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The Problems of Singularity Particle Horizon and Flatness in Quantum Cosmology

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Classical relativistic cosmology is known to have the space-time singularity as an inevitable feature. The standard big bang models have very small particle horizons in the early stages which make it difficult to understand the observed homogeneity in the universe. The relatively narrow range of the observed matter density in the neighbourhood of closure density requires highly fine tuning of the early universe. In this paper it is argued that these three problems can be satisfactorily resolved in quantum cosmology. It is shown that it is extremely unlikely that the universe evolved to the present state from quantum states with singularity and particle horizon. Similarly, it is shown that of all possible states the Robertson–Walker model of flat spatial sections is the most likely state for the universe to evolve out of a quantum fluctuation. To demonstrate these results a suitable formalism for quantum cosmology is first developed.

1. INTRODUCTION

While the discovery of the microwave background, the work on primordial nucleosynthesis and the recent applications of grand unified theories to the early universe have given a boost to the standard big bang cosmology, serious difficulties with this picture are also being increasingly emphasized. In this paper we discuss three problems of a fundamental nature which at present seem to be beyond the ability of classical cosmology to resolve. These problems are known under the headings (i) space-time singularity; (ii) particle horizon and (iii) flatness.

The problem of singularity has been with relativistic cosmology from the early days in 1922 when Friedmann [1] first constructed the model of an evolving universe. As later shown rigorously by Robertson [2] and Walker [3], the Friedmann models are maximally symmetric. In the comoving spherical polar coordinates (r, θ, ϕ) and the cosmic time t, the metric of the spacetime is given by

$$ds^2 = c^2\,dt^2 - S^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\,d\phi^2)\right]. \tag{1}$$

Here c is the speed of light, which we shall henceforth take as unity, $k = +1$, 0 or -1 is the curvature parameter and $S(t)$ is the scale factor for expansion. We will refer to (1) as the FRW line element, after the three authors associated with it.

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Einstein's field equations for an isotropic energy tensor T^i_k then become

$$2\frac{\ddot{S}}{S}+\frac{\dot{S}^2+k}{S^2}=-8\pi p, \tag{2}$$

$$3\frac{\dot{S}^2+k}{S^2}=8\pi\varepsilon. \tag{3}$$

Note that we have set the Newtonian gravitational constant $G=1$. The pressure p and energy density ε are related by an equation of state. For the dust models$p=0$ while for the early radiation dominated models $p=\varepsilon/3$. For $p \geqslant 0$, $\varepsilon>0$, we see that at a turning point of S, $\ddot{S}$ cannot be positive. Hence the presently expanding universe must have had $S=0$ at some epoch in the past. We will choose t such that $t=0$ at this epoch.

In the 1950s it was believed that the space-time singularity implied by $S=0$ was a consequence of the high degree of symmetry assumed in the FRW model. Work by Raychaudhuri [4] and Heckmann and Schücking [5] seemed to suggest that a spinning universe might be nonsingular. This hope was, however, laid to rest by the general theorems of Penrose, Hawking and Geroch [6] which showed that provided the energy tensor satisfied well specified positivity conditions and provided certain other reasonable topological conditions were met, the space-time singularity was inevitable in relativistic cosmology.

The particle horizon appears in the FRW models in the following way. At any cosmic time t, a typical observer at $r=0$ receives signals from observers within the sphere $r \leqslant r(t)$, where

$$\int_0^{r(t)}\frac{dr}{\sqrt{1-kr^2}}=\int_0^t\frac{du}{S(u)}<\infty. \tag{4}$$

The integral on the right-hand side is convergent for the standard big bang models and tends to zero as $t\to 0$. That is,

$$\lim_{t\to 0} r(t)=0. \tag{5}$$

The particle horizon at $r=r(t)$ therefore shrinks to zero as $t\to 0$.

This geometrical property causes concern when one observes the large-scale isotropy of the microwave background radiation at the present epoch [7]. The big bang explanation of this radiation implies, however, that it was fully thermalized prior to the epoch of redshift $\sim 10^3$, i.e., when S was about a thousandth of its present value. Clearly, unless the radiation was created homogeneously all over the universe, it cannot appear so isotropic today. Had there been no particle horizons, we could have argued that large scale homogenization took place subsequent to creation. This alternative would have been more attractive since it does away with special initial conditions.

In the late 1960s Misner [8] proposed that early anisotropy of expansion could lead to horizon-free directions. For example, the Bianchi type I model with dust given by

$$ds^2 = dt^2 - \frac{t^2}{(t+t_0)^{2/3}}\,dx^2 - (t+t_0)^{4/3}\,(dy^2+dz^2), \qquad t_0 = \text{constant} \tag{6}$$

has no particle horizon in the x-direction, although horizons exist in the y and z directions. In Misner's mixmaster universe the horizon free direction was varied rapidly and randomly in the hope that regions in all directions would have the opportunity to mix. It was shown by Chitre [9], however, that this mixing cannot occur. Like the singularity, the particle horizon also remains an unsolved problem for relativistic cosmology.

