Gravitational collapse of Type II fluid in higher dimensional space-times

S. G. Ghosh*
Department of Mathematics, Science College, Congress Nagar, Nagpur-440 012, INDIA

Naresh Dadhich†
Inter-University Center for Astronomy and Astrophysics, Post Bag 4, Ganeshkhind, Pune - 411 007, INDIA

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We find the general solution of the Einstein equation for spherically symmetric collapse of Type II fluid (null strange quark fluid) in higher dimensions. It turns out that the nakedness and curvature strength of the shell focusing singularities carry over to higher dimensions. However, there is shrinkage of the initial data space for a naked singularity of the Vaidya collapse due to the presence of strange quark matter.

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I. INTRODUCTION

The gravitational collapse of sufficiently massive star, under fairly general conditions, will result into a singularity is the fact established by the singularity theorems of Hawking and Ellis [1]. However these theorems do not indicate whether the singularity will be hidden behind the event horizon or whether it will be visible to an outside observer. The singularity will not be visible is indicated in the relativistic literature by the phrase *cosmic censorship conjecture* (CCC). It will not be visible to an observer outside the event horizon is the weak form while it will not be visible to any observer, even the one who is sitting on the collapsing star, is the strong form of the conjecture. This conjecture was articulated by Penrose [2] (see [3] for reviews). The weak form essentially states that gravitational collapse from a regular initial data never creates the space-time singularity visible to distant observers, i.e., any singularity that forms must be hidden within a black hole. For the strong form on the other hand, nothing is supposed to emanate out of the singularity and hence it is not visible to anyone even if one is infinitesimally close to it.

There is no general theory of nature and visibility of singularities. There do exist a number of exact solutions of the Einstein equation which admit depending upon the initial data black holes (BH) or naked singularities (NS) (see, e.g., Joshi [3]). In particular the Vaidya solution [4] is extensively used to show that the end state of collapse for a regular initial data results into a naked singularity. In the absence of a general result, the study of various examples of collapse becomes pertinent to examine the validity of the phenomenon of occurrence of naked singularities. The two most investigated cases are that of inhomogeneous dust and the Vaidya null fluid. A more general matter distribution should be considered. Undoubtedly, as singularity is approached, matter would be in highly dense and exotic state. The strange quark matter is the densest state of matter known which is produced in quark-hadron phase transition in the early Universe or at ultra high energy neutron stars converting into strange quark matter stars [5]. It would therefore be interesting to study the collapse of the mixture of the Vaidya fluid with the strange matter (SQM), which is a Type II fluid [1]. This has been studied in the usual 4-dimensional space-time [6] and here we wish to study it in higher dimensions. The presence of SQM tends to shrink the initial data window leading to NS. As is expected, the increase in dimensions would also tend to enhance the shrinkage because of gravity getting stronger in higher dimensions [7].

Recent developments in string theory indicate that gravity may be truly higher-dimensional(HD) interaction, becoming effectively $4D$ at lower energies. The main question is how to bring this theoretical development on the anvil of observations? One of the ways is to study its effects in astrophysical settings. That is where the phenomenon of gravitational collapse in HD attains relevance. In this study we would be finding the effects of HD as well as SQM on the end product of collapse of null SQM. For this, we first obtain the exact solution describing collapse of null SQM fluid sphere, and then investigate their effect in the context of CCC.

In section II, we obtain the generalized Vaidya solution for null quark fluid in higher dimensional spherically symmetric space-time. For definiteness we shall refer this metric as HD generalized Vaidya metric. We adapt this solution to study end state of collapse in section III . Finally we shall conclude with a discussion on the effect of SQM on HD Vaidya collapse.

II. TYPE II FLUID COLLAPSE IN HIGHER DIMENSIONS

Let us begin with the general $(n+2)$-dimensional spherically symmetric space-time, in advanced Edding-

