# Hierarchical search strategy for inspiraling compact binaries

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## Abstract

Here we present a detection strategy, called a *two step search*, that utilizes a hierarchy of template banks. We find that for a family of Newtonian signals, an on-line two step search is at most $\simeq 8$ times faster than an on-line one step search (for initial LIGO). We find that for the typical case of a post$^{1.5}$-Newtonian family of templates and signals, an on-line two step search requires $\sim 1/25$ the computing power that would be required for the corresponding on-line one step search, given a false alarm probability of one per year, detection probability of 0.95 and the noise power spectral density of the LIGO both initial and advanced.

## 1 Introduction

Generically, the wave form from an inspiraling binary is an amplitude and phase modulated sinusoid both of whose instantaneous frequency and amplitude increase as the two bodies proceed towards merger. The signal becomes "visible" in the output of a detector when its instantaneous frequency exceeds the lower frequency cutoff of the output's bandwidth.

Thus, if matched filtering is used for such signals, it would be necessary to employ a bank of filters (or template wave forms) corresponding to different values of the signal parameters. One would then compare the maximum over all the filtered outputs with a threshold. This strategy is usually called a *one-step search.* Even though the time of arrival, initial phase and distance can be handled easily, a one-step search would still be a computationally expensive proposition. Estimates [6, 1] of the computational power that is required for an on-line one-step search using post-Newtonian templates turn out to be $\sim 200$ Gflops (Giga flops) or higher. Therefore, it is desirable to have computationally less expensive detection strategies without, however, compromising too much on the performance afforded by matched filtering.

The main results of this paper are summarized in Tables 1, 2, 3 and 4. Tables 1, 2 display results for the Newtonian waveform, while Tables 3 and 4 correspond to the post$^{1.5}$-Newtonian wave form. In Tables 1, 2 and 3, the noise power spectral density (p.s.d.) expected for the initial LIGO [4] has been used while in Table 4, the p.s.d. used is that which is expected for the advanced LIGO [2]. The detection probability is kept at 0.95 for all signals having a *strength* $S_{\rm min}$ (as given in the captions of the tables) and an average false alarm rate is fixed at 1 false event/y. These values of the computational requirement have been obtained for various lengths of the input data segment which are tabulated in the first column. The range of the binary masses, $m_1$ and $m_2$ is chosen to be $0.5 \leq m_1, m_2 \leq 30.0\, M_\odot$. These results show that a two step hierarchical search can reduce computational requirements by about a factor of at most 8 for the Newtonian case and $\sim 25$ for the post-Newtonian one.

## 2 Maximum likelihood detection of signals

### 2.1 The post$^{1.5}$-Newtonian signal

For the post$^{1.5}$-Newtonian order (spin parameter $\sigma = 0$) inspiral waveform of compact binaries the signal parameters are, $\Theta = (\mathcal{A},\ t_a,\ \Phi,\ \tau_0,\ \tau_{1.5})$. $\mathcal{A}$ is the (nearly) constant part of the amplitude of the signal. The time of arrival of the signal is denoted by $t_a$. The phase of the wave form at $t = t_a$ is denoted by $\Phi$. The remaining parameters have the dimension of time and depend on the masses of the binary components.

The restricted post$^{1.5}$-Newtonian signal is,

$$h(t;\, \Theta) = \mathcal{A}\, a(t - t_a;\, \tau_0) \cos\left[\int_{-\infty}^{t-t_a} dt'\, f(t';\, \tau_0, \tau_{1.5}) + \Phi\right], \tag{1}$$

where,

$$a(t) = \left(1 - \frac{t}{\tau_0}\right)^{-1/4}, \tag{2}$$

and $f(t;\, \tau_0, \tau_{1.5})$, the instantaneous frequency of the signal, is given by an implicit equation,

$$t = \tau_0 + \tau_1 - \tau_{1.5} - \left(\tau_0 x^{-8/3} + \tau_1 x^{-2} - \tau_{1.5} x^{-5/3}\right) , \tag{3}$$

where $x = f(t;\, \tau_0, \tau_{1.5})/f(t_a;\, \tau_0, \tau_{1.5})$. The chirp times are given by the following expressions ($G = c = 1$),

$$\begin{aligned}
\tau_0 &= \frac{5}{256}\mathcal{M}^{-5/3}(\pi f_a)^{-8/3} , && (4)\\
\tau_1 &= \frac{5}{192\mu(\pi f_a)^2}\left(\frac{743}{336} + \frac{11}{4}\eta\right) , && (5)\\
\tau_{1.5} &= \frac{1}{8\mu}\left(\frac{M}{\pi^2 f_a^5}\right)^{1/3} , && (6)
\end{aligned}$$

where, $M$ is the total mass of the binary, $\mu$ is the reduced mass, $\eta = \mu/M$ and $\mathcal{M} = (\mu^3 M^2)^{1/5}$ is the *chirp mass*. We have chosen $\tau_0$ and $\tau_{1.5}$ as our independent parameters.