Finally, we mention the flatness problem, first highlighted by Dicke and Peebles [10] and more recently by Guth [11]. We can appreciate this problem with the help of the dust models. Denote by subscript zero the values at the present epoch and write

$$H(t) = \dot{S}/S, \qquad H_0 = H(t_0). \tag{7}$$

For $k = 0$, the value of ε defines the closure density which we denote by

$$\varepsilon_c = 3H^2/8\pi. \tag{8}$$

Define the flatness parameter Ω by

$$\Omega = \varepsilon/\varepsilon_c. \tag{9}$$

Observations show that the present value of Ω lies in the range $0.01 \lesssim \Omega_0 < 10$. Precise future observations may easily narrow this range. For $\Omega > 1$ we have closed models ($k = 1$) while for $\Omega < 1$ we have open models ($k = -1$).

From (3) and (9) we get

$$\Omega - 1 = (\dot{S}_0/\dot{S}^2)(\Omega_0 - 1). \tag{10}$$

This equation is valid for any $\varepsilon - p$ relation. If we consider its implications for the past epochs when $\dot{S}$ was higher than now, we find that $|\Omega - 1| \ll |\Omega_0 - 1|$. Dicke and Peebles showed that the present range in Ω_0 corresponds to $|\Omega - 1| \ll \sim 10^{-14}$ at $t = 1$. As Guth has pointed out, if we apply this relation to even earlier epochs when baryons are supposed to have formed, the fine tuning of Ω near the value 1 for the flat $k = 0$ model has to be as good as 1 part in 10^{80}. Such finely tuned initial condition is difficult to understand.

Guth's remedy of an inflationary universe does account for this fine tuning. However, as discussed by Barrow and Turner [12] this scenario has many problems. Its successor the new inflationary universe has also run into difficulties on account of monopole production [13].

We have discussed at length the three difficulties of classical cosmologies because we wish to show that their solution may very well lie in quantum cosmology. In the following section we give a brief account of our approach to quantum cosmology. We then show that the first two problems—of singularity and horizon—are related and can both be resolved together. Finally we outline our solution of the flatness problem, again with the help of the quantum techniques developed in Section 2.

Our analysis to be given below is restricted on two counts. First, we work entirely within the Robertson–Walker models. These are homogeneous and isotropic and describe the *present* state of the universe fairly well. The problems of singularity and horizon are, however, applicable to more general space-times, space-times which are neither homogeneous nor isotropic. Our present analysis does not cover these general models. Second, we do not have a complete theory of quantum gravity, Our excursion into this field is a limited one, confined to quantizing the conformal degree of freedom. Nevertheless the dividends yielded by these simplifications hold out hopes of dealing with the more general situations by future techniques of quantum gravity more sophisticated than ours. In this sense the proposed approach may be looked upon as a "first attempt" rather than "the last word" on the subject.

2. Quantum Cosmology through Path Integrals

Quantum gravity is still a developing area and experts differ in their assessment of what is required of a quantum theory of gravity. In the spectrum of current activity ranging from the purely formal approaches of Penrose and MacCallum [14] or DeWitt [15] to the flat space perturbation theory of Feynman [16], the approach to be adopted in this paper falls somewhere in between. For details of the present approach see Refs. [17–20]. In this section we present an outline of this approach highlighting the results which will be needed in later sections.

Consider first the classical problem of general relativity from a dynamical angle. Suppose that we have a space-time manifold $\mathscr{M}$ foliated by spacelike hypersurfaces $\{\Sigma\}$ and that in the coordinate system (x^i, $i = 0, 1, 2, 3$), $t \equiv x^0$ is constant along a typical hypersurface. Let $t = t_1$ and $t = t_2$ denote two such hypersurfaces Σ_1 and Σ_2 and let $\mathscr{V}$ be the space-time 4-volume sandwiched between Σ_1 and Σ_2. The classical Einstein–Hilbert problem then consists of extremizing the action

$$J = \frac{1}{16\pi} \int R \sqrt{-g}\, d^4x + J_m \tag{11}$$

for specified data on Σ_1 and Σ_2. J_m is the action for the physical contents of space-time. For example, for a system of particles a, b,..., with rest masses m_a, m_b,..., and elements of proper time ds_a, ds_b,..., the action is

$$J_m = \sum_a \int m_a\, ds_a. \tag{12}$$

The volume term in the Hilbert action should be accompanied by a surface term suggested by Hawking and Gibbons [35]. As discussed in [17] this term may be ignored here.

What data are to be specified on Σ_1 and Σ_2? In the early discussions of geometrodynamics Wheeler had advocated the specification of 3-geometries [21]. Later Isenberg and Wheeler [22] showed that there is ambiguity as well as lack of uniqueness in specifying data this way and a better prescription would be to specify the conformal part of 3-geometry and the extrinsic curvature instead.

In our approach to quantization we will confine ourselves to conformal degrees of freedom while freezing out the other degrees. Thus we will label the initial and final hypersurfaces with time coordinates while dispensing with the prescription of the extrinsic curvature (which also can be used to specify the time coordinate). As will become clear in the work that follows, the conformal function will vary with time and can be specified arbitrarily on Σ_1 and Σ_2.

This method gives us the clue towards a path integral approach to quantum gravity. Instead of a unique classical trajectory $\bar{\Gamma}$ specified by

$$\delta J = 0 \tag{13}$$

we now sum over all trajectories Γ with the specified end-conditions the quantity $\exp iJ/\hbar$. What do we mean by a trajectory?

