel. Grav.* **29**, 1585.
- Abramowicz, M.A., Andersson, N., Bruni, M., Gosh, P. and Sonego, S. (1997b), *Class. Quant. Grav.* **14**, L189.
- Abreu, J.P., Crawford, P. and Mimoso, J.P. (1994), *Class. Quant. Grav.* **11**, 1919.
- Accetta, F.S. and Trester, J.S. (1989), *Phys. Rev. D* **39**, 2854.
- Accetta, F.S., Zoller, D.J. and Turner, M.S. (1985), *Phys. Rev. D* **31**, 3046.
- Accioly, A.J. and Pimentel, B.M. (1990), *Can. J. Phys.* **68**, 1183.
- Accioly, A.J., Vaidya, A.N. and Som, M.M. (1983), *Phys. Rev. D* **27**, 2282.
- Accioly, A.J., Wichowski, U.F., Kwok, S.F. and Pereira da Silva, N.L. (1993), *Class. Quant. Grav.* **10**, L215.
- Allen, B. (1983), *Nucl. Phys. B* **226**, 282.
- Alonso, J.S., Barbero, F., Julve, J. and Tiemblo, A. (1994), *Class. Quant. Grav.* **11**, 865.
- Alvarez, E. and Belén Gavela, M. (1983), *Phys. Rev. Lett.* **51**, 931.
- Amendola, L., Bellisai, D. and Occhionero, F. (1993), *Phys. Rev. D* **47**, 4267.
- Amendola, L., Capozziello, S., Occhionero, F. and Litterio, M. (1992), *Phys. Rev. D* **45**, 417.
- Amendola, L., Litterio, M. and Occhionero, F. (1990), *Int. J. Mod. Phys. A* **5**, 3861.
- Appelquist, T. and Chodos, A. (1983), *Phys. Rev. Lett.* **50**, 141.
- Appelquist, T., Chodos, A. and Freund, P.G.O. (Eds.) (1987), *Modern Kaluza-Klein Theories*, Addison-Wesley, Menlo Park.
- Bailin, D. and Love, A. (1987), *Rep. Prog. Phys.* **50**, 1087.
- Barros, A. and Romero, C. (1997), preprint gr-qc/9712080.
- Barroso, A., Casayas, J., Crawford, P., Moniz, P. and Nunes, A. (1992), *Phys. Lett. B* **275**, 264.
- Barrow, J.D. (1993), *Phys. Rev. D* **47**, 5329.
- Barrow, J.D. and Cotsakis, S. (1988), *Phys. Lett. B* **214**, 515.
- Barrow, J.D. and Kunze, K.E. (1997), preprint hep-th/9710018.
- Barrow, J.D. and Maeda, K. (1990), *Nucl. Phys. B* **341**, 294.
- Barrow, J.D., Mimoso, J.P. and de Garcia Maia, M.R. (1993), *Phys. Rev. D* **48**, 3630.
- Bassett, A.B. and Liberati, S. (1997), preprint hep-ph/9709417, to appear in *Phys. Rev.*

D.

- Batakis, N.A. (1995), *Phys. Lett. B* **353**, 450.
- Batakis, N.A. and Kehagias, A.A. (1995), *Nucl. Phys. B* **449**, 248–264;
- Bateman, M. (1910), *Proc. Lon. Math. Soc.* **8**, 223.
- Bekenstein, J.D. (1974), *Ann. Phys. (NY)* **82**, 535.
- Bekenstein, J.D. and Meisels, A. (1978), *Phys. Rev. D* **18**, 4378.
- Belinskii, V.A. and Khalatnikov, I.M. (1973), *Sov. Phys. JETP* **36**, 591.
- Bennet *et al.* (1996), *Astrophys. J. (Lett.)* **464**, L1.
- Berezin, V.A., Domenech, G., Levinas, M.L., Lousto, C.O. and Umérez, N.D. (1989), *Gen. Rel. Grav.* **21**, 1177.
- Bergmann, P.G. (1968), *Int. J. Theor. Phys.* **1**, 25.
- Berkin, A.L. and Maeda, K. (1991), *Phys. Rev. D* **44**, 1691.
- Berkin, A.L., Maeda, K. and Yokoyama, J. (1990), *Phys. Rev. Lett.* **65**, 141.
- Bertolami, O. (1987), *Phys. Lett. B* **186**, 161.
- Bicknell, G. (1974), *J. Phys. A* **7**, 1061.
- Biesiada, M. (1994), *Class. Quant. Grav.* **11**, 2589.
- Billyard, A. and Coley, A. (1997), *Mod. Phys. Lett. A* **12**, 2121.
- Birrell, N.D. and Davies, P.C. (1980), *Phys. Rev. D* **22**, 322.
- Birrell, N.D. and Davies, P.C. (1982), *Quantum Fields in Curved Space*, Cambridge University Press, Cambridge.
- Bleyer, U., Rainer, M. and Zhuk, A. (1994), preprint gr-qc/9405011.
- Bombelli, L., Koul, R.K., Kunstatter, G., Lee, J. and Sorkin, R.D. (1987), *Nucl. Phys. B* **289**, 735.
- Bonanno A. (1995), *Phys. Rev. D* **52**, 969.
- Bonanno, A. and Zappalà, D. (1997), *Phys. Rev. D* **55**, 6135.
- Bose, S. and Lohiya, D. (1997), preprint IUCAA 44/97.
- Brans, C.H. (1988), *Class. Quant. Grav.* **5**, L197.
- Brans, C.H. (1997), preprint gr-qc/9705069.
