

# Tales of tails in cosmology

Valerio Faraoni<sup>1</sup> and Edgard Gunzig<sup>2,3</sup>

<sup>1</sup> *Inter-University Centre for Astronomy and Astrophysics  
Post Bag 4, Ganeshkhind P.O., Pune 411 007, India*

<sup>2</sup> *RggR, Faculté des Sciences  
Campus Plaine, Université Libre de Bruxelles  
Boulevard du Triomphe, CP231  
1150 Bruxelles, Belgium*

<sup>3</sup> *Instituts Internationaux de Chimie et de Physique Solvay*

## Abstract

Late time mild inflation (LTMI) proposes to solve the age of the universe problem and the discrepancy between locally and globally measured values of the Hubble parameter. However, the mechanism proposed to achieve LTMI is found to be physically pathological by applying the theory of tails for the solutions of wave equations in curved spaces. Alternative mechanisms for LTMI are discussed, and the relevance of scalar wave tails for cosmology is emphasized.

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# 1 Introduction

The late time mild inflationary (LTMI) scenario has recently been proposed [1] to solve the age of the universe problem, and the puzzle of the discrepancy between the locally and globally measured values of the Hubble parameter.

In this paper, we discuss the homogeneous Klein–Gordon equation for a scalar field from the point of view of the Huygens’ principle and wave tails, with respect to its cosmological applications. In the LTMI scenario, the scalar field is allowed to couple explicitly to the Ricci curvature of spacetime (see Eq. (2.2) below). The physical reasons to consider a nonminimal coupling term are many, and are summarized in Ref. [2]; indeed, the nonminimal coupling is forced upon us by the physics of the scalar field [2]. The study of the Huygens’ principle and of scalar tails leads to unexpected physics [3, 4, 5, 6]. The main goal of the present paper is the application of these results to cosmology, in particular to LTMI, continuing the program initiated in Refs. [7, 2, 8, 9].

An interesting issue in the physics of wave propagation is the validity of the Huygens’ principle. A field satisfying a linear wave equation can propagate “sharply” along the characteristic surfaces, or with “tails” of radiation, reverberations that degrade the information carried by a initially delta–like pulse, and that violate the Huygens’ principle. To be clear, we adopt the physical definition of Huygens’ principle due to Hadamard [10]. Assume that a delta–like pulse of radiation (light, for example) is emitted by a point–like source in  $P$ , at the time  $t = 0$ . If, at the time  $t > 0$ , the radiation is entirely confined to the surface of the sphere of center  $P$  and radius  $r = ct$  (where  $c$  is the speed of light), one says that the Huygens’ principle is satisfied. If, on the contrary, there is radiation at radii  $r$  such that  $r < ct$ , there are *tails* of radiation: the waves are spread at any radius. A precise mathematical definition<sup>1</sup> is found in Sec. 2.

It is known that, given a wave equation in a curved spacetime  $(M, g_{ab})$ , the Huygens’ principle is generally violated by its solutions, due to the following possibilities: the presence of a mass term in the wave equation satisfied by the field, the dimensionality of spacetime, and backscattering off the background curvature of spacetime [10]–[13]. The first of these causes is trivial and well known. Moreover, in this paper there are no tails due to the spacetime dimensionality. Backscattering off the spacetime curvature, on the other hand, is nontrivial and the presence or absence of tails for scalar, electromagnetic, and gravitational waves has been established only for a handful of spacetime metrics  $g_{ab}$ .

It is not surprising that the study of violations of the Huygens’ principle has fruitful

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<sup>1</sup>Unfortunately, the terminology commonly used in the literature is misleading; it would be more appropriate to refer to the “Huygens’ property” instead of the “Huygens’ principle”.

applications to cosmology, in view of the fact that scalar fields are widely used in this area, especially in inflationary theories of the early universe, and as candidates for dark matter in today's universe. Also, it is worth reminding the reader that tails of gravitational waves due to the spacetime curvature near compact sources have received attention in conjunction with the data analysis of the large interferometric detectors of gravitational waves [14]. The relevance of tails for cosmological gravitational waves is, instead, unclear.