The Newtonian waveform is obtained by setting $\tau_1 = \tau_{1.5} = 0$ in the above equations. In conformity with the earlier literature [3] we will call $\tau_0$ as $\xi$ when we are dealing only with the Newtonian case.

## 2.2 The test statistic and the ambiguity function

Given a data segment $x(t)$, the maximum likelihood detection strategy for post$^{1.5}$-Newtonian signals would consist of the computation of a test statistic $\Lambda$ given by,

$$\Lambda = \max_{\Theta}[\langle x(t),\, h(t;\, \Theta)\rangle - 1/2\langle h(t;\, \Theta),\, h(t;\, \Theta)\rangle] . \tag{7}$$

where,

$$\langle u(t),\, v(t)\rangle = \int_{-\infty}^{\infty} \frac{df}{S_n(f)}\, \tilde{u}(f)\tilde{v}^*(f) , \tag{8}$$

for any two functions $u(t)$ and $v(t)$ and $S_n(f)$ is the one sided power spectral density of the noise. For the sake of convenience in the following, we adopt the notation

$$\begin{aligned}
\theta' &= (t_a, \tau_{1.5}, \tau_0) , && (9)\\
\theta &= (\tau_{1.5}, \tau_0) . && (10)
\end{aligned}$$

We can write the R.H.S. of Eq. (1) as

$$h(t;\, \Theta) = \mathcal{A}h_0(t - t_a;\, \tau_0, \tau_{1.5})\cos\Phi + \mathcal{A}h_{\pi/2}(t - t_a;\, \tau_0, \tau_{1.5})\sin\Phi, \tag{11}$$

Then the maximization over the parameters $\mathcal{A}$ and $\Phi$ can be carried out analytically. In the stationary phase approximation,

$$\Lambda = \max_{\theta'} \left[C_0^2(x;\, \theta') + C_{\pi/2}^2(x;\, \theta')\right]^{1/2} , \tag{12}$$

$$C_0(x;\, \theta') = \langle\, x(t),\, q_0(t - t_a;\, \theta)\,\rangle\ , \tag{13}$$

$$C_{\pi/2}(x;\, \theta') = \langle\, x(t),\, q_{\pi/2}(t - t_a;\, \theta)\,\rangle\ . \tag{14}$$

where the $q_0(t;\, \theta)$ and $q_{\pi/2}(t;\, \theta)$ are normalised templates corresponding to $h_0$ and $h_{\pi/2}$ respectively. In the stationary phase approximation they are also orthogonal to each other.

The *strength* $S$ of a signal [3, 5]is defined as,

$$S = \langle\, h(t;\, \Theta),\, h(t;\, \Theta)\rangle^{1/2}. \tag{15}$$

The marginal probability density function of a rectified output sample $d(\theta')$ is a Rician $Ri(z)$ when a signal is present and a Rayleigh $R(z)$ in the absence of a signal,

$$Ri(z) = z \exp\left[-\frac{1}{2}(z^2 + d^2)\right] I_0(zd)\ , \tag{16}$$

$$R(z) = z \exp\left[-\frac{z^2}{2}\right]\ , \tag{17}$$

where,

$$d^2 = d^2(\theta') = \overline{C}_0^2(\theta') + \overline{C}_{\pi/2}^2(\theta')\ , \tag{18}$$

and $I_0(x)$ is the modified Bessel function of the first kind (of order zero). The overbar denotes ensemble average.

Let

$$\begin{aligned} H(\theta'_p,\, \theta'_q) = {} & \Big[\langle\, q_0(t - t_a^p;\, \theta_p),\, q_0(t - t_a^q;\, \theta_q)\,\rangle^2 \\ & + \langle\, q_0(t - t_a^p;\, \theta_p)\,,\, q_{\pi/2}(t - t_a^q;\, \theta_q)\,\rangle^2\Big]^{1/2} . \end{aligned} \tag{19}$$

$$d(\theta') = S\, H(\theta',\, \theta'_s)\ . \tag{20}$$

The *intrinsic ambiguity function* $\mathcal{H}(\tau_{1.5}^a, \tau_0^a;\, \tau_{1.5}^b, \tau_0^b)$ is defined as,

$$\mathcal{H}(\theta_a,\, \theta_b) = \max_{t_a^a - t_a^b} H(\theta'_a,\, \theta'_b)\ . \tag{21}$$

In other words, this is the maximum value that the rectified output of a template $\theta_a$ will have if the input consists of only a signal $\theta_b$ having $S = 1$. For $S \neq 1$, the maximum value will simply be $S\mathcal{H}$. The ambiguity function plays a vital role in spacing of the templates.