- Brans, C.H. and Dicke, R.H. (1961), *Phys. Rev.* **124**, 925.
- Brown, J.D., Henneaux, M. and Teitelboim, C. (1986), *Phys. Rev. D* **33**, 319.
- Bruckman, W.F. and Velazquez, E.S. (1993), *Gen. Rel. Grav.* **25**, 901.
- Buchbinder, I.L. (1986), *Fortschr. Phys.* **34**, 605.
- Buchbinder, I.L. and S.D. Odintsov, S.D. (1983), *Sov. J. Nucl. Phys.* **40**, 848.
- Buchbinder, I.L. and Odintsov, S.D. (1985), *Lett. Nuovo Cimento* **42**, 379.
- Buchbinder, I.L., Odintsov, S.D. and Lichzier, I.M. (1989), *Class. Quant. Grav.*, **6**, 605.
- Buchbinder, I.L., Odintsov, S.D. and Shapiro, I.L. (1986), in *Group-Theoretical Methods in Physics*, Markov, M. (Ed.), Moscow. p. 115.

- Buchbinder, I.L., Odintsov, S.D. and Shapiro, I.L. (1992), *Effective Action in Quantum Gravity*, IOP, Bristol.
- Buchmüller, W. and N. Dragon, N. (1989), *Nucl. Phys. B* **321**, 207.
- Callan, C.G. Jr., Coleman, S. and Jackiw, R. (1970), *Ann. Phys. (NY)* **59**, 42.
- Callan, C.G., Friedan, D., Martinec, E.J. and Perry, M.J. (1985), *Nucl. Phys. B* **262**, 593.
- Campbell, B.A., Linde, A.D. and Olive, K. (1991), *Nucl. Phys. B* **355**, 146.
- Canuto, V., Adams, P.J., Hsieh, S.-H. and Tsiang, E. (1977), *Phys. Rev. D* **16**, 1643.
- Capovilla, R. (1992), *Phys. Rev. D* **46**, 1450.
- Capozziello, S. and de Ritis, R. (1993), *Phys. Lett. A* **177**, 1.
- Capozziello, S. and de Ritis, R. (1996), preprint astro-ph/9605070.
- Capozziello, S. and de Ritis, R. (1997a), *Int. J. Mod. Phys. D* **6**, 491.
- Capozziello, S. and de Ritis, R. (1997b), *Gen. Rel. Grav.* **29**, 1425.
- Capozziello, S., de Ritis, R. and Marino, A.A. (1997), *Class. Quant. Grav.* **14**, 3243.
- Capozziello, S., de Ritis, R. and Rubano, C. (1993), *Phys. Lett. A* **177**, 8.
- Capozziello, S., Occhionero, F. and Amendola, L. (1993), *Int. J. Mod. Phys. D* **1**, 615.
- Casas, Garcia-Bellido, J. and M. Quirós, M. (1991), *Nucl. Phys. B* **361**, 713.
- Cecotti, S. (1987), *Phys. Lett. B* **190**, 86.
- Chamseddine, A.H. (1981), *Nucl. Phys. B* **185**, 403.
- Chan, K.C.K., Creighton, J.D.E. and Mann, R.B. (1996), *Phys. Rev. D* **54**, 3892.
- Chatterjee, S. and Banerjee, A. (1993), *Class. Quant. Grav.* **10**, L1.
- Chernikov, N.A. and Tagirov, E.A. (1968), *Ann. Inst. H. Poincaré A* **9**, 109.
- Cho, Y.M. (1987), *Phys. Lett. B* **199**, 358.
- Cho, Y.M. (1990), *Phys. Rev. D* **41**, 2462.
- Cho, Y.M. (1992), *Phys. Rev. Lett.* **68**, 3133.
- Cho, Y.M. (1994), in *Evolution of the Universe and its Observational Quest*, Yamada, Japan 1993, Sato, H. (Ed.), Universal Academy Press, Tokyo, p. 99.
- Cho, Y.M. (1997), *Class. Quant. Grav.* **14**, 2963.
- Cho, Y.M. and Keum, Y.Y. (1998), *Mod. Phys. Lett. A* **13**, 109.
- Cho, Y.M. and Park, D.H. (1991), *Gen. Rel. Grav.* **23**, 741.
- Cho, Y.M. and J.H. Yoon, J.H. (1993), *Phys. Rev. D* **47**, 3465.
- Collins, P.D.B., Martin, A.D. and Squires, E.J. (1989), *Particle Physics and Cosmology*, J. Wiley, New York, p. 293.
- Copeland, E.J., Lahiri, A. and Wands, D. (1994), *Phys. Rev. D* **50**, 4868.
- Copeland, E.J., Lahiri, A. and Wands, D. (1995), *Phys. Rev. D* **51**, 1569.
- Cotsakis, S. (1993), *Phys. Rev. D* **47**, 1437; *errata* (1994), *Phys. Rev. D* **49**, 1145.
- Cotsakis, S. (1995), *Phys. Rev. D* **52**, 6199.

- Cotsakis, S. and Flessas, G. (1993), *Phys. Rev. D* **48**, 3577.
- Cotsakis, S. and Saich, P.J. (1994), *Class. Quant. Grav.* **11**, 383.
- Coule, D.H. (1992), *Class. Quant. Grav.* **9**, 2352.
- Crittenden, R. and Steinhardt, P.J. (1992), *Phys. Lett. B* **293**, 32.
- Cunningham, E. (1909), *Proc. Lon. Math. Soc.* **8**, 77.
- Cvetič, M. (1989), *Phys. Lett. B* **229**, 41.