The plan of the paper is as follows: in Sec. 2, a theorem valid for massive fields of spin  $s \geq 1/2$  satisfying wave equations is recalled, and analogous results are derived for the massive spin 0 field. The importance of a correct formulation of the Huygens' principle for physical applications is emphasized. Then, we proceed to study an "ultrapathological" case of wave propagation for a scalar field in a curved space. In Sec. 3, which is the most relevant to cosmology, the late time mild inflationary scenario of the universe is studied, and it is shown that this scenario essentially coincides with the ultrathological space of Sec. 2. Alternative mechanisms to achieve late time mild inflation are discussed. Section 4 presents the conclusions.

## 2 Massive fields in curved spaces and the tail-free property

Massive fields of arbitrary spin satisfying wave equations in a curved space have been studied for a long time, both from the mathematical and the physical (classical and quantum) point of view. In this paper, we restrict ourselves to the classical aspects of the physics of wave propagation, in particular the violation of the Huygens' principle and the occurrence of tails of radiation for a field satisfying a wave equation [10]–[13].

It is required that the fields considered live in the spacetime  $(M, g_{ab})$ , where  $M$  is a four-dimensional smooth manifold,  $g_{ab}$  is the metric tensor, and  $\nabla_a$  is the associated covariant derivative operator<sup>2</sup>.

We begin by considering massive fields with spin  $s \geq 1/2$ , which have recently been the subject of renewed interest [15]; the following theorem is valid (we refer the reader to Ref. [16] for the relevant equations and a proof):

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<sup>2</sup>The metric signature is  $-+++$ . The speed of light and Planck's constant assume the value unity. The Ricci tensor is given by  $R_{\mu\rho} = \Gamma_{\mu\rho,\nu}^\nu - \Gamma_{\nu\rho,\mu}^\nu + \Gamma_{\mu\rho}^\alpha \Gamma_{\alpha\nu}^\nu - \Gamma_{\nu\rho}^\alpha \Gamma_{\alpha\mu}^\nu$  in terms of the Christoffel symbols  $\Gamma_{\alpha\beta}^\delta$ , and  $R = R_\mu{}^\mu$ . The abstract index notation is used.

**Theorem 1:** *A solution of the homogeneous wave equation for a massive field with spin  $s \geq 1/2$  on the spacetime  $(M, g_{ab})$  obeys the Huygens' principle if and only if  $(M, g_{ab})$  is a spacetime of constant curvature and the Ricci scalar satisfies*

$$R = \frac{6m^2}{s}, \quad (2.1)$$

where  $m$  is the mass of the field.

The formulation of the Huygens' principle used in Theorem 1 and in its proof is crucial. In fact, although the Huygens' principle for the solutions of a wave equation was formulated by Hadamard [10] in a clear and physically meaningful way as the absence of tails of radiation, several other definitions have been introduced in the literature over the years: characteristic propagation property, progressing-wave propagation, etc. These definitions are *a priori* inequivalent, and they are all loosely referred to as the "Huygens' principle". This improper terminology is often a source of confusion and misinterpretations of mathematical results (see Refs. [17, 18] for a clarification of the relationships between at least some of the various definitions proposed in the literature). In the following, we consider the analogue of Theorem 1 for the case of the massive scalar field ( $s = 0$ ). To this end, we first provide a unambiguous definition of the Huygens' principle.