The classical solution of Einstein's may be looked upon as a sequence of 3-geometries on $\{\Sigma\}$ beginning with the details given on Σ_1 and ending with those on Σ_2. This sequence we call $\bar{\Gamma}$. Any other sequence with the same details on Σ_1 and Σ_2 but *not* satisfying $\delta J = 0$, will be called a nonclassical trajectory. The sum

$$K[2;1] = \sum_{\Gamma} \exp\{iJ(\Gamma)/\hbar\} \tag{14}$$

is called the probability amplitude for the specified geometrical details on Σ_2 given the geometrical details on Σ_1. The sum Σ is formally a functional integral. The full theory of quantum gravity is contained in (14).

There are many conceptual and technical difficulties in the way of the actual computation of (14). We have, however, the following argument for justifying a simplification that we propose to introduce. Suppose we wish to preserve the causal structure of space-time for all the paths Γ that we choose, including $\bar{\Gamma}$. In that case we need all space-time geometries that have the same light cone structure as the classical geometry. Such geometries are conformal to one another. In other words, if $\bar{g}_{ik}$ is the metric tensor for the classical geometry, a nonclassical geometry will be given by a metric tensor

$$g_{ik} = \Omega^2 \bar{g}_{ik}, \tag{15}$$

where Ω is a C^2 function of space-time coordinates. If we chose any other nonclassical trajectory, we would necessarily have a different light cone structure and hence alter the causal connections between space-time points.

Why preserve the causal connections? We have done so to simplify the physical implications of quantizing gravity. Unlike quantizing flat-space theories which leave the background space-time unaltered, quantizing gravity does imply disturbing the structure of space-time geometry. Since all the rest of physics described in J_m is supposed to preserve causality, it is natural, as a first step to look for those quantization schemes which leave the causal connections unaltered. Indeed, an overall consistency may require even the full theory of quantum gravity to take note of this issue and this may very well reduce the full range of nonclassical Γ to those given by (15).

We therefore assume that the classical problem has been solved for given Isenberg–Wheeler conditions on Σ_1 and Σ_2 and that the only permissible nonclassical trajectories are those given by (15), with $\Omega = \Omega_1$ on Σ_1 and $\Omega = \Omega_2$ on Σ_2. The sum (14) is then replaced by a functional integral over Ω; with

$$K[\Omega_2, \Sigma_2; \Omega_1; \Sigma_1] = \int \exp\left\{\frac{iJ(\Omega)}{\hbar}\right\} \mathscr{D}\Omega \tag{16}$$

$$= \int \mathscr{D}\Omega(x) \exp\left\{\frac{i}{\hbar}\frac{1}{16\pi}\int (R\Omega^2 - 6\Omega^i\Omega_i)\sqrt{-g}\, d^4x\right\}.$$

As is well known the conformal factor enters with the "wrong" sign in the action creating serious problems when transition to Euclidean section is made. Various methods have been suggested in literature as to how a meaning can be ascribed to this path integral. [These methods range from cancelling $\Omega(x)$ with ghost fields [32] to working with an analytically continued imaginary $\Omega(x)$.] However, in our model, $\Omega(x)$ is *the only* quantum degree of freedom. In such a circumstance, it is possible to get physically meaningful results out of Eq. (16) as shown by the authors in their previous works. We define the quantity

$$\phi = \Omega - 1 \tag{17}$$

the quantum conformal fluctuation (QCF).

Next suppose that the quantum state of the universe is defined by a wavefunctional $\Psi(\phi_1, t_1)$ on Σ_1. Here ϕ_1 is a scalar function of the space coordinates x^μ ($\mu = 1, 2, 3$) and Ψ is defined over the space of such functions. The label t_1 serves to remind us that we are concerned with conditions on Σ_1. Then the propagator K gives us the state of the universe on Σ_2:

$$\Psi(\phi_2, t_2) = \int K[\Omega_2, t_2; \Omega_1, t_1]\, \Psi(\phi_1, t_1)\, \mathscr{D}\phi_1. \tag{18}$$

Our procedure so far is no different from the usual extension of the path integral formalism from quantum mechanics to quantum field theory [23]. In quantum field theory it is convenient at this stage to introduce second quantisation and to talk of states with given occupation numbers, creation and annihilation operators, etc. The type of questions that we wish to ask, however, can be best answered by continuing along the present route.

Quantising the conformal part of the 3-space metric was considered previously by

DeWitt [33], in the light of superspace quantisation. In this and other conventional approaches to quantum gravity the constraint equations are used to impose restrictions on the quantum state of geometry. Our approach in which only the conformal part of the 4-geometry is quantised is different on this account. The relationship between our path integral approach and the approach based on superspace is spelled out in detail elsewhere [34].

In particular, we note that the explicit form of K can be computed for most cases of interest to quantum cosmology. For example, in (12) we have the matter action for a system of particles not interacting with each other except through gravity. The classical picture of the early universe just after the Planck time (cf. [24], for example) contains particles which are highly relativistic and noninteracting because of asymptotic freedom. We may therefore take (12) as the suitable form of action to be included in (11). The computation of K under these conditions has been given by one of us [20] in explicit form. Changing from Ω to ϕ we obtain with $\hbar = 1$

$$K[\phi_2, t_2; \phi_1, t_1] = F(t_1, t_2) \exp \frac{3i}{8\pi} \left[\int \bar{\phi}\bar{\phi}_{,i}\, d\Sigma^i \right]_{t_2}^{t_1}, \tag{19}$$

where the exponent in the right-hand side is the difference in the values of the integral evaluated at t_1 and t_2. The function $\bar{\phi}$ is the solution of the wave equation

$$\Box\bar{\phi} + \tfrac{1}{6}\bar{R}\bar{\phi} = 0 \tag{20}$$

with the boundary conditions

$$\bar{\phi} = \phi_1 \text{ on } t_1, \qquad \bar{\phi} = \phi_2 \text{ on } t_2. \tag{21}$$

The coefficient $F(t_1, t_2)$ is a van Vleck determinant whose properties have been discussed by DeWitt [25]. In most cases of interest it can be evaluated from the transitivity property of the kernel K. Its explicit value is not relevant to our argument.