*To whom all correspondence should be addressed.
†Electronic address: nkd@iucaa.ernet.in

ton time coordinate v, described by the metric [8, 9]:

$$ds^2 = -e^{\psi(v,r)}dv\ \left[f(v,r)e^{\psi(v,r)}dv + 2dr\right] + r^2 d\Omega_n^2 \quad (1)$$

where $0 \leq r \leq \infty$ is the proper radial coordinate, $-\infty \leq v \leq \infty$ is an avanced time coordiante, and

$$d\Omega^2 = \sum_{i=1}^{n}\left[\prod_{j=1}^{i-1}\sin^2\theta_j\right]d\theta_i^2 = d\theta_1^2 + sin^2\theta_1 d\theta_2^2 + sin^2\theta_1 sin^2\theta_2 d\theta_3^2 + \ ... + sin^2\theta_1 sin^2\theta_2 ...sin^2\theta_{n-1}d\theta_n^2 \quad (2)$$

is the metric on n-sphere in spherical polar coordinates. It proves useful to introduce a local mass function $m(v,r)$ defined by $f = 1 - 2m/r$ [10]. Here $m(v,r)$ is an arbitrary function of advanced time v and radial coordinate r. When $m = m(v)$, it is the Vaidya solution in higher dimensions [7, 8]. The usual Vaidya solution in 4-spacetime follows for $m = m(v)$ and $n = 2$.

We wish to find the general solution of the Einstein equation for the matter field given by Eq.(3) for the metric (1), which contains two arbitrary functions. Hence it is general in the retarded coordinates used. It is the field equation $G_1^0 = 0$ that leads as in the $4D$ case as well as in the present HD case to $e^{\psi(v,r)} = g(v)$ [6]. This could be absorbed by writing $d\tilde{v} = g(v)dv$. Hence, without loss of generality, the metric (2) takes the form,

$$ds^2 = -\left[1 - \frac{2m(v,r)}{(n-1)r^{n-1}}\right]dv^2 + 2dvdr + r^2 d\Omega_n^2 \quad (3)$$

The energy-momentum tensor for the Type II (null SQM) fluid is given by [11, 12],

$$T_{ab} = \mu l_a l_b + (\rho + p)(l_a\eta_b + l_b\eta_a) + p g_{ab} \quad (4)$$

$$l_a = \delta_a^0,\ n_a = \frac{1}{2}\left[1 - \frac{2m(v,r)}{r}\right]\delta_a^0 - \delta_a^1$$
$$l^a = \delta_1^a,\ n^a = -\delta_0^a - \frac{1}{2}\left[1 - \frac{2m(v,r)}{r}\right]\delta_1^a$$
$$l_a l^a = n_a n^a = 0 \ \ l_a n^a = -1. \quad (5)$$

Here $\rho,\ p$ are the strange quark matter energy density and thermodynamic pressure while μ is the energy density of the Vaidya null radiation. The null vector l_a is a double null eigenvector of T_{ab}. Physically occurring distribution is null radiation flowing in the radial direction corresponding to $\rho = p = 0$, the Vaidya space-time of radiating star. When $\mu = 0$, T_{ab} reduces to degenerate Type I fluid [8], and it represents string dust for $\mu = 0 = p$. The energy condition for such a distribution are as follows [1]:
(i) the weak and strong energy conditions,

$$\mu > 0,\ \rho \geq 0,\ p \geq 0 \quad (6)$$

(ii) the dominant energy condition,

$$\mu > 0\ \rho \geq p \geq 0. \quad (7)$$

By choosing properly the mass function $m(v,r)$, it is possible to satisfy the energy conditions. In the particular case of $m = m(v)$, the energy condition reduces to $\mu \geq 0$.

The field equations are [8],

$$8\pi\mu = \frac{n\dot{m}}{(n-1)r^n}, \quad (8a)$$

$$8\pi\rho = \frac{nm'}{(n-1)r^n}, \quad (8b)$$

$$8\pi p = -\frac{m''}{(n-1)r^{n-1}}. \quad (8c)$$

Here and in what follows dash and dot denote derivative respectively $\partial/\partial r$ and $\partial/\partial v$. The part $\mu l_a l_b$ of T_{ab} in (4) is the component of matter field that moves along the null hypersurface $v = const.$. In particular when $p = \rho = 0$ we have the Vaidya solution in higher dimensions. Thus the distribution in (4) represents Vaidya radiating star in Type II fluid universe in higher dimensions. Note that when $\mu = 0$, it is static perfect fluid space-time, which will yield regular space-time only when $m \propto r^{n+1}$ which is the de Sitter space. As in the $4D$ case, the equation of state for the zero temperature quark matter is taken [13]:

$$p = \frac{1}{k}(\rho - 4B) \quad (9)$$

where $k > 0$ is a constant. Inserting the above equation in the field equations, we find that