A scalar field  $\phi$  in a source-free region of spacetime satisfies the homogeneous Klein-Gordon equation

$$g^{ab}\nabla_a\nabla_b\phi - m^2\phi - \xi R\phi = 0, \quad (2.2)$$

where the dimensionless constant  $\xi$  describes the direct coupling between the field  $\phi$  and the Ricci curvature  $R$  of spacetime. The formal solution of Eq. (2.2) is given by a Green function representation in a normal domain  $\mathcal{N}$  of spacetime not containing sources as

$$\phi(x) = \int_{\partial\mathcal{N}} dS^{a'}(x') G(x', x) \overleftrightarrow{\nabla}_{a'} \phi(x'), \quad (2.3)$$

where  $\partial\mathcal{N}$  is the boundary of the normal domain  $\mathcal{N}$ ,  $dS^{a'}(x')$  is the oriented volume element on the hypersurface  $\partial\mathcal{N}$  at  $x'$ , and

$$f_1 \overleftrightarrow{\nabla} f_2 \equiv f_1 \nabla f_2 - f_2 \nabla f_1 \quad (2.4)$$

for any pair of differentiable functions  $(f_1, f_2)$ . For physical reasons, we restrict ourselves to the consideration of the retarded Green function  $G_R(x', x)$ , which is a solution of the wave equation (2.2) with an impulsive source located at  $x$ ,

$$\left[ g^{a'b'}(x') \nabla_{a'} \nabla_{b'} - m^2 - \xi R(x') \right] G(x', x) = -\delta(x', x). \quad (2.5)$$

$\delta(x', x)$  is the delta function on spacetime such that, for each test function  $f$ ,

$$\int d^4x' \sqrt{-g(x')} f(x') \delta(x', x) = f(x). \quad (2.6)$$

The retarded Green function  $G_R(x', x)$  admits the decomposition [10]–[12]

$$G_R(x', x) = \Sigma(x', x) \delta_R(\Gamma(x', x)) + V(x', x) \Theta_R(-\Gamma(x', x)). \quad (2.7)$$

$\Gamma(x', x)$  is the square of the proper distance between  $x'$  and  $x$  computed along the unique geodesic connecting  $x'$  and  $x$  in the normal domain  $\mathcal{N}$ ;  $\Gamma = 0$  corresponds to the light cones.  $\delta_R$  and  $\Theta_R$  are, respectively, the Dirac delta distribution and the Heaviside step function with support in the past of  $x'$ .

The functions  $\Sigma$  and  $V$  are uniquely determined in a given spacetime metric [11, 12]. The non-vanishing of  $V(x', x)$  corresponds to the presence of wave tails propagating *inside* the light cone [10]–[12], while the first contribution to  $G_R$ , weighted by the coefficient  $\Sigma(x', x)$  describes sharp propagation *along* the light cone. The structure (2.7) of the retarded Green function is qualitatively the same for the wave equations satisfied by fields of higher spin in a curved space. Here, we omit writing these equations and the corresponding Green functions explicitly, for the sake of brevity. However, it is important to remember that the formulation of the Huygens' principle used in Theorem 1 corresponds to the absence of tails (i.e.  $V(x', x) = 0$  for all spacetime points  $x', x$  in Eq. (2.7)). Following Ref. [3], one considers a neighborhood  $U(x)$  of the spacetime point  $x \in M$ , and one Taylor-expands  $G_R(x', x)$ , obtaining

$$\Sigma(x', x) = \frac{1}{4\pi} + r_1(x', x), \quad (2.8)$$

$$V(x', x) = -\frac{1}{8\pi} \left[ m^2 + \left( \xi - \frac{1}{6} \right) R(x) \right] + r_2(x', x), \quad (2.9)$$

where the remainders  $r_{1,2}(x', x) \rightarrow 0$  as  $x' \rightarrow x$ . When the neighborhood  $U(x')$  has a small diameter ( $x' \rightarrow x$ ), there is a tail ( $V(x', x) \neq 0$ ) unless the effective mass  $m_{eff}(x)$  given by

$$m_{eff}^2(x) = m^2 + \left( \xi - \frac{1}{6} \right) R(x) \quad (2.10)$$

vanishes. We introduce

**Definition 1:** the field  $\phi$  obeying Eq. (2.2) satisfies the Huygens' principle at the spacetime point  $x$  if  $V(x', x) \rightarrow 0$  for  $x' \rightarrow x$  in a normal neighborhood of  $x$ .