- D'Hoker, E. and Jackiw, R. (1982), *Phys. Rev. D* **26**, 3517.
- Damour, T. and Esposito-Farèse, G. (1992), *Class. Quant. Grav.* **9**, 2093.
- Damour, T., Gibbons, G. and Gundlach, C. (1990), *Phys. Rev. Lett.* **64**, 123.
- Damour, T. and Gundlach, C. (1991), *Phys. Rev. D* **43**, 3873.
- Damour, T. and Nordvedt, K. (1993a), *Phys. Rev. Lett.* **70**, 2217.
- Damour, T. and Nordvedt, K. (1993b), *Phys. Rev. D* **48**, 3436.
- Damour, T. and Polyakov, A.M. (1994a), *Nucl. Phys. B* **423**, 532.
- Damour, T. and Polyakov, A.M. (1994b), *Gen. Rel. Grav.* **26**, 1171.
- Damour, T. and Vilenkin, A. (1996), *Phys. Rev. D* **53**, 2981.
- del Campo, S. (1992), *Phys. Rev. D* **45**, 3386.
- Delgado, V. (1994), preprint ULLFT-1/94, hep-ph/9403247.
- Demianski, M., de Ritis, R., Marmo, G., Platania, G., Rubano, C., Scudellaro, P. and Stornaiolo, P. (1991), *Phys. Rev. D* **44**, 3136.
- de Ritis, R., Marmo, G., Platania, G., Rubano, C., Scudellaro, P. and Stornaiolo, C. (1990), *Phys. Rev. D* **42**, 1091.
- Deruelle, N., Garriga, J. and Verdaguer, E. (1991), *Phys. Rev. D* **43**, 1032.
- Deruelle, N. and Madore, J. (1987), *Phys. Lett. B* **186**, 25.
- Deruelle, N. and Spindel, P. (1990), *Class. Quant. Grav.* **7**, 1599.
- Deser, S. (1984), *Phys. Lett. B* **134**, 419.
- de Witt, B. and Nicolai, H. (1986), *Nucl. Phys. B* **274**, 363.
- Dick, R. (1988), *Gen. Rel. Grav.* **30**, 435.
- Dicke, R.H. (1962), *Phys. Rev.* **125**, 2163.
- Dine, M., Rohm, R., Seiberg, N. and Witten, E. (1985), *Phys. Lett. B* **156**, 55.
- Dirac, P.A.M. (1973), *Proc. R. Soc. Lond. A* **333**, 403.
- Dita, P. and Georgescu, V. (Eds.) (1989), *Conformal Invariance and String Theory*, Proceedings, Poiana Brasov, Romania 1987, Academic Press, Boston.
- Dolgov, A.D. (1983), in *The Very Early Universe*, Gibbons, G.W., Hawking, S.W. and Siklos, S.T.C. (Eds.), Cambridge University Press, Cambridge.
- Duff, M.J. (1981), in *Quantum Gravity 2: A Second Oxford Symposium*, Isham, C.J., Penrose, R. and Sciama, D.W. (Eds.), Oxford University Press, Oxford.
- Eardley, D.M. (1975), *Astrophys. J. (Lett.)* **196**, L59.

- Easter, R. (1994), preprint NZ-CAN-RE-94/1, astro-ph/9405034.
- Elizalde, E. and Odintsov, S.D. (1994), *Phys. Lett. B* **333**, 331.
- Ellis, J. *et al.* (1989), *Phys. Lett. B* **228**, 264.
- Epstein, H., Glaser, V. and Jaffe, A. (1965), *Nuovo Cimento* **36**, 1016.
- Evans, M. and McCarthy, J.G. (1985), *Phys. Rev. D* **31**, 1799.
- Fabris, J.C. and Martin, J. (1993), *Phys. Lett. B* **316**, 476.
- Fabris, J.C. and Sakellariadou, M. (1997), *Class. Quant. Grav.* **14**, 725.
- Fabris, J.C. and Tossa, J. (1997), *Gravit. Cosmol.* **3**, 165.
- Fakir, R. and Habib, S. (1993), *Mod. Phys. Lett. A* **8**, 2827.
- Fakir, R., Habib, S. and Unruh, W.G. (1992), *Astrophys. J.* **394**, 396.
- Fakir, R. and Unruh, W.G. (1990a), *Phys. Rev. D* **41**, 1783.
- Fakir, R. and Unruh, W.G. (1990b), *Phys. Rev. D* **41**, 1792.
- Faraoni, V. (1996a), *Astrophys. Lett. Comm.* **35**, 305.
- Faraoni, V. (1996b), *Phys. Rev. D* **53**, 6813.
- Faraoni, V. (1997a), in *Proceedings of the 7th Canadian Conference on General Relativity and Relativistic Astrophysics*, Calgary, Canada 1997, Hobill, D. (Ed.), in press.
- Faraoni, V. (1997b), *Gen. Rel. Grav.* **29**, 251.
- Faraoni, V. (1998), preprint IUCAA 22/98, gr-qc/9805057, to appear in *Phys. Lett. A*.
- Faraoni, V., Cooperstock, F.I. and Overduin, J.M. (1995), *Int. J. Mod. Phys. A* **4**, 387.
- Faraoni, V. and Gunzig, E. (1998a), *Astron. Astrophys.* **332**, 1154.
- Faraoni, V. and Gunzig, E. (1998b), preprint IUCAA 23/98.
- Ferraris, M. (1986), in *Atti del 6° Convegno Nazionale di Relatività Generale e Fisica della Gravitazione*, Firenze 1984, Modugno, M. (Ed.), Tecnoprint, Bologna, p. 127.