$$m'' = -\frac{nm'}{kr} + \frac{32\pi B}{k}r^{n-1}. \quad (10)$$

If we choose the function $m(v,r)$ such that

$$m(v,r) = m_0(v,r) + \frac{\Lambda}{(n+1)}r^{n+1}, \quad (11)$$

where $m_0(v,r)$ is an unknown function, then we find that

$$m_0'' = -\frac{n}{k}m_0', \qquad B = \frac{32\pi\Lambda}{n(k+1)}. \quad (12)$$

It admits the general solution

$$m_0 = S(v)r^{-n/k+1} + M(v). \quad (13)$$

Now, one can easily calculate m, μ, ρ and p, which are

$$m = M(v) + S(v)r^{-n/k+1} + \frac{\Lambda}{n+1}r^{n+1} \quad (14)$$

$$\mu = \frac{n}{8\pi(n-1)r^n}\left[\dot{M}(v) + \dot{S}(v)r^{-n/k+1}\right] \quad (15)$$

$$\rho = \frac{n}{8\pi(n-1)r^n}\left[\frac{k-n}{k}S(v)r^{-n/k} + \Lambda r^n\right] \quad (16)$$

$$p = \frac{n}{8\pi(n-1)r^n}\left[\frac{(k-n)}{k^2}S(v)r^{-n/k+1} - \Lambda r^n\right] \quad (17)$$

TABLE I: Variation of critical parameter α_c and y_0 with D in HD Vaidya and HD null SQM Collapse

$D = n + 2$	HD Vaidya Collapse		HD null SQM Collapse ($\beta = 0.001$, $k = 3$)	
	α_C^v	Double Roots(y_0)	α_C^s	Double Roots(y_0)
4	0.125	4	0.12437012771	4.00339
5	0.037037	3	0.036037037037	3.0
6	0.0131836	2.66667	0.01179754129	2.65859
7	0.00512	2.5	0.0032850449803	2.47188

This is the general solution of collapsing null strange quark fluid in higher dimensions. The metric explicitly reads as

$$ds^2 = -\left[1 - \frac{2M(v)}{(n-1)r^{(n-1)}} - \frac{2S(v)}{(n-1)r^{[(n-2)k+n]/k}} - \frac{\Lambda}{n^2-1}r^2\right]dv^2 + 2dvdr + r^2 d\Omega^2. \tag{18}$$

Clearly, all the energy conditions would be satisfied for $n \geq 2$ because it would ensure $\rho \geq 0$ and $p \geq 0$, while $\mu \geq 0$ would be taken care of when we choose the mass functions for both null radiation and SQM. The case, $S(v) = 0$, corresponds to the HD Vaidya de Sitter solution [7, 8], whereas the Vaidya solution is obtained by taking $S(v) = B = 0$. The solutions in Refs. [6, 13] can be recovered by setting $n = 2$. In the next section, we shall adapt the solution obtained to study the end state of the collapse for $k = 3$ in Eq. (9), for which the matter is quark plasma.

III. EXISTENCE OF A NAKED SINGULARITY

The physical scenario here is that of a radial influx of null fluid in an initially empty region of the higher dimensional non-flat but empty space. The first shell arrives at $r = 0$ at time $v = 0$ and the final at $v = T$. A central singularity of growing mass is developed at $r = 0$. For $v < 0$ we have $m(v) = 0$, i.e., the higher dimensional de Sitter like space, and for $v > T$, $\dot{m}(v) = 0$, $m(v)$ is positive definite. The metric for $v = 0$ to $v = T$ is the higher dimensional generalized Vaidya metric, and for $v > T$ we have the higher dimensional Schwarzschild solution. For the HD Vaidya region, we choose,

$$2M(v) = \alpha(n-1)v^{n-1} \ (\alpha > 0), \tag{19a}$$

$$2S(v) = \beta(n-1)v^{[(n-2)k+n]/k} \ (\beta > 0), \tag{19b}$$

Radial null geodesics of the metric (3) must satisfy

$$\frac{dr}{dv} = \frac{1}{2}\left[1 - \frac{2M(v)}{(n-1)r^{(n-1)}} - \frac{2S(v)}{(n-1)r^{[(n-2)k+n]/k}} - \frac{\Lambda}{n^2-1}r^2\right]. \tag{20}$$