**Definition 2:** the field  $\phi$  obeying Eq. (2.2) satisfies the Huygens' principle if the latter is satisfied at every spacetime point  $x$ .

Then, a straightforward consequence of Eqs. (2.7), (2.9) is

**Lemma:** *The solution of Eq. (2.2) with  $\xi \neq 1/6$  in the spacetime  $(M, g_{ab})$  satisfies the Huygens' principle in  $x$  if and only if*

$$R(x) = \frac{6m^2}{1 - 6\xi}. \quad (2.11)$$

The case  $\xi = 1/6$  is special; in this case there are no tails if and only if  $m = 0$ , irrespective of the curvature (the value  $\xi = 1/6$  is of physical significance – see below). We also have

**Theorem 2:** *A sufficient condition for a solution of Eq. (2.2) with  $\xi \neq 1/6$  to satisfy the Huygens' principle in the spacetime  $(M, g_{ab})$  is that the latter is a constant curvature space and  $R = 6m^2/(1 - 6\xi)$ .*

So far, our considerations have been limited to the mathematical aspects of the propagation of a scalar field in a curved space. At this point, it is interesting to examine the subject from the physical point of view. The physical reasons for the occurrence of tails are [3, 18]:

*i):* The field is massive ( $m \neq 0$ ). For example, the solutions of the Klein–Gordon equation (2.1) in the four–dimensional Minkowski space  $(R^4, \eta_{ab})$  have tails whenever  $m \neq 0$ .

*ii):* The dimensionality of spacetime. For example, the solutions of Eq. (2.2) in the  $k$ –dimensional Minkowski space have tails for odd  $k$ , but not for even  $k > 2$  [10]. In this paper, we restrict ourselves to the case of a four–dimensional manifold.

*iii):* Backscattering of the waves off a potential and/or the spacetime curvature. This is the most interesting case and in this section we consider only a non self–interacting field, hence the potential is absent and we are concerned solely with the backscattering off the background curvature. The extension to a self–interacting field is straightforward [3]. Moreover, in this paper the dimension of spacetime is fixed to four, and we are not concerned with tails due to odd spacetime dimension.

Although the study of the conditions for the absence of wave tails for massive fields of arbitrary spin (e.g. Refs. [16, 19]) is legitimate from the mathematical point of view, it is not easy to justify from the physical perspective. In fact, a field with  $m \neq 0$  will have a tail due the fact that it is massive (this tail is present even in flat space) and due to the backscattering off the background curvature of spacetime. The absence of tails means that the two effects exactly cancel each other. This situation corresponds to a field with nonzero intrinsic mass that propagates sharply along the light cone, a phenomenon that has no experimental or observational support. A wave tail is indeed a desirable feature for a massive field; there is not much point in requiring that the Huygens' principle be satisfied on a curved space and in deriving the conditions under which this "principle" is satisfied. As a matter of fact, these conditions are very restrictive, as is suggested by Theorems 1 and 2. In other words, the Huygens' principle is not a fundamental principle like, say, the equivalence principle, and its violation is very realistic.

We conclude this section with an example relevant for cosmology, which will be used later in Sec. 3. In this example, the balance between tails due to a mass term and those due to backscattering off the background curvature is achieved exactly at *every* spacetime point. Keeping in mind Theorem 2, we consider the de Sitter space of constant curvature  $R$ , and a test scalar field satisfying Eq. (2.2), with a mass given by

$$m = \left[ R \left( \frac{1}{6} - \xi \right) \right]^{1/2} \quad (2.12)$$

for  $\xi < 1/6$ . Then  $V(x', x) = 0$  and the field propagates sharply along the light cone at every spacetime point. However, its intrinsic mass  $m$  can be made arbitrarily large by suitably choosing the Ricci curvature (or the constant  $\xi$ , or both), while the effective mass given by Eq. (2.10) vanishes. We will call this example the *ultrapathological* spacetime. Of course, one could also consider its counterpart obtained by using the anti-de Sitter space and  $\xi > 1/6$ .