We describe below in detail the hierarchical strategy for the Newtonian case, since it is much easier to illustrate, and then further briefly describe the results for the post-Newtonian case.

# 3 The Newtonian signal

## 3.1 The one step search

We begin by stating our *one step template placement criteria* : The bank of templates should be chosen in such a way that **(i)** every waveform, having a strength $S$ greater than a given minimum strength $S_{min}$, should have a detection probability greater than a given minimum detection probability $Q_{d,min}$, and **(ii)** The false alarm should stay below a specified level, $Q_{0,max}$. For the Newtonian case the p.s.d. used will be that of the initial LIGO.

It can be shown that for our purpose it is sufficient to use a two template formula for detection probability,

$$Q_d(\eta;\, S, \xi) = 1 - \int_0^{\eta} Ri(x, S_m)dx \int_0^{\eta} Ri(x, S_{m+1})dx \;. \qquad (22)$$

(Remember that $\tau_0$ has been replaced by $\xi$.)

The detection probability can be expected to be the smallest for signals having a strength $S_{min}$ and chirp time $\xi = (\xi_m + \xi_{m+1})/2$ for $\xi_m \in \{\, \xi_j \,\}$. Such signals will have a detection probability given by $Q_d(\eta;\, S_{min}, (\xi_m + \xi_{m+1})/2)$. To satisfy criterion (i) above, all that needs to be done, given a threshold $\eta$, is to ensure that all such minimum detection probability signals have, $Q_d(\eta;\, S_{min}, (\xi_m + \xi_{m+1})/2) = Q_{d,min}$ . It follows from the location independence of $\mathcal{H}$ that only $\xi_{m+1} - \xi_m$ will enter into the calculation of the detection probability and not $\xi_m$ and $\xi_{m+1}$ separately. We call this quantity the *spacing* of the templates and denote it by $\delta$. The whole template bank can now be constructed, using $\delta$, as $\xi_k = \xi_{min} + k\delta$ ( $k = 0, 1, \dots$ ) till $\xi_{max}$ is reached. We denote by $B_1 = \{\, \xi_k \,\}$ the one step template bank.

## 3.2 The two step hierarchical search

The basic idea behind a two step search is the use of two banks of templates $B_1$ and $B_2$, where the first stage bank $B_1$ is coarser than the second stage bank $B_2$. The maximum over a rectified output is, first, computed for each chirp time in a bank $B_1 = \{ \xi_j \}$, which has a template spacing of $\delta^{(1)}$ and the number of templates is $n_t^{(1)} = (\xi_{max} - \xi_{min})/\delta^{(1)}$. If for some template in $B_1$ having a chirp time $\xi_m$, it happens that the statistic crosses $\eta^{(1)}$, set at a lower level, then we call this event a *crossing* of $\eta^{(1)}$ produced by the chirp time $\xi_m$. Given a crossing, the next step involves using the template bank $B_2$ with a spacing $\delta^{(2)} < \delta^{(1)}$. We take $\delta^{(1)}/\delta^{(2)} = n$ to be an integer.

The set of chirp times used in $B_2$ will be located symmetrically around $\xi_m$ ( except when $\xi_m = \xi_{min}$ or $\xi_{max}$, but these can be ignored ), as $\xi_p = \xi_m + p\delta^{(2)}$, where $-n+1 \leq p \leq n-1$. For every crossing of the first stage threshold we will employ a fixed number $M = 2(n-1)$ templates from $B_2$. For $n_c$ crossings in the first stage templates in $B_1$, $n_c M$ *second stage* templates will be used.

The *two step template placement criteria* are : **(i)** Every signal with a strength greater than a given minimum strength $S_{min}$ should produce, with a probability $Q_{d,min}$, at least one crossing among the two templates which lie on either side of it. It should also be detected with a probability of $Q_{d,min}$ when the second stage templates corresponding to the above crossings are employed. **(ii)** The false alarm should be less than a specified level $Q_{0,max}$.