- Ferraris, M., Francaviglia, M. and Magnano, G. (1988), *Class. Quant. Grav.* **5**, L95.
- Ferraris, M., Francaviglia, M. and Magnano, G. (1990), *Class. Quant. Grav.* **7**, 261.
- Fierz, M. (1956), *Helv. Phys. Acta* **29**, 128.
- Fonarev, O.A. (1994), *Class. Quant. Grav.* **11**, 2597.
- Ford, L.H. (1987), *Phys. Rev. D* **35**, 2339.
- Ford, L.H. and Roman, T.A. (1992), *Phys. Rev. D* **46**, 1328.
- Ford, L.H. and Roman, T.A. (1993), *Phys. Rev. D* **48**, 776.
- Ford, L.H. and Roman, T.A. (1995), *Phys. Rev. D* **51**, 4277.
- Freedman, D.Z., Muzinich, I.J. and Weinberg, E.J. (1974), *Ann. Phys. (NY)* **87**, 95.
- Freedman, D.Z. and Weinberg, E.J. (1974), *Ann. Phys. (NY)* **87**, 354.
- Freund, P.G.O. (1982), *Nucl. Phys. B* **209**, 146.
- Frolov, A.V. (1998), preprint gr-qc/9806112.
- Froyland, J. (1982), *Phys. Rev. D* **25**, 1470.
- Fujii, Y. (1997), preprint gr-qc/9708010.

- Fujii, Y. and Nishioka, T. (1990), *Phys. Rev. D* **42**, 361.
- Fukuyama, T., Hatakeyama, M., Miyoshi, M., Morikawa, M. and Nakamichi, A. (1997), *Int. J. Mod. Phys. D* **6**, 69.
- Fulton, T., Rohrlich, F. and Witten, L. (1962a), *Rev. Mod. Phys.* **34**, 442.
- Fulton, T., Rohrlich, F. and Witten, L. (1962b), *Nuovo Cimento* **26**, 652.
- Futamase, T. and Maeda, K. (1989), *Phys. Rev. D* **39**, 399.
- Futamase, T., Rothman, T. and Matzner, R. (1989), *Phys. Rev. D* **39**, 405.
- Futamase, T. and Tanaka, M. (1997), preprint OCHA-PP-95, hep-ph/9704303.
- Garay, L. and Garcia-Bellido, J. (1993), *Nucl. Phys. B* **400**, 416.
- Garcia-Bellido, J. and Linde, A.D. (1995), *Phys. Rev. D* **51**, 429.
- Garcia-Bellido, J. and Linde, A.D. (1995), *Phys. Rev. D* **52**, 6730.
- Garcia-Bellido, J. and Quirós, M. (1990), *Phys. Lett. B* **243**, 45.
- Garcia-Bellido, J. and Quirós, M. (1992), *Nucl. Phys. B* **368**, 463.
- Garcia-Bellido, J. and Wands, D. (1995), *Phys. Rev. D* **52**, 5636.
- Garfinkle, D., Horowitz, G. and Strominger, A. (1991), *Phys. Rev. D* **43**, 3140; *erratum* (1992), *Phys. Rev. D* **45**, 3888.
- Gasperini, M. (1994), *Phys. Lett. B* **327**, 214.
- Gasperini, M., Maharana, J. and Veneziano, G. (1991), *Phys. Lett. B* **272**, 277.
- Gasperini, M. and Ricci, R. (1993), *Class. Quant. Grav.* **12**, 677.
- Gasperini, M., Ricci, R. and Veneziano, G. (1993), *Phys. Lett. B* **319**, 438.
- Gasperini, M. and Veneziano, G. (1992), *Phys. Lett. B* **277**, 256.
- Gasperini, M. and Veneziano, G. (1994), *Phys. Rev. D* **50**, 2519.
- Geyer, B. and Odintsov, S.D. (1996), *Phys. Rev. D* **53**, 7321.
- Gibbons, G.W. and Maeda, K. (1988), *Nucl. Phys. B* **298**, 741.
- Gott, S., Schmidt, H.-J. and Starobinsky, A.A. (1990), *Class. Quant. Grav.* **7**, 893.
- Gottlöber, S., Müller, V. and A.A. Starobinsky, A.A. (1991), *Phys. Rev. D* **43**, 2510.
- Green, A.M. and Liddle, A.R. (1996), *Phys. Rev. D* **54**, 2557.
- Green, B., Schwarz, J.M. and Witten, E. (1987), *Superstring Theory*, Cambridge University Press, Cambridge.
- Grib, A.A. and Poberii, E.A. (1995), *Helv. Phys. Acta* **68**, 380.
- Grib, A.A. and Rodrigues, W.A. (1995), *Gravit. Cosmol.* **1**, 273.
- Gross, D.J. and Perry, M.J. (1983), *Nucl. Phys. B* **226**, 29.
- Guendelman, E.I. (1992), *Phys. Lett. B* **279**, 254.
- Gunzig, E. and Nardone, P. (1984), *Phys. Lett. B* **134**, 412.
- Guth, A.H. and Jain, B. (1992), *Phys. Rev. D* **45**, 426.
- Guth, A.H. and Pi, S.-Y. (1985), *Phys. Rev. D* **32**, 1899.
- Hammond, R.T. (1990), *Gen. Rel. Grav.* **7**, 2107.

- Hammond, R.T. (1996), *Class. Quant. Grav.* **13**, L73.
- Harrison, E.R. (1972), *Phys. Rev. D* **6**, 2077.