### 3 Late time mild inflation

In this section, we proceed to apply to cosmology the previous considerations on scalar wave tails. In Ref. [3] it was argued that, if the Einstein equivalence principle [20] is valid (i.e. in any metric theory of gravity in which the nature of  $\phi$  is nongravitational), then in the limit  $x' \rightarrow x$  the solutions of Eq. (2.2) and the corresponding Green functions must have the same structure as in flat space. This corresponds to the local approximation of the spacetime  $(M, g_{ab})$  with its tangent Minkowski space. The flat space retarded Green

function

$$G_R^{(M)}(x', x) = \frac{1}{4\pi} \delta_R(\Gamma(x', x)) - \left( \frac{m^2}{8\pi} + r_3(x', x) \right) \Theta_R(-\Gamma(x', x)), \quad (3.1)$$

where  $r_3(x', x) \rightarrow 0$  as  $x' \rightarrow x$ , must be reproduced in the  $x' \rightarrow x$  limit, and this requirement leads to the prescription  $\xi = 1/6$  for the value of the coupling constant [3]. This result was rederived and confirmed in [4, 5] and it can be physically interpreted as the fact that, in the absence of a scalar field mass, no scale must appear in the local solution to the wave equation, in analogy with the flat space situation<sup>3</sup>. The prescription  $\xi = 1/6$  has many consequences for cosmological inflation. In fact, the success of many inflationary scenarios strongly depends from the fine tuning of the parameter  $\xi$ , which is impossible once the value of  $\xi$  is fixed to the conformal value  $1/6$  ([21, 2]).

If inflation is driven by a quantum scalar field, the Einstein equivalence principle probably cannot be imposed. The equivalence principle is likely to be violated at the quantum level, and the prescription  $\xi = 1/6$  is not applicable in the quantum regime. However, it is a common belief that inflation is a classical phenomenon [22]. Moreover, there are other prescriptions for the value of the coupling constant  $\xi$  (see references in [2]) that are valid for quantum fields, and they differ according to the physical nature of the field  $\phi$ . The existence of tails of radiation, and the issue of the value of  $\xi$  are relevant also for other areas of cosmology and of theoretical physics [23, 7, 24, 25, 2].

Currently, cosmology faces two problems raised by recent observations: the age of the universe problem (the age of certain globular clusters is larger than the age of the universe inferred from the method of Cepheid variables [26, 27, 28]), and the discrepancy between the local and the global (based on the Zeldovich–Sunyaev effect [29, 30]) measures of the Hubble parameter  $H_0$ . In order to reconcile theory and observations, it has been proposed that the universe undergoes short periods of piece-wise exponential expansion that interrupt the matter-dominated era after star formation (“late time mild inflation” or LTMI) [1].

### 3.1 The proposed mechanism for LTMI is physically pathological

The mechanism proposed in [1] to achieve LTMI is based on a classical, massive, non self-interacting scalar field nonminimally coupled to the Ricci curvature of spacetime,

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<sup>3</sup>Note that this is not guaranteed by setting  $\xi = 0$ ; in this case the curvature scale would survive in the Green function, which is the solution for an impulsive source used in the physical definition of the Huygens’ principle given by Hadamard [10].

and satisfying Eq. (2.2), in the context of general relativity. The authors of Ref. [1] assume a Einstein–de Sitter universe and a baryon density of order  $\Omega_m = 0.01$  (in units of the critical density  $\rho_c = 3H^2/8\pi G$ ). The Einstein equations for a mixture of dust and a scalar field are

$$H^2 = \frac{8\pi G}{3} (\rho_m + \rho_\phi) , \quad (3.2)$$

$$\dot{H} + H^2 = -\frac{4\pi G}{3} (\rho_m + \rho_\phi + 3P_\phi) , \quad (3.3)$$