In [20] it was explicitly shown that if Σ_2 approaches a singular hypersurface, the kernel K tends to a degenerate form of product of two functionals

$$K[\phi_2, t_2; \phi_1, t_1] \sim f_2[\phi_2, t_2] f_1[\phi_1, t_1] \tag{22}$$

with the result that in (18) the final wavefunctional $\Psi(\phi_2, t_2)$ gets essentially decoupled from the initial wavefunctional $\Psi(\phi, t_1)$. This result was interpreted as the divergence of quantum uncertainty at the classical space-time singularity.

Reversing the role of t_1 and t_2 as the initial and final epochs, we can reinterpret (18) in this way. Suppose $t = 0$ is the singular epoch, and let $t_1 > t_2 > 0$. Then with K defined as a backward propagator, (18) describes the state $\Psi(\phi_2, t_2)$ from which the universe could have evolved to the present state $\Psi(\phi_1, t_1)$ at $t = t_1$. If the present state is nearly classical, Ψ would be peaked at $\phi_1 = 0$ and would have a narrow spread about this value. Then (18) shows that the spread around $\phi_2 = 0$ at $t = t_2$ was larger than what it is now. Moreover, this spread becomes infinite as $t_2 \to 0$. The classical solution is therefore no longer reliable as the description of the very early universe.

Given the full range $\mathscr{C}$ of conformal transforms (15) of the classical metric we may ask the following question. For $\Omega = 1$, i.e., for the classical solution we have a space-time singularity. Even for $\Omega \neq 1$ some metrics would describe space-times singular at $t = 0$. Let us denote their class by $\mathscr{C}_S$. The remaining class

$$\mathscr{C}_{NS} \equiv \mathscr{C} - \mathscr{C}_S \tag{23}$$

describes space-times nonsingular at $t = 0$. With the help of the kernel can we estimate the probability that the present state of the universe came out of $\mathscr{C}_S$? If this probability turns out to be nonzero, we can argue that although the classical singular solution is no longer reliable, its conclusion of the existence of singularity is still probably valid.

We have not yet succeeded in answering this question in its most general form. However, if we restrict our attention to homogeneous and isotropic models, we are able to give an explicit answer to this question. This we now propose to demonstrate.

3. Did the Universe Have a Singular Origin?

To conform with our notation of Section 2 we rewrite the classical solution of Einstein's equations for a homogeneous and isotropic universe in the form

$$d\bar{s}^2 = dt^2 - \bar{S}^2(t)[dr^2/(1 - kr^2) + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)]. \tag{24}$$

The existence of a singularity at $t = 0$ is expressed by

$$\lim_{t \to 0+} \bar{S}(t) = 0. \tag{25}$$

It is not difficult to see that a conformal transform of (24) which preserves the homogeneity and isotropy of the universe as implicit in Weyl's postulate and the cosmological principle [26] must have Ω depending on t only. Writing

$$d\tau = \Omega(t)\, dt, \qquad S(t) = \bar{S}(t)\, \Omega(t) \tag{26}$$

we can transform

$$ds^2 = \Omega^2 d\bar{s}^2 \tag{27}$$

to the form (1) with τ replacing t. Clearly, the class $\mathscr{C}_S$ of singular metrics is now made of those $\Omega(t)$ for which

$$\lim_{t \to 0+} S(t) \equiv \lim_{t \to 0+} \Omega(t)\, \bar{S}(t) = 0. \tag{28}$$

Note that since Ω depends on t only, our problem of computation of the kernel is considerably simpler, as it involves evaluating ordinary rather than functional integrals. For the explicit case of dust and $k = 0$, the kernel was evaluated in

Ref. [17]. We give here a more general solution which is applicable for any classical solution of (2) and (3).

We first restate the problem posed in (19)–(21) of Section 2. for the present simpler case. To evaluate $K[\phi_2, t_2; \phi_1, t_1]$ we have to solve the differential equation

$$\frac{1}{\bar{S}^3}\frac{d}{dt}\left(\bar{S}^3\frac{d\bar{\phi}}{dt}\right)+\left(\frac{\ddot{\bar{S}}}{\bar{S}}+\frac{\dot{\bar{S}}^2+k}{\bar{S}^2}\right)\bar{\phi}=0 \tag{29}$$

for the boundary conditions

$$\bar{\phi}(t_1)=\phi_1\equiv\Omega_1-1, \qquad \bar{\phi}(t_2)=\phi_2\equiv\Omega_2-1. \tag{30}$$

Then

$$K[\phi_2,t_2;\phi_1,t_1]=F(t_1,t_2)\exp\frac{3iV}{8\pi}\,[\bar{\phi}\dot{\bar{\phi}}\bar{S}^3]_{t_2}^{t_1}, \tag{31}$$

where V is the spatial coordinate volume of the space-time region $\mathscr{V}$. For the closed universe V may be its entire spatial volume ($=2\pi^2$) while for open models we have to restrict $\mathscr{V}$ to a compact subspace.