- Hawking, S.W. and Horowitz, G.T. (1996), *Class. Quant. Grav.* **13**, 1487.
- Hehl, E.W., von der Heyde, P., Kerlick, G.D. and Nester, J.M. (1976), *Rev. Mod. Phys.* **48**, 393.
- Higgs, P.W. (1959), *Nuovo Cimento* **11**, 816.
- Hill, C.T. and Salopek, D.S. (1992), *Ann. Phys. (NY)* **213**, 21.
- Hill, C.T., Steinhardt, P.J. and Turner, M.S. (1990), *Phys. Lett. B* **252**, 343.
- Hirai, T. and Maeda, K. (1993), preprint WU-AP/32/93.
- Hirai, T. and Maeda, K. (1994), *Astrophys. J.* **431**, 6.
- Hirai, T. and Maeda, K. (1997), in *Proceedings of the 7th Marcel Grossman Meeting*, Stanford, USA 1994, World Scientific, Singapore, p. 477.
- Hiscock, W.A. (1990), *Class. Quant. Grav.* **7**, L35.
- Hochberg, D. and Kephart, T.W. (1991), *Phys. Rev. Lett.* **66**, 2553.
- Hochberg, D. and Kephart, T.W. (1995), *Phys. Rev. D* **51**, 2687.
- Holman, R., Kolb, E.W., Vadas, S. and Wang, Y. (1991), *Phys. Rev. D* **43**, 995.
- Holman, R., Kolb, E.W. and Wang, Y. (1990), *Phys. Rev. Lett.* **65**, 17.
- Horowitz, G. (1990), in *Proceedings of the 12th International Conference on General Relativity and Gravitation*, Boulder, USA 1989, N. Ashby, D. Bartlett and W. Wyss eds. (Cambridge University Press, Cambridge).
- Hosotani, Y. (1985), *Phys. Rev. D* **32**, 1949.
- Hu, Y., Turner, M.S. and Weinberg, E.J. (1994), *Phys. Rev. D* **49**, 3830.
- Hwang, J. (1990), *Class. Quant. Grav.* **7**, 1613.
- Hwang, J. (1996), *Phys. Rev. D* **53**, 762.
- Hwang, J. (1997a), *Class. Quant. Grav.* **14**, 1981.
- Hwang, J. (1997b), *Class. Quant. Grav.* **14**, 3327.
- Iorio, A., O’Raifeartaigh, L., Sachs, I. and Wiesendanger, C. (1997), *Nucl. Phys. B* **495**, 433.
- Ishikawa, J. (1983), *Phys. Rev. D* **28**, 2445.
- Jakubiec, A. and Kijowski, J. (1988), *Phys. Rev. D* **37**, 1406.
- Jetzer, P. (1992), *Phys. Rep.* **220**, 163.
- Jordan, P. (1949), *Nature* **164**, 637.
- Jordan, P. (1955), *Schwerkraft und Weltall*, F. Vieweg und Sohn, Braunschweig.
- Jordan, P. (1959), *Z. Phys.* **157**, 112.
- Kaiser, D.I. (1995a), preprint astro-ph/9507048.
- Kaiser, D.I. (1995b), *Phys. Rev. D* **52**, 4295.
- Kalara, S., Kaloper, N. and Olive, K.A. (1990), *Nucl. Phys. B* **341**, 252.

- Kaloper, N. and Olive, K.A. (1998), *Phys. Rev. D* **57**, 811.
- Kasper, U. and Schmidt, H.-J. (1989), *Nuovo Cimento B* **104**, 563.
- Klimcik, C. (1993), *J. Math. Phys.* **34**, 1914.
- Klimcik, C.K. and Kolnik, P. (1993), *Phys. Rev. D* **48**, 616.
- Kolb, E.W., Salopek, D. and Turner, M.S. (1990), *Phys. Rev. D* **42**, 3925.
- Kolb, E.W. and Turner, M.S. (1990), *The Early Universe*, Addison-Wesley, Reading, Mass.
- Kolitch, S.J. and Eardley, D.M. (1995), *Ann. Phys. (NY)* **241**, 128.
- Kubyshin, Yu. and Martin, J. (1995), preprint UB-ECM-PF 95/13, LGCR 95/06/05, DAMPT R95, gr-qc/9507010.
- Kubyshin, Y., Rubakov, V. and Tkachev, I. (1989), *Int. J. Mod. Phys. A* **4**, 1409.
- Kunstatter, G., Lee, H.C. and Leivo, H.P. (1986), *Phys. Rev. D* **33**, 1018.
- La, D. and Steinhardt, P.J. (1989), *Phys. Rev. Lett.* **62**, 376.
- Lafrance, R. and Myers, R.C. (1995), *Phys. Rev. D* **51**, 2584.
- Laycock, A.M. and Liddle, A.R. (1994), *Phys. Rev. D*, **49**, 1827.
- Levin, J.J. (1995a), *Phys. Rev. D* **51**, 462.
- Levin, J.J. (1995b), *Phys. Rev. D* **51**, 1536.
- Liddle, A.R. (1996), preprint SUSSEX-AST 96/12-1, astro-ph/9612093, to appear in Proceedings, *From Quantum Fluctuations to Cosmological Structures*, Casablanca, Morocco 1996.
- Liddle, A.R. and Lyth, D.H. (1993), *Phys. Rep.* **231**, 1.
- Liddle, A.R. and Madsen, M.S. (1992), *Int. J. Mod. Phys.* **1**, 101.
- Liddle, A.R. and Wands, D. (1992), *Phys. Rev. D* **45**, 2665.