where  $H = \dot{a}/a$  is the Hubble parameter,  $a(t)$  is the scale factor of the Einstein–de Sitter line element, and a overdot denotes differentiation with respect to the comoving time  $t$ .  $\rho_m$  is the energy density of dust,  $P_m = 0$ , and the energy density and pressure of the scalar field component of the cosmic fluid are given by

$$\rho_\phi = (1 - 8\pi G\xi\phi^2)^{-1} \left[ \frac{(\dot{\phi})^2}{2} + F(\phi) + 6\xi H\phi\dot{\phi} \right] , \quad (3.4)$$

$$P_\phi = (1 - 8\pi G\xi\phi^2)^{-1} \left[ \left( \frac{1}{2} - 2\xi \right) \dot{\phi}^2 - F(\phi) - 2\xi\phi\ddot{\phi} - 4\xi H\phi\dot{\phi} \right] , \quad (3.5)$$

respectively, where  $F(\phi) = m^2\phi^2/2$ . The Klein–Gordon equation becomes

$$\ddot{\phi} + 3H\dot{\phi} + m^2\phi + 6\xi (\dot{H} + 2H^2) \phi = 0 . \quad (3.6)$$

A period of LTMI corresponds to the particular solution

$$H_* = \left( \frac{m^2}{12|\xi|} \right)^{1/2} , \quad \phi_*^2 = \frac{1}{8\pi G|\xi|} , \quad (3.7)$$

for which  $\rho_\phi \geq 0$ ,  $P_\phi = -\rho_\phi$ . Due to the onset of instabilities, the exponential expansion soon stops and is followed by a oscillatory decay (due to the fact that  $m \neq 0$ ). The values of the parameters  $m$  and  $\xi$  have to be adjusted in order to fit the observations; in particular, a negative value of  $\xi$  and a rather large (compared to unity) value of its modulus are essential for a successful LTMI [1]. The authors of Ref. [1] chose  $\xi = -80$  and  $m = 10^{-31}$  eV (although this is more an example than a best fit of the observational data, it gives an idea of the orders of magnitude of the parameters  $\xi, m$  needed for an interesting LTMI).

In the light of the result of Ref. [3] explained at the beginning of this section, the value of the coupling constant  $\xi$  is fixed to  $1/6$  in general relativity, and the LTMI

scenario does not work. It is in principle possible that LTMI can be achieved in the context of a theory of gravity and of the boson field in which the prescription  $\xi = 1/6$  does not apply. In this case, one still has to deal with the other prescriptions for the value of  $\xi$  existing in the literature (see [2] for a review). A more serious problem is that, even ignoring the prescription  $\xi = 1/6$  coming from the Einstein equivalence principle, each phase of LTMI is extremely close to the ultrathological spacetime described at the end of the previous section. In fact, LTMI corresponds to the vanishing of the effective mass given by

$$\mu^2 = m^2 + 6\xi (\dot{H} + 2H^2) , \quad (3.8)$$

while the ultrathological space corresponds to the vanishing of  $m_{eff}$  given by Eq. (2.10),

$$m_{eff}^2 = m^2 + (6\xi - 1) (\dot{H} + 2H^2) . \quad (3.9)$$

For  $\xi \gg 1$ ,  $m_{eff} \simeq \mu$  and LTMI essentially reproduces the ultrathological space. Using the value  $\xi = -80$  of Ref. [1], one obtains from Eq. (3.8) that  $H_*^2 \simeq 1.0416 \cdot 10^{-3} m^2$ , while the ultrathological case corresponds to  $H_*^2 \simeq 1.0395 \cdot 10^{-3} m^2$ . A very substantial part of the tail of  $\phi$  due to the intrinsic mass  $m$  is cancelled by the tail due to the backscattering off the background curvature of spacetime. The cancellation becomes more and more precise as  $-\xi$  increases, which makes inflation more and more pronounced [1].