It can be verified that $\bar{\phi}(t)$ is given by

$$Q(t_2,t_1)\,\bar{S}(t)\,\bar{\phi}(t)=\phi_1\bar{S}(t_1)\,Q(t_2,t)+\phi_2\bar{S}(t_2)\,Q(t,t_1), \tag{32}$$

where

$$Q(x,y)=\int_y^x\frac{dt}{\bar{S}(t)}. \tag{33}$$

The propagator K is given by the formula

$$K[\phi_2,t_2;\phi_1,t_1]=\frac{3Vi}{8\pi^2}\frac{\bar{S}_1\bar{S}_2}{Q(t_1,t_2)}\exp\left[\frac{3Vi}{8\pi}\{A_{11}\phi_1^2-2A_{12}\phi_1\phi_2+A_{22}\phi_2^2\}\right], \tag{34}$$

where $\bar{S}_1=\bar{S}(t_1)$, $\bar{S}_2=\bar{S}(t_2)$ and

$$\begin{aligned}A_{11}&=\{Q(t_1,t_2)^{-1}-\dot{\bar{S}}(t_1)\}\bar{S}_1^2,\\ A_{22}&=\{Q(t_1,t_2)^{-1}+\dot{\bar{S}}(t_2)\}\bar{S}_2^2,\\ A_{12}&=Q(t_1,t_2)^{-1}\,\bar{S}_1\bar{S}_2.\end{aligned} \tag{35}$$

We now consider application of (18) to our problem. Let us suppose as in Section 2 that $0<t_2<t_1$ and that the state of the universe at t_1 is very nearly classical, i.e., given by a wavepacket peaked at $\phi_1=0$ and with a small dispersion Δ_1. Thus we take the state wavefunction at t_1 to be

$$\psi(\phi_1,t_1)=(2\pi\Delta_1^2)^{-1/4}\exp\left(-\frac{\phi_1^2}{4\Delta_1^2}\right). \tag{36}$$

We apply (18) to ask this question: What was the state wavefunction at t_2 from which the quantum universe would have evolved to (36)? Using (34) and (36) into (18) (which is now an ordinary integral), we find that $\psi(\phi_2, t_2)$ was also a wavepacket centred on $\phi_2 = 0$ but with a dispersion Δ_2 given by

$$\Delta_2 = \frac{2\pi Q(t_1, t_2)}{3V\bar{S}_1 \bar{S}_2 \Delta_1} \left[1 + \left\{ \frac{3V\Delta_1^2 \bar{S}_1^2}{2\pi} \left(Q(t_1 - t_2)^{-1} - \dot{\bar{S}}(t_1) \right) \right\}^2 \right]^{1/2}. \tag{37}$$

We will show in Section 4 that $Q(t_1, 0)$ is finite and hence as $t_2 \to 0$ (37) gives the result

$$\bar{S}_2 \Delta_2 \sim \text{constant} \qquad (= a, \text{ say}). \tag{38}$$

We thus recover the result stated in Section 2 that quantum uncertainty around the classical state diverges at the classical singular epoch.

It is now easy to estimate the probability for the singular class $\mathscr{C}_S$. We define the incomplete error function $E(x)$ by the integral

$$E(x) = \frac{1}{\sqrt{2\pi}} \int_{-x}^{x} \exp\left(-\frac{y^2}{2}\right) dy. \tag{39}$$

At $t = t_2$ we have by definition

$$\phi_2 = \Omega_2 - 1 = \frac{S_2 - \bar{S}_2}{\bar{S}_2}, \tag{40}$$

where $S_2 = S(t_2)$. From (38) and (40) we see that as $t_2 \to 0$ the unit Gaussian variable

$$x \equiv \frac{\phi_2}{\Delta_2} \sim \frac{(S_2 - \bar{S}_2)}{a}. \tag{41}$$

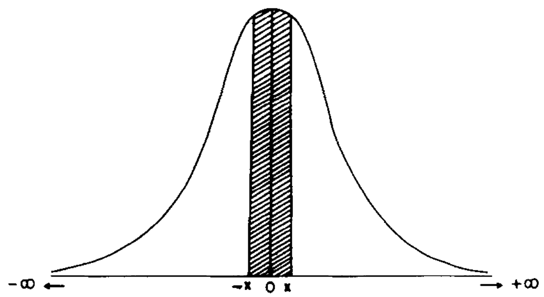

FIG. 1. The probability curve shows the singular models as confined to the shaded region which has vanishingly small area as $x \to 0$.

The solutions belonging to the singular class have $S_2 \to 0$ as $t_1 \to 0$. Since S_2 also tends to zero as $t_2 \to 0$, we have

$$\lim_{t_2 \to 0} x = 0. \tag{42}$$

The probability for such solutions being given by $E(x)$ therefore becomes varnishingly small as $t_2 \to 0$, as shown in Fig. 1. In other words, the probability that the universe could have evolved from a singular state is zero. The classical singularity therefore does not pose any threat to quantum cosmology.

4. Horizon-Free Cosmologies

Having established the improbability of a singular origin, we now show that particle horizons are also equally improbable in the regime of quantum conformal fluctuations. At first sight, the following argument might suggest that conformal transforms of the classical solution are not going to be free from horizons.