- Lidsey, E.J. (1992), *Class. Quant. Grav.* **9**, 149.
- Lightman, A.P. Press, W.H., Price, R.H. and Teukolsky, S.A. (1975), *Problem Book in Relativity and Gravitation*, Princeton University Press, Princeton NJ, p. 85.
- Linde, A.D. (1990), *Particle Physics and Inflationary Cosmology*, Hardwood, Chur, Switzerland.
- Linde, A.D. (1994), *Phys. Rev. D* **49**, 748.
- Lorentz, H.A. (1937), *Collected Papers*, Nijhoff, The Hague, vol. 5, p. 363.
- Lorentz-Petzold, D. (1984), in *Lecture Notes in Physics*, Vol. 105, C. Hoenselaers, C. and W. Dietz, W. (Eds.), Springer, Berlin.
- Lu, H.Q. and Cheng, K.S. (1996), *Astrophys. Sp. Sci* **235**, 207.
- Madsen, M.S. (1988), *Class. Quant. Grav.* **5**, 627.
- Madsen, M.S. (1993), *Gen. Rel. Grav.* **25**, 855.
- Maeda, K. (1986a), *Class. Quant. Grav.* **3**, 651.
- Maeda, K. (1986b), *Phys. Lett. B* **166**, 59.

- Maeda, K. (1987), *Phys. Lett. B* **186**, 33.
- Maeda, K. (1989), *Phys. Rev. D* **39**, 3159.
- Maeda, K. (1992), in *Relativistic Astrophysics and Cosmology*, Proceedings, Potsdam 1991, Gottlöber, S., Mückel, J.P. and Müller, V. (Eds.), World Scientific, Singapore, p. 157.
- Maeda, K. and Pang, P.Y.T. (1986), *Phys. Lett. B* **180**, 29.
- Maeda, K., Stein-Schabes, J.A. and Futamase, T. (1989), *Phys. Rev. D* **39**, 2848.
- Magnano, G. 1995, in *Proceedings of the XI Italian Conference on General Relativity and Gravitation*, Trieste, Italy 1994, in press (preprint gr-qc/9511027).
- Magnano, G., Ferraris, M. and Francaviglia, M. (1990), *J. Math. Phys.* **31**, 378.
- Magnano, G. and Sokolowski, L.M. (1994), *Phys. Rev. D* **50**, 5039.
- Majumdar, A.S. (1997), *Phys. Rev. D* **55**, 6092.
- Makino, N. and Sasaki, M. (1991), *Progr. Theor. Phys.* **86**, 103.
- MAP homepage (1998) <http://map.gsfc.nasa.gov/>
- Mashoon, B. (1993), in *Quantum Gravity and Beyond, Essays in Honour of Louis Witten on His Retirement*, Mansouri, F. and Scario, J. (Eds.), World Scientific, Singapore.
- Mazenko, G.F. (1985a), *Phys. Rev. Lett.* **54**, 2163.
- Mazenko, G.F. (1985b), *Phys. Rev. D* **34**, 2223.
- Mazenko, G.F., Unruh, W.G. and Wald, R.M. (1985), *Phys. Rev. D* **31**, 273.
- McDonald, J. (1993a), *Phys. Rev. D* **48**, 2462.
- McDonald, J. (1993b), *Phys. Rev. D* **48**, 2573.
- Mignemi, S. and Schmidt, H.-J. (1995), *Class. Quant. Grav.* **12**, 849.
- Mignemi, S. and Whiltshire, D. (1992), *Phys. Rev. D* **46**, 1475.
- Mimoso, J.P. and Wands, D. (1995a), *Phys. Rev. D* **51**, 477.
- Mimoso, J.P. and Wands, D. (1995b), *Phys. Rev. D* **52**, 5612.
- Miritzis, J.M. and Cotsakis, S. (1996), *Phys. Lett. B* **383**, 377.
- Mollerach, S. and Matarrese, S. (1992), *Phys. Rev. D* **45**, 1961.
- Moniz, P., Crawford, P. and Barroso, A. (1990), *Class. Quant. Grav.* **7**, L143.
- Morikawa, M. (1990), *Astrophys. J. (Lett.)* **362**, L37.
- Morikawa, M. (1991), *Astrophys. J.* **369**, 20.
- Mukhanov, V.F., Feldman, H.A. and Brandenberger, R.H. (1992), *Phys. Rep.* **215**, 203.
- Muta, T. and Odintsov, S.D. (1991), *Mod. Phys. Lett. A* **6**, 3641.
- Mychelkin, E.G. (1991), *Astrophys. Sp. Sci.* **184**, 235.
- Nambu, Y. and Sasaki, M. (1990), *Progr. Theor. Phys.* **83**, 37.
- Nelson, B. and Panangaden, P. (1982), *Phys. Rev. D* **25**, 1019.
- Nishioka, T. and Fujii, Y. (1992), *Phys. Rev. D* **45**, 2140.
- Noonan, T.W. (1995), *Class. Quant. Grav.* **12**, 1087.

- Nordvedt, K. (1970), *Astrophys. J.* **161**, 1059.
- Novello, M. (1982), *Phys. Lett. A* **90**, 347.
- Novello, M. and Elbaz, E. (1994), *Nuovo Cimento* **109**, 741.
- Novello, M. and Heintzmann, H. (1984), *Gen. Rel. Grav.* **16**, 535.
- Novello, M., Pereira, V.M.C. and Pinto-Neto, N. (1995), *Int. J. Mod. Phys. D* **4**, 673.
- Novello, M. and J.M. Salim, J.M. (1979), *Phys. Rev. D* **20**, 377.