### 3.2 Alternative mechanisms for LTMI

The mechanism used in Ref. [1] to achieve LTMI is clearly unphysical. Is there a realistic mechanism that works? In order to answer this question, one possibility is adding a nontrivial (i.e. not a pure mass term) potential  $F(\phi)$  to the picture. However, one then defines the intrinsic mass of the scalar field in the late time mild inflationary state  $(H_*, \phi_*)$  as  $m_\phi = d^2 F/d\phi^2(\phi_*)$ , and one is facing again the problem of the cancellation between the tail due to the intrinsic mass  $m$  and the tail due to the backscattering off the background curvature.

A way out of this dilemma could be the consideration of the linear potential  $F(\phi) = \lambda\phi$ , for which  $m_\phi = 0$ . In this case, the Einstein equations (3.2), (3.3), supplemented by the expressions for the energy density and pressure of the scalar field (3.4), (3.5) admit, for  $\xi < 0$ , the de Sitter solution

$$H_{**}^2 = \left( \frac{\pi G}{6|\xi|} \right)^{1/2} \lambda , \quad \phi_{**}^2 = \frac{1}{24\pi G|\xi|} . \quad (3.10)$$

In principle, one can stop a late time inflation of this kind; in the original mechanism for LTMI proposed in Ref. [1], the exit from the exponential expansion was due to the Ljapunov instability of the de Sitter solution against small perturbations. For a field in a linear potential the de Sitter solution is also unstable. In fact, consider the universe in a state that is a small perturbation of the  $(H_{**}, \phi_{**})$  inflationary state,

$$\phi = \phi_{**}(1 + x), \quad H = H_{**}(1 + y), \quad (3.11)$$

where  $x$  and  $y$  are small compared to unity. After straightforward calculations, one obtains the evolution equations for the perturbations,

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad (3.12)$$

where

$$a_1 = \alpha, \quad (3.13)$$

$$a_2 = a_4 = -4\alpha, \quad (3.14)$$

$$a_3 = -\frac{2\alpha}{3\xi - 2}, \quad (3.15)$$

$$\alpha = \left( \frac{\pi G}{6|\xi|} \right)^{1/4} \lambda^{1/2}, \quad (3.16)$$

or

$$\dot{\underline{x}} = \alpha M \underline{x}. \quad (3.17)$$

The matrix  $M$  has real eigenvalues

$$s_{1,2} = \frac{3}{2} \left( -1 \pm \sqrt{1 - \frac{16\xi}{2 - 3\xi}} \right), \quad (3.18)$$

where the discriminant  $\Delta = (18 - 75\xi)(2 - 3\xi)^{-1} > 0$  for  $\xi < 0$ . Since  $s_1, s_2$  have opposite signs,  $(H_{**}, \phi_{**})$  is a saddle point, and describes an unstable equilibrium. A perturbation can grow and break the exponential expansion.

Another possibility that one can naturally think of in order to avoid the ultrathological spacetime, consists in requiring that, during LTMI, the growth of the scale factor be accelerated, but not exponential. For example, one can search for piece-wise power-law inflationary solutions  $a = a_0 t^p$ , where  $p > 1$ . Then, the Ricci curvature is not constant and the exact cancellation of mass and curvature tails can occur at most

at a single instant in the history of the universe; because of the monotonic behaviour of the Ricci curvature  $R = 6p(2p - 1)t^{-2}$ , the equation  $m_{eff} = 0$  has only one root. This instant of pathological behaviour can be avoided by making the piece-wise period of inflation sufficiently short. A power-law inflationary solution for a universe driven by a nonminimally coupled scalar field with the potential

$$F(\phi) = A\phi^n, \quad n > 6, \quad (3.19)$$

was found in Ref. [21]. One has, for this solution,

$$p = 2 \frac{1 + (n - 10)\xi}{(n - 4)(n - 6)|\xi|}. \quad (3.20)$$

In general relativity, the prescription  $\xi = 1/6$  yields  $p = 2/(n - 6)$ , which corresponds to *i*) accelerated expansion if  $6 < n < 8$ , *ii*) to a coasting universe if  $n = 8$ , and *iii*) to a decelerated universe which still expands faster than  $a(t) = a_0 t^{2/3}$  if  $n < 9$ . Hence, in principle, one can achieve periods of LTMI in general relativity, with the potential (3.19). The detailed analysis of these alternative mechanisms for LTMI and their comparison with the cosmological observations are beyond the scope of this paper, which focuses on tails of radiation. In addition, it would be desirable to identify the scalar field in the potential (3.19) with some known field from high energy physics, which is not done in the phenomenological approach to LTMI.