Recall from (4) that the existence of particle horizons is due to the finiteness of the integral

$$\bar{I}(t) = \int_0^t \frac{dt'}{\bar{S}(t')}. \tag{43}$$

Making a conformal transformation with $\Omega(t)$ and using (26) to go over to a new cosmic time τ, we get the corresponding integral as

$$I(\tau) = \int_{\tau_0}^{\tau} \frac{d\tau'}{S} = \int_0^t \frac{dt'}{\bar{S}(t')} = \bar{I}(t). \tag{44}$$

In (44) we have assumed that $\tau = \tau_0$ corresponds to $t = 0$. The horizon integral appears to be unaltered by a conformal transformation. This is not unexpected since a conformal transformation leaves light cones unchanged globally.

There is, however, a loophole in this argument. We would be justified in terminating the τ-integral at the lower limit τ_0 if the conformally transformed space-time were singular at τ_0. If, however, the singularity has been eliminated by the conformal transformation, i.e., if

$$\lim_{t \to 0+} \Omega(t)\, \bar{S}(t) = b > 0, \tag{45}$$

then the integral can very well be continued beyond τ_0, to $\tau < \tau_0$, and made divergent. A familiar example of this is the case (shown in Fig. 2) $k = 0$, $\Omega(t) = \bar{S}(t)^{-1}$ when the conformally transformed manifold is the flat Minkowski space-time. This space time has $\tau_0 = 0$ at $t = 0$; but since there is no singularity at $\tau_0 = 0$, we may continue the integral to $\tau < 0$.

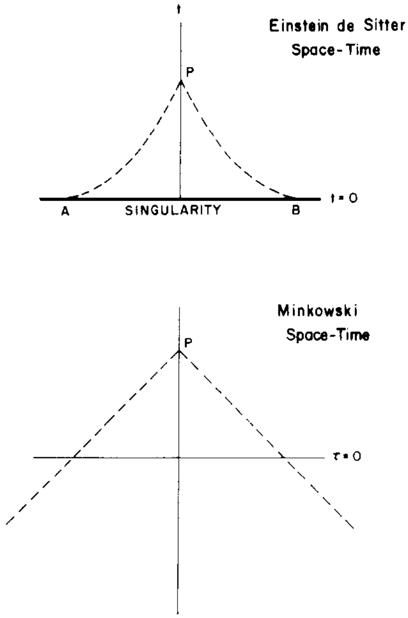

FIG. 2. The upper figure shows the space-time for the Einsein–de Sitter model ($k = 0$, $\bar{S} \propto t^{2/3}$). It has a singularity at $t = 0$. The past light cone of a typical point P intersects the singular hypersurface in a spherical region of diameter AB, which is the particle horizon. In the lower figure we see the Minkowski space-time obtained by a conformal transformation of the Einstein–de Sitter model. This has no singularity at $t = 0$ and no particle horizon. The space-time can be continued to the part $t < 0$. (The dotted curves show light tracks.)

Of course, for this loophole to exist in general we need τ_0 to be finite, i.e., $\tau_0 > -\infty$. It is only then that continuation beyond $\tau < \tau_0$ is possible. We now show that this condition is satisfied by all $\Omega(t)$ which make the conformally transformed manifold nonsingular.

First note that for such space-times (45) holds and hence close to $t = 0$,

$$d\tau = \Omega(t)\, dt \sim b \frac{dt}{\bar{S}(t)}. \tag{46}$$

Hence for a finite τ_0 we need the integral

$$Q(t_1, 0) = \int_0^{t_1} \frac{dt}{\bar{S}(t)} \tag{47}$$

to be finite for any constant $t_1 > 0$.

Of course this condition is none other than the finiteness of $I(t)$ defined in (43). However, it is instructive to show this explicitly using Eqs. (2) and (3), when applied to $\bar{S}(t)$.

From (2) and (3) it follows that

$$\frac{d}{d\bar{S}}(\bar{S}^3\varepsilon) = -3p\bar{S}^2 \leqslant 0, \tag{48}$$

for $p \geqslant 0$. Hence $\bar{S}^3\varepsilon$ is a non increasing function of $\bar{S}$. Thus for $\bar{S} < \bar{S}_1 = \bar{S}(t_1)$, $\bar{S}^3\varepsilon \geqslant \bar{S}_1^3\varepsilon_1$, where $\varepsilon(t_1) = \varepsilon_1$. Thus for Eq. (3) we get

$$\dot{\bar{S}}^2\bar{S}^2 = -k\bar{S}^2 + \frac{8\pi}{3}\varepsilon\bar{S}^4 \geqslant -k\bar{S}^2 + \frac{8\pi}{3}\varepsilon_1\bar{S}_1^3\bar{S}.$$

For t_1 sufficiently close to 0, the second term on the right side of the inequality dominates, so that for a suitable constant λ

$$\dot{\bar{S}}\bar{S} > \lambda\bar{S}^{1/2}. \tag{49}$$

Therefore

$$Q(t_1, 0) = \int_0^{\bar{S}_1} \frac{d\bar{S}}{\dot{\bar{S}}\bar{S}} < \lambda^{-1}\int_0^{\bar{S}_1} \frac{d\bar{S}}{\bar{S}^{1/2}} < \infty, \tag{50}$$

which proves the result.

Hence particle horizons can be eliminated for all those conformally transformed models which are nonsingular at $t = 0$.

This result taken together with our earlier result of Section 3 shows that it is highly unlikely that the universe had particle horizons sufficiently close to the classical singular epoch.