Clearly the above differential equation has a singularity at $r = 0$, $v = 0$. We note that a singularity forms when a shell hits the origin at $r = 0$, where the density diverges. If the singularity is at least locally naked (for brevity we have addressed it as naked in this paper), there must be a light ray coming out from it. The critical ray is the Cauchy horizon defined as the first outgoing radial null geodesic emerging from the singularity. Therefore, by investigating the behavior of radial null geodesics near the singularity, one can find out if outgoing null geodesics meet the singularity in the past. In order to determine the nature of the limiting value of y at $r = 0$, $v = 0$ on a singular geodesic, we let $y_0 = \lim_{r\to 0\ v\to 0} y = \lim_{r\to 0\ v\to 0} v/r$. Using Eq. (20) and L'Hôpital's rule we get

$$y_0 = \lim_{r\to 0\ v\to 0} y = \lim_{r\to 0\ v\to 0} \frac{v}{r} = \lim_{r\to 0\ v\to 0} \frac{dv}{dr} = \lim_{r\to 0\ v\to 0} \frac{2}{1 - \alpha y^{n-1} - \beta y^{n(k+1)/k} - \frac{\Lambda}{n^2-1}r^2} \tag{21}$$

which can be rearranged as

$$\beta y_0^{[(n-2)k+n]/k+1} + \alpha y_0^n - y_0 + 2 = 0 \tag{22}$$

This algebraic equation is the key equation which governs the behavior of the tangent vector near the singular point.

If the singularity is naked, Eq. (22) must have one or more positive roots y_0, i.e., at least one outgoing geodesic that will terminate in the past at the singularity. Hence in the absence of positive roots, the collapse will always lead to a black hole. Thus, the occurrence of positive roots implies that the strong CCC is violated, though not necessarily the weak one.

The limit $\beta \to 0$ corresponds to HD Vaidya collapse [7], and in that case, Eq. (22) admits positive roots for $\alpha \leq$

TABLE II: Variation of β_T with D. For $\beta > \beta_T$, the end state of collapse is a black hole for all α ($\alpha_T^C = 10^{-6}$) and $k = 3$

$D = n + 2$	Critical Value β_T	Equal Roots y_0
4	0.20519540362596	4.99999
5	0.037036037037	3.0
6	0.009547734436	2.6
7	0.002807259781	2.42857
8	0.00088477111	2.33334

α_C^v. Hence singularities are naked for $\alpha \in (0, \alpha_C^v]$, and black holes form otherwise. Thus $\alpha = \alpha_C^v$ is the critical value at which the transition occurs, the end state of collapse switches from naked singularities to black holes. This is in agreement with earlier work (see, e.g., [7]).

We want to investigate the changes, at least qualitatively, in the above picture of HD Vaidya collapse in the presence of a SQM. It is verified numerically that as n increases, the threshold value β_T decreases (see Table II). For $\beta > \beta_T$, it is BH for all α. Further, the critical parameter α_C^s for HD SQM is smaller than corresponding critical parameter α_C^v for HD Vaidya , i.e, the naked singularity spectrum of the collapsing HD Vaidya region gets covered and hence the initial data space $(0, \alpha_C^v]$ for a naked singularity in HD Vaidya collapse contracts due to the presence of SQM. Thus, at least at a qualitative level, the presence of SQM facilitates the formation of black holes. However, it is interesting to see that for each β, there exists a α_C^s such that singularities are always naked for all $\alpha \in (0, \alpha_C^s]$. That is for each β there exists a non zero measure set of α values giving rise to NS and consequently violating CCC (see Tables I, II).

A. Other Cases

a. Case $k = n$.
This corresponds to

$$\mu = \frac{n}{8\pi(n-1)r^n}\left[\dot{M}(v) + \dot{S}(v)\right], \qquad \rho = -p = \frac{n\Lambda}{8\pi(n-1)} \tag{23}$$

and the algebraic equation takes the form

$$(\alpha+\beta)y_0^n - y_0 + 2 = 0 \tag{24}$$

The metric in this case takes the form of the HD Vaidya - de Sitter metric, i.e., the HD collapse of null fluid in an expanding de Sitter back ground where Λ is generated by the bag constant. The singularity is naked for

$$(\alpha+\beta) \leq \lambda_C = \frac{1}{n}\left(\frac{n-1}{2n}\right)^{n-1}$$

The equal roots at λ_C is $y_o = 2n/(n-1)$. It is remarkable to note that both critical parameters and tangents to outgoing geodesics are dependent on dimension of space-time. Further, we note that y_0 is bounded below by value 2, $y_0 \to 2$ as $\lambda \to 0$ or $D \to \infty$.