We quantify the computational requirement of a given two step search by the average of total number of templates, $n_t^{av}$. We denote the average value of $n_c$ in the absence of a signal by $n_c^{av}$. The computational requirement can then be expressed as,

$$n_t^{av} = n_c^{av} \times M + n_t^{(1)} \,, \qquad n_c^{av} = Q_0(\eta^{(1)}) \times n_t^{(1)} \,, \tag{23}$$

where $Q_0(\eta^{(1)})$ is the probability of a crossing for a single template.

### 3.3 Computational power required for an on-line analysis

The total number of flops required for the whole template bank on the average is therefore $N_{flop} = n_t^{av}(6N \log_2 N + 4N_p)$, where $N$ is the number data train samples and $N_p$ samples correspond to the padding. Thus, for an on-line implementation of a two step search, $N_{flop}$ operations would have to be performed in $N_p\Delta$ sec., where $\Delta$ is the sampling rate. The average computational power required, $C_{online}$, is then,

$$C_{online} = \frac{N_{flop}}{N_p\Delta} \times 10^{-6} \ \text{MFlops}. \tag{24}$$

Table 1: Minimum $C_{online}$ as a function of $N$ for : $S_{min} = 8.8$, $\xi_{min} = 2.0$ sec, $\xi_{max} = 32.0$ sec, $Q_{d,min} = 0.95$. $\eta^{(2)} = 7.92$, $\delta^{(2)} = 0.0325$ sec.

| $T$(sec) | $k$ | $\eta^{(1)}$ | $n_c^{\rm av}$ | $n_t^{\rm av}$ | $C_{\rm online}^{(2)}(MFlops)$ | $C_{\rm online}^{(1)}(MFlops)$ | $C_{\rm gain}$ |
|---|---|---|---|---|---|---|---|
| 64.0 | 11 | 5.58 | 1 | 97 | 41.3 | 392.9 | 9.5 |
| 128.0 | 10 | 5.75 | 1 | 107 | 32.6 | 279.6 | 8.6 |
| 256.0 | 9 | 5.92 | 1 | 115 | 31.6 | 253.7 | 8 |
| 512.0 | 8 | 6.11 | 1 | 124 | 33.6 | 249.3 | 7.4 |
| 1024.0 | 8 | 6.11 | 1 | 134 | 36.7 | 253.2 | 6.9 |

Table 2: Minimum $C_{online}$ as a function of $N$ for : $S_{min} = 9.0$, $\xi_{min} = 2.0$ sec, $\xi_{max} = 138.0$ sec, $Q_{d,min} = 0.95$. $\eta^{(2)} = 8.10$, $\delta^{(2)} = 0.0335$ sec.

| $T$(sec) | $k$ | $\eta^{(1)}$ | $n_c^{\rm av}$ | $n_t^{\rm av}$ | $C_{\rm online}^{(2)}(MFlops)$ | $C_{\rm online}^{(1)}(MFlops)$ | $C_{\rm gain}$ |
|---|---|---|---|---|---|---|---|
| 256.0 | 10 | 5.84 | 3 | 455 | 234.2 | 2092.1 | 8.9 |
| 512.0 | 9 | 6.03 | 3 | 502 | 172.9 | 1400.8 | 8.1 |
| 1024.0 | 8 | 6.21 | 3 | 545 | 167.1 | 1245.5 | 7.5 |
| 2048.0 | 8 | 6.21 | 6 | 588 | 175.2 | 1211.5 | 6.9 |

We compute the value of $C_{online}$, as a function of $N$, for two different ranges of the chirp time. For each range, the minimum values of $n_t^{av}$ is found for a few representative values of $N_p$, keeping $S_{min}$ fixed. This process is then repeated for progressively lower values of $S_{min}$ till $\delta^{(2)}$ becomes $\sim 0.030$ sec.

In each table, the minimum value of $n_t^{av}$ is computed, for several values of $N$. The value of $C_{online}$ is then found at each such minimum. We also list the corresponding values of $\eta^{(1)}$, $\delta^{(1)}$, $n_t^{av}$ and $n_c^{av}$( the last two are rounded to the nearest whole number ).