5. Resolution of the Flatness Problem

The picture that the early universe was dominated by conformal fluctuations leads to an interesting explanation of the flatness problem. We will proceed from the assumption that the empty Minkowski space-time is unstable to quantum fluctuations. An explicit demonstration of this instability has been given by one of us [27]. Others, from different standpoints, have advocated similar ideas [28–30]. Below we show how to compute the probability of arriving at different Robertson–Walker universes via quantum fluctuations. Our results will show that the $k = 0$ model is definitely preferred.

We wish to point out one major difference between the present approach and that of the inflationary scenarios. Inflationary scenarios try to offer a nonquantum gravitational solution to the flatness problem. They attempt to show that whatever may be the parameters at around 10^{17} GeV, the universe will have ε of the order of ε_c after the inflation. In our approach we begin at the still earlier epoch of Planck time when the quantum gravitational fluctuations dominate. We take the point of view that the universe "came into being" as a result of the instabilities of the flat space-time. Thus starting from "nothing"—flat space-time, vacuum—one generates the universe by a random fluctuation. In such a case even though all three types of models ($k = 1$, 0, -1) could arise through conformal fluctuations, the $k = 0$ model happens to be most probable. We shall indicate the calculation below.

First note that the line element (1) is conformally flat. Infeld and Schild [31] have given explicit transformations which express (1) in the form

$$ds^2 = \Omega_0^2 [dt^2 - dx^2 - dy^2 - dz^2]. \tag{51}$$

The function Ω_0 depends on all four coordinates $\mathbf{r} \equiv (x, y, z)$ and t for $k = \pm 1$ while for $k = 0$, Ω_0 depends on t only.

Since we are considering transitions from the empty Minkowski space-time we may use (16) with that background metric $\bar{g}_{ik}$ and with $J_m = 0$. We then get

$$K[\Omega_2, \Sigma_2; \Omega_1, \Sigma_1] = \int \exp\left\{ -\frac{3}{8\pi} \int_{\gamma} \Omega_i \Omega^i \, d^4x \right\} \mathscr{D}\Omega, \tag{52}$$

where $\Omega_i \equiv \Omega_{,i}$. Our initial state is specified by a wavefunctional $\Psi_I[\Omega_1]$ sharply peaked at $\Omega = 1$, and the final state by $\Psi_F[\Omega_2]$ strongly peaked at Ω_0. The transition amplitude is given by

$$\langle F | I \rangle = \iint \Psi_F^*[\Omega_2] \, K[\Omega_2, t_2; \Omega_1, t_1] \, \Psi_I[\Omega_1] \, \mathscr{D}\Omega_2 \mathscr{D}\Omega_1. \tag{53}$$

Note that in the present approach wherein we are using the "Schrodinger-representation" for field theory this Hilbert space product is well defined and has the standard interpretation of transition amplitude.

It is convenient to use Fourier transforms to compute the amplitude. The method of computation is shown in the Appendix. We quote the formula (A–10) obtained there for the probability of transition from a state with wavefunctional peaked at Ω_I to a state with wavefunctional peaked at Ω_F:

$$|\langle F | I \rangle|^2 = N \exp\left\{ -\frac{3}{8\pi} W \right\}, \tag{54}$$

where N is a constant and

$$W = \iint \left\{ \frac{\nabla\Omega_I(\mathbf{r}_1) \cdot \nabla\Omega_I(\mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|^2} + \frac{\nabla\Omega_F(\mathbf{r}_1) \cdot \nabla\Omega_F(\mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|^2} - \frac{2\nabla_I(\mathbf{r}_1) \cdot \nabla\Omega_F(\mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|^2 - (t_1 - t_2)^2} \right\} d^3\mathbf{r}_1 \, d^3\mathbf{r}_2. \tag{55}$$

For $\Omega_I \equiv 1$ and $\Omega_F \equiv \Omega_0$ we get

$$W = \iint \frac{\Omega_0(\mathbf{r}_1) \cdot \nabla\Omega_0(\mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|^2} \, d^3\,\mathbf{r}_1 \, d^3\,\mathbf{r}_2. \tag{56}$$

It is clear from (54) and (56) that the probability is a maximum when W is least. The least value of W is obtained when $\nabla\Omega_0 = \mathbf{0}$, i.e., when Ω_0 depends on t only. From our earlier discussion this corresponds to the $k = 0$ case.

A qualitative interpretation of the above result is the following: In the Einstein action, written in a $(3 + 1)$ form, the curvature of the spatial hypersurfaces occurs only as a potential term. Thus direct quantum transitions between three geometries with different three curvatures is (crudely speaking) forbidden, leading to a "conservation of three curvature." This argument, of course, is incorrect in a general space-time because 3-curvature is not a single dynamical degree of freedom. But in the Robertson–Walker space-times, 3-curvature is directly related to the expansion factor and contains the same amount of information. Thus starting from flat space where spacelike hypersurfaces have zero curvature, one is most likely to end up with a spatially flat Robertson–Walker model.

6. Conclusion

We have shown that quantum conformal fluctuations have a significant role to play in understanding the fundamental problems in cosmology. The space-time singularity and particle horizons of standard classical cosmology turn out to be exceptions rather than the rule in the wider framework of quantum cosmology and the near flatness ($k = 0$) of the universe is seen as the most likely outcome of its quantum origin.

For reasons given in Section 2 we have limited ourselves to quantizing the conformal degrees of freedom only. This limitation is compensated for by the fact that exact calculations of various quantum effects become possible. We are therefore confident that a future more complete theory of quantum gravity (if it ever comes about in a *workable* form) will confirm the conclusions of our limited approach.