b. Case $k \to \infty$.
We have

$$\mu = \frac{n}{16\pi(n-1)r^2}[\alpha y^{n-2} + \beta y^{n-3}], \qquad p = 0,$$
$$\rho = \frac{n\beta}{16\pi r^2}y^{n-2} \tag{25}$$

and the algebraic equation becomes:

$$\alpha y_0^n + \beta y_0^{n-1} - y_0 + 2 = 0,$$

For $n = 2$, this would admit a positive root for $\alpha \leq 1/8(\beta-1)^2$, giving the range for naked singularity as obtained in [14]. This is simply the null fluid collapse in the background of constant potential which is characterized by $T_0^0 = T_1^1 = const./r^2$, as is the case in Eq.(25) above. The condition for naked singularity modifies to $\beta^2 < 4\alpha + 108\alpha^2 + 36\alpha\beta + \beta^2$ when $n = 3$.

c. Other k
We also note that as k increases, so does the threshold value β_T. However, to conserve the space we shall not present the details.

IV. DISCUSSION

Motivated by the development of superstring and other field theories, there is considerable interest in the models with extra dimensions from viewpoint of both cosmology [15] and gravitational collapse [16]. In this paper, We have shown that the 4D spherically symmetric solution describing null SQM fluid go over to $(n+2)$-dimensional spherically symmetric solution and essentially retaining its physical behavior. The solution obtained includes many previously known solutions to Einstein equations, such as, HD Vaidya de Sitter solution. We have also used this solution to study the end state of collapsing star and showed that there exists a regular initial data which leads to naked singularity. Higher dimensions tend to shrink the naked singularity initial data space alternatively enlargement of black hole initial data space, of corresponding $4D$ collapse. We have also studied the effect of SQM on Vaidya collapse (both in $4D$ and HD). The naked singularity spectrum of Vaidya collapse shrinks further with introduction of SQM and thus atleast qualitatively SQM favors black hole in comparison to naked singularities. It is also interesting to see that there exists threshold value β_T, which decreases with increase in n, such that for $\beta > \beta_T$, the end state of the collapse is always black hole for all α. However the threshold touches zero only as $n \to \infty$. That means for finite dimensions, there would always be a window howsoever narrow in the initial data leading to NS. However for sufficiently large n, it would practically be black hole and CCH would be obeyed. This case includes several previous cases of spherical collapse in $4D$ and HD.

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[1] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Spacetime* (Cambridge University Press, Cambridge, 1973)

[2] R. Penrose, Riv. Nuovo Cimento **1**, 252 (1969); in *General Relativity - An Einstein Centenary Survey*, edited by S. W. Hawking and W. Israel (Cambridge University Press, Cambridge, 1979) pp 581-638.

[3] P. S. Joshi, *Global Aspects in Gravitation and Cosmology* (Clendron, Oxford, 1993); C. J. S. Clarke, Class. Quantum Grav. **10**, 1375 (1993); R. M. Wald, gr-qc/9710068; S. Jhingan and G. Magli, *Preprint* gr-qc/9903103; T. P. Singh, J. Astrophys. Astron. **20**, 221 (1999); P. S. Joshi, Pramana **55**, 529 (2000).

[4] P. C. Vaidya, Proc Indian Acad Sci **A33** (1951) 264.

[5] E. Witten, Phys. Rev. D **30**, 272 (1984).

[6] S. G. Ghosh and N. Dadhich, IUCAA Preprint 1/2002.

[7] S. G. Ghosh and N. Dadhich, Phys. Rev. D **64**, 047501 (2001).

[8] L. K. Patel and N. Dadhich, gr-qc/9909068.

[9] J. F. V. Rocha, gr-qc/0105095.

[10] C. Barrabes and W. Israel, Phys. Rev. D **43**, 1129 (1991).

[11] A. Wang and Y. Wu, Gen. Rel. Grav. **31**, 107 (1999).

[12] V. Husain, Phys. Rev. D **53**, 1759 (1996).

[13] T. Harko and K. S. Chang, Phys. Lett. A **266**, 249 (2000).

[14] S. Jhingan, N. Dadhich and P. S. Joshi, Phys. Rev. D **63**, 044010 (2001).

[15] L. Randall and R. Sundram, Phys. Rev. Lett. **83**, 3370 (1999); **83**, 4690 (1999).

[16] J. Soda and K. Hirata, Phys. Lett. B **387**, 271 (1996); A. llha and J. P. S. Lemos, Phys. Rev. D **55**, 1788 (1997); A. llha, A. Kleber and J. P. S. Lemos, J. Math. Phys. **40**, 3509 (1999); A. V. Frolov, Class. Quantum Grav. **16**, 407 (1999); J. F. V. Rocha and A. Wang, Class. Quantum Grav. **17**, 2589 (2000).