## 4 The two step hierarchical search for the 1.5 post-Newtonian waveform

In this case, a similar approach can be followed to space the first stage templates as was done for the Newtonian case. Here we have a two dimensional parameter space corresponding to $\tau_0$ and $\tau_{1.5}$ (see section 2.1). The area of the space of interest for the mass ranges, $0.5M_\odot \leq m_1, m_2 \leq 30M_\odot$ is,

$$A = \begin{cases} 50.174 \text{ sec}^2 & \text{for initial LIGO} \\ 20389.542 \text{ sec}^2 & \text{for advanced LIGO} \end{cases} . \tag{25}$$

Here, there will be two spacings to fix, namely, along the minor and major axes of the one-step unit cells, and these can be chosen as integral multiples of the corresponding one step spacings $l_1$ and $l_2$. The first stage unit cell has side lengths $k_1 \times l_1$ and $k_2 \times l_2$. The number of second stage templates we employ per first stage are $4(k_1 - 1)(k_2 - 1) + 2(k_1 - 1) + 2(k_2 - 1)$.

We present our results in the form of Table 3, for initial LIGO, and Table 4 for advanced LIGO. The computing costs are in Gflops and $N_T^t$ is the number of one step templates.

Table 3: Minimum $C^{(2)}_{on-line}$ as a function of $T$ for : $S_{min} = 9.98$, $\xi_{max} = 140.482$ sec, $Q_{d,min} = 0.95$, $\eta^{(2)} = 8.314$, $l_1 = 0.022$ sec, $l_2 = 0.144$ sec, $N_T^t = 13279$.

| $T$(sec) | $k_1$ | $k_2$ | $\eta^{(1)}$ | $n_c^{\rm av}$ | $n_t^{\rm av}$ | $C^{(2)}_{\rm online}$ | $C^{(1)}_{\rm online}$ | $C_{\rm gain}$ |
|---|---|---|---|---|---|---|---|---|
| 256.0 | 8 | 9 | 6.056 | 0 | 360 | 0.192 | 7.07 | 36.82 |
| 512.0 | 8 | 6 | 6.283 | 0 | 441 | 0.155 | 4.65 | 30.00 |
| 1024.0 | 8 | 5 | 6.484 | 0 | 490 | 0.152 | 4.12 | 27.11 |
| 2048.0 | 8 | 4 | 6.649 | 0 | 620 | 0.187 | 3.99 | 21.34 |
| 4096.0 | 8 | 4 | 6.649 | 1 | 682 | 0.207 | 4.02 | 19.42 |
| 8192.0 | 8 | 3 | 6.866 | 0 | 733 | 0.228 | 4.12 | 18.07 |

Table 4: Minimum $C^{(2)}_{on-line}$ as a function of $T$ for : $S_{min} = 10.34$, $\xi_{max} = 5621.51$ sec, $Q_{d,min} = 0.95$, $\eta^{(2)} = 8.658$, $l_1 = 0.116$ sec, $l_2 = 0.560$ sec, $N_T^t = 300796$.

| $T$(sec) | $k_1$ | $k_2$ | $\eta^{(1)}$ | $n_c^{\rm av}$ | $n_t^{\rm av}$ | $C^{(2)}_{\rm online}$ | $C^{(1)}_{\rm online}$ | $C_{\rm gain}$ |
|---|---|---|---|---|---|---|---|---|
| 8192.0 | 5 | 9 | 6.649 | 11 | 10188 | 9.771 | 288.48 | 29.52 |
| 16384.0 | 4 | 7 | 7.002 | 6 | 13490 | 6.476 | 144.39 | 22.30 |
| 32768.0 | 4 | 7 | 7.002 | 16 | 14390 | 5.709 | 119.34 | 20.90 |
| 65536.0 | 4 | 7 | 7.002 | 35 | 16100 | 6.014 | 112.36 | 18.68 |
| 131072.0 | 4 | 6 | 7.060 | 56 | 18935 | 7.004 | 111.27 | 15.89 |

## 5 Conclusions

We have investigated the performance of a two step hierarchical search for the detection of gravitational wave signals emitted during the inspiral of a compact binary.

We find that a two-step search brings about a significant reduction in computational requirements. For the Newtonian case the computational gain is at most 8, falling to about 5 for a minimal match [6] of 0.97. For the post-Newtonian case the results are as follows: For the (i) initial LIGO noise p.s.d., a two-step search is $\sim 27.0$ times faster than the corresponding one-step search and (ii) for the advanced LIGO noise p.s.d., a two-step search is $\sim 23.0$ times faster than the corresponding one-step search. The range used for the masses $m_1$ and $m_2$ is $0.5 \leq m_1, m_2 \leq 30.0\, M_\odot$.

The results of this paper show that the use of hierarchical methods of detection can be very useful for the case of coalescing binary signals and provide a strong motivation for more detailed investigations. Such methods would be indispensable if the number of signal parameters required becomes large.

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