- Occhionero, F. and Amendola, L. (1994), *Phys. Rev. D* **50**, 4846.
- Odintsov, S.D. (1991), *Fortschr. Phys.* **39**, 621.
- Oukuiss, A. (1997), *Nucl. Phys. B* **486**, 413.
- Overduin, J.M. and Wesson, P.S. (1997), *Phys. Rep.* **283**, 303.
- Padmanabhan, T. (1988), in *Highlights in Gravitation and Cosmology*, Proceedings, Goa, India 1987, Iyer, B.R., Kembhavi, A.K., Narlikar, J.V. and Vishveshwara, C.V. (Eds.), Cambridge University Press, Cambridge, p. 156.
- Page, L. (1936a), *Phys. Rev.* **49**, 254.
- Page, L. (1936b), *Phys. Rev.* **49**, 946.
- Page, L. and Adams, N.I. (1936), *Phys. Rev.* **49**, 466.
- Park, C.J. and Yoon, Y. (1997), *Gen. Rel. Grav.* **29**, 765.
- Parker, L. and Toms, D.J. (1985), *Phys. Rev. D* **32**, 1409.
- Pauli, W. (1955), quoted in *Schwerkraft und Weltall*, F. Vieweg und Sohn, Braunschweig.
- Pauli, W. (1958), *Theory of Relativity*, Pergamon Press, New York, p. 224.
- Perlick, V. (1990), *Class. Quant. Grav.* **7**, 1849.
- Pi, S.-Y. (1985), *Nucl. Phys. B* **252**, 127.
- Piccinelli, G., Lucchin, F. and Matarrese, S. (1992), *Phys. Lett. B* **277**, 58.
- Pimentel, L.O. and Stein-Schabes, J. (1989), *Phys. Lett. B* **216**, 27.
- PLANCK homepage (1998) <http://astro.estec.esa.nl/SA-general/Projects/Planck>
- Pollock, M.D. (1982), *Phys. Lett. B* **108**, 386.
- Rainer, M. and Zuhk, A. (1996), *Phys. Rev. D* **54**, 6186.
- Reasenber, R.D. *et al.* (1979), *Astrophys. J. (Lett.)* **234**, L219.
- Reuter, M. (1994), *Phys. Rev. D* **49**, 6379.
- Rothman, T. and Anninos, P. (1991), *Phys. Rev. D* **44**, 3087.
- Sadhev, D. (1984), *Phys. Lett. B* **137**, 155.
- Salgado, M., Sudarsky, D. and Quevedo, H. (1996), *Phys. Rev. D* **53**, 6771.
- Salgado, M., Sudarsky, D. and Quevedo, H. (1997), *Phys. Lett. B* **408**, 69.
- Salopek, D.S. (1992), *Phys. Rev. Lett.* **69**, 3602.
- Salopek, D.S., Bond, J.R. and Bardeen, J.M. (1989), *Phys. Rev. D* **40**, 1753.
- Sasaki, M. (1986), *Progr. Theor. Phys.* **76**, 1036.
- Shapiro, L.L. and Takata, H. (1995), *Phys. Lett. B* **361**, 31.

- Scheel, M.A., Shapiro, S.L. and Teukolsky, S.A. (1995), *Phys. Rev. D* **51**, 4236.
- Scherk, J. and Schwarz, J.H. (1979), *Nucl. Phys. B* **153**, 61.
- Schmidt, H.-J. (1987), *Astr. Nachr.* **308**, 183.
- Schmidt, H.-J. (1988), *Phys. Lett. B* **214**, 519.
- Schmidt, H.-J. (1990), *Class. Quant. Grav.* **7**, 1023.
- Schmidt, H.-J. (1995), *Phys. Rev. D* **52**, 6196.
- Schneider, P., Ehlers, J. and Falco, E.E. (1992), *Gravitational Lenses*, Springer, Berlin.
- Semenoff, G. and Weiss, N. (1985), *Phys. Rev. D* **31**, 699.
- Shapiro, I.L. and Takata, H. (1995), *Phys. Lett. B* **361**, 31.
- Smoot, G.F. *et al.* (1992), *Astrophys. J. (Lett.)* **396**, L1.
- Sokolowski, L. (1989a), *Class. Quant. Grav.* **6**, 59.
- Sokolowski, L. (1989b), *Class. Quant. Grav.* **6**, 2045.
- Sokolowski, L.M. (1997), in *Proceedings of the 14th International Conference on General Relativity and Gravitation*, Firenze, Italy 1995, M. Francaviglia, G. Longhi, L. Lusanna and E. Sorace eds. (World Scientific, Singapore), P. 337.
- Sokolowski, L.M. and Carr, B. (1986), *Phys. Lett. B* **176**, 334.
- Sokolowski, L.M. and Golda, Z.A. (1987), *Phys. Lett. B* **195**, 349.
- Sokolowski, L.M., Golda, Z.A., Litterio, A.M. and Amendola, L. (1991), *Int. J. Mod. Phys. A* **6**, 4517.
- Sonego, S. and Faraoni, V. (1992), *J. Math. Phys.* **33**, 625.
- Sonego, S. and Faraoni, V. (1993), *Class. Quant. Grav.* **10**, 1185.
- Sonego, S. and Massar, M. (1996), *Mon. Not. R. Astr. Soc.* **281**, 659.
- Stahlofen, A.A. and Schramm, A.J. (1989), *Phys. Rev. A* **40**, 1220.
- Starobinsky, A.A. (1980), *Phys. Lett. B* **91**, 99.