## 4 Discussion and conclusions

The violation of the Huygens' principle and the presence of tails of radiation have been studied for many years in the context of mathematical physics. Only recently it has been realized that tails of radiation have important physical applications. An example in astrophysics is given by the tails in the gravitational radiation emitted by compact objects. These tails are relevant for the correct data analysis (matched filtering) in the large laser interferometric detectors of gravitational waves (*LIGO*, *VIRGO*, *GEO600*, *TAMA*, ...) [14].

In classical field theory in curved spaces, a counterintuitive result is that the absence of pathologies in the propagation of scalar fields fixes to  $1/6$  the value of the coupling constant  $\xi$  of the scalar field with the Ricci curvature [3]. This prescription has far-reaching consequences for cosmological inflation ([2] and references therein), for the cosmic no-hair theorems [31, 32], and possibly for other areas of cosmology and of theoretical physics [8, 2, 25].

In the present paper, we have considered the idea of LTMI, recently proposed to solve the age of the universe problem and the puzzle of the discrepancy between the local and global measures of the Hubble parameter. While the idea of LTMI appears to be very valuable, unfortunately the mechanism employed to achieve it (a classical, massive, non self-interacting scalar field nonminimally coupled to the Ricci curvature) is not viable, because it corresponds to the extremely pathological physics discussed in Sec. 2. In fact, the LTMI scenario almost exactly reproduces the ultrathological spacetime. Alternative mechanisms to generate LTMI are discussed in Sec. 3, and the possibility of having LTMI with a negatively coupled, self-interacting scalar field is not ruled out in general relativity. However, one must be willing to pay the price of introducing a suitable scalar field potential and obtaining a less-than-exponential expansion of the universe during LTMI.

The approach to LTMI is purely phenomenological, and no serious attempt is made to identify the scalar field with a known field from a high energy physics theory. The hypothetical possibility of identifying the scalar field with a superlight Proca field [1] clearly does not work if the field is self-interacting (apart from the problem that a homogeneous vector field would introduce anisotropy, and a nontrivial distribution of this vector field would need to be considered [1]). The analysis of the LTMI scenario would have to be redone if a vector field instead of a scalar one was used as a source term in the right hand side of the Einstein equations. On the other hand, fields of different spin in the same background metric have different behaviour with respect to tails. For example, the Maxwell field satisfies the Huygens' principle in a Friedmann-Robertson-Walker space; in fact, the latter is conformally flat, and the Maxwell equations are conformally invariant. The tail-free property of the Maxwell field in Minkowski space is then transferred to the Friedmann-Robertson-Walker space [17, 33]. Hopefully, a viable mechanism will be found which is capable of successfully implementing the idea of LTMI. Work in this direction is in progress.

The model of the universe analogous to that of LTMI, but with  $\xi > 0$ , does not give rise to inflation and was considered in Ref. [34] in order to explain the reported periodicity in the redshift of galaxies [35]. A look at the latter model with the knowledge of scalar field tails is also instructive [8], and leads to information on the nature of the correct theory of gravity, should the reported redshift periodicity turn out to be genuine and not an artifact of incomplete or faulty statistics.

Previous literature [8] and the present paper show that tails of scalar fields and nonminimal coupling to the Ricci curvature are very relevant for cosmology, and not only for inflationary theories. Moreover, tails and nonminimal ( $\xi \neq 0$ ) coupling are forced upon us in almost all situations of physical interest. Thus, it is seen that the

study of these phenomena is not optional, rather it is necessary in cosmology.

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