- Starobinski, A.A. (1981), *Sov. Astron. Lett.* **7**, 36.
- Starobinsky, A.A. (1986), *Sov. Astr. Lett.* **29**, 34.
- Starobinsky, A.A. (1987), in *Proceedings of the 4th Seminar on Quantum Gravity*, Markov, M.A. and Frolov, V.P. (Eds.), World Scientific, Singapore.
- Steinhardt, P.J. and Accetta, F.S. (1990), *Phys. Rev. Lett.* **64**, 2740.
- Streater, R. and Wightman, A. (1964), *PCT, Spin and Statistics, and All That*, Benjamin, New York.
- Sudarsky, D. (1992), *Phys. Lett. B* **281**, 281.
- Suen, W.-M. and Will, C.M. (1988), *Phys. Lett. B* **205**, 447.
- Sunahara, K., Kasai, M. and Futamase, T. (1990), *Progr. Theor. Phys.* **83**, 353.
- Suzuki, Y. and Yoshimura, M. (1991), *Phys. Rev. D* **43**, 2549.
- Synge, J.L. (1955), *Relativity: the General Theory*, North Holland, Amsterdam.
- Tanaka, T and Sakagami, M. (1997), preprint OU-TAP 50, kucp0107, gr-qc/9705054.

- Tao, Z.-J. and Xue, X. (1992), *Phys. Rev. D* **45**, 1878.
- Taylor, T.R. and Veneziano, G. (1988), *Phys. Lett. B* **213**, 450.
- Teyssandier, P. (1995), *Phys. Rev. D* **52**, 6195.
- Teyssandier, P. and Tourrenc, P. (1983), *J. Math. Phys.* **24**, 2793.
- Tkacev, I. (1992), *Phys. Rev. D* **45**, 4367.
- Tseytlin, A.A. (1993), *Phys. Lett. B* **317**, 559.
- Turner, M.S. (1993), in *Recent Directions in Particle Theory - From Superstrings and Black Holes to the Standard Model*, Proceedings of the Theoretical Advanced Study Institute in Elementary Particle Physics, Boulder, Colorado 1992, Harvey, J. and Polchinski, J. (Eds.), World Scientific, Singapore (preprint FERMILAB-Conf-92/313-A, astro-ph/9304012).
- Turner, M.S. and Weinberg, E.J. (1997), *Phys. Rev. D* **56**, 4604.
- Turner, M.S. and Widrow, L.M. (1988), *Phys. Rev. D* **37**, 2743.
- Van den Bergh, N. (1980), *Gen. Rel. Grav.* **12**, 863.
- Van den Bergh, N. (1982), *Gen. Rel. Grav.* **14**, 17.
- Van den Bergh, N. (1983a), *Gen. Rel. Grav.* **15**, 441.
- Van den Bergh, N. (1983b), *Gen. Rel. Grav.* **15**, 449.
- Van den Bergh, N. (1983c), *Gen. Rel. Grav.* **15**, 1043.
- Van den Bergh, N. (1983d), *Gen. Rel. Grav.* **16**, 2191.
- Van den Bergh, N. (1986a), *J. Math. Phys.* **27**, 1076.
- Van den Bergh, N. (1986b), *Lett. Math. Phys.* **11**, 141.
- Van den Bergh, N. (1986c), *Gen. Rel. Grav.* **18**, 649.
- Van den Bergh, N. (1986d), *Gen. Rel. Grav.* **18**, 1105.
- Van den Bergh, N. (1986e), *Lett. Math. Phys.* **12**, 43.
- Van den Bergh, N. (1988), *J. Math. Phys.* **29**, 1451.
- Van den Bergh, N. and Tavakol, R.K. (1993), *Class. Quant. Grav.* **10**, 183.
- Van der Bij, J.J. and Gleiser, M. (1987), *Phys. Lett. B* **194**, 482.
- Voloshin, M.B. and Dolgov, A.D. (1982), *Sov. J. Nucl. Phys.* **35**, 120.
- Wagoner, R.V. (1970), *Phys. Rev. D* **1**, 3209.
- Wald, R.M. (1984), *General Relativity*, Chicago University Press, Chicago.
- Wands, D. (1994), *Class. Quant. Grav.* **11**, 269.
- Weinstein, S. (1996), *Phil. Sci.* **63**, S63.
- Weyl, H. (1919), *Ann. Phys. (Leipzig)* **59**, 101.
- Whitt, B. (1984), *Phys. Lett. B* **145**, 176.
- Will, C.M. (1977), *Astrophys. J.* **214**, 826.
- Will, C.M. (1993), *Theory and Experiment in Gravitational Physics* (revised edition), Cambridge University Press, Cambridge.

- Will, C.M. and Eardley, D.M. (1977), *Astrophys. J. (Lett.)* **212**, L9.  
Will, C.M. and P.J. Steinhardt, P.J. (1995), *Phys. Rev. D* **52**, 628.  
Will, C.M. and Zaglauer, H.W. (1989), *Astrophys. J.* **346**, 366.  
Witten, E. (1982), *Nucl. Phys. B* **195**, 481.  
Wood, R.W., Papini, G. and Cai, Y.Q. (1989), *Nuovo Cimento B* **104**, 653.  
Wu, A. (1992), *Phys. Rev. D* **45**, 2653.  
Xanthopoulos, B.C. and Dialynas, T.E. (1992), *J. Math. Phys.* **33**, 1463.  
Yokoyama, J. (1988), *Phys. Lett. B* **212**, 273.  
Yoon, J.H. and Brill, D.R. (1990), *Class. Quant. Grav.* **7**, 1253.