Appendix

We indicate here briefly the derivation of the Eqs. (54) and (55) in the text.

Since the action in Eq. (52) contains $\Omega(\mathbf{x}, t)$ only in the combination $\Omega_i \Omega^i$, it is convenient to use the Fourier decomposition,

$$\Omega(\mathbf{x}, t) = \int \frac{d^3\mathbf{K}}{(2\pi)^3} \, q_{\mathbf{K}}(t) \exp i\,(\mathbf{K} \cdot \mathbf{x}). \tag{A.1}$$

The action

$$J = -\frac{3}{8\pi}\int \Omega_i \Omega^i d^4x \tag{A.2}$$

now becomes

$$J = -\frac{3}{8\pi}\int \frac{d^3\mathbf{K}}{(2\pi)^3}\int_{t_1}^{t_2} dt\, [|\dot{q}_\mathbf{K}|^2 - |\mathbf{K}|^2\, |q_\mathbf{K}|^2], \tag{A.3}$$

which is the action functional for an infinite set of independent harmonic oscillators labelled by $\mathbf{K}$. We are interested in the transition probability between two states Ψ_I and Ψ_F, peaked at $\Omega_I(\mathbf{x})$ and $\Omega_F(\mathbf{x})$, respectively (see Eq. (53)). Let the functions $\Omega_I(\mathbf{x})$ and $\Omega_F(\mathbf{x})$ have the Fourier decompositions,

$$\Omega_I(\mathbf{x}) = \int \frac{d^3\mathbf{K}}{(2\pi)^3} A_\mathbf{K} \exp i\, (\mathbf{K}\cdot\mathbf{x}), \tag{A.4}$$

$$\Omega_F(\mathbf{x}) = \int \frac{d^3\mathbf{K}}{(2\pi)^3} B_\mathbf{K} \exp i\, (\mathbf{K}\cdot\mathbf{x}). \tag{A.5}$$

The wavefunctional Ψ_I peaked at $\Omega_I(\mathbf{x})$ is equivalent, in the Fourier space, to a wave functional Φ_I peaked at $A_\mathbf{K}$. We shall assume that each of the harmonic oscillators is in a minimum uncertainty wavepacket state. Thus the initial state, in Fourier space is,

$$\Phi_I[\{q_\mathbf{K}\}] = N \exp\left\{-\frac{3}{8\pi}\int \frac{d^3\mathbf{K}}{(2\pi)^3} |q_\mathbf{K} - A_\mathbf{K}|^2\right\}. \tag{A.6}$$

The transition amplitude in Eq. (53) can now be considered to be a product of transition amplitudes $T_\mathbf{K}$ for each harmonic oscillator. (That is, $T_\mathbf{K}$ gives the probability amplitude for the $\mathbf{K}$th harmonic oscillator to make the transition from $A_\mathbf{K}$ to $B_\mathbf{K}$ under the influence of the harmonic oscillator kernel). Using this expression for $T_\mathbf{K}$, we get the total transition amplitude as,

$$\langle F|I\rangle = N \exp\left\{-\int \frac{d^3\mathbf{K}}{(2\pi)^3} U_\mathbf{K}(B_\mathbf{K}; A_\mathbf{K})\right\}, \tag{A.7}$$

where

$$U_\mathbf{K} = \frac{3\,|\mathbf{K}|}{16\pi}\{|A_\mathbf{K}|^2 + |B_\mathbf{K}|^2 - (A_\mathbf{K} B_\mathbf{K}^* + A_\mathbf{K}^* B_\mathbf{K})\, e^{-i|\mathbf{K}|(t_2 - t_1)}\}. \tag{A.8}$$

Note that the real part of $U_\mathbf{K}$ (which will appear in $|\langle F|I\rangle|^2$) is positive definite. Equations (A.7) and (A.8) give the necessary transition amplitude in terms of the Fourier transformed variables $A_\mathbf{K}$ and $B_\mathbf{K}$. Using Eqs. (A.4) and (A.5) we can express

A_K and B_K in terms of $\Omega_I(\mathbf{x})$ and $\Omega_F(\mathbf{x})$. Substituting for A_K and B_K in Eq. (A.8) in terms of $\Omega_I(\mathbf{x})$ and $\Omega_F(\mathbf{x})$ leads to the following final expression, quoted in the text:

$$|\langle F|I\rangle|^2 = N \exp\left\{-\frac{3}{8\pi} W\right\}, \tag{A.9}$$

$$W = \iint \left\{ \frac{\nabla\Omega_I(\mathbf{r}_1)\cdot\nabla\Omega_I(\mathbf{r}_2)}{|\mathbf{r}_1-\mathbf{r}_2|^2} + \frac{\nabla\Omega_F(\mathbf{r}_1)\cdot\nabla\Omega_F(\mathbf{r}_2)}{|\mathbf{r}_1-\mathbf{r}_2|^2} - \frac{2\nabla\Omega_I(\mathbf{r}_1)\cdot\nabla\Omega_F(\mathbf{r}_2)}{|\mathbf{r}_1-\mathbf{r}_2|^2-(t_2-t_1)^2} \right\} d^3\mathbf{r}_1\, d^3\mathbf{r}_2. \tag{A.10}$$

All through the discussion we have not bothered to keep track of the normalization constants (denoted by N). This is permissible because we are only interested in the relative probabilities and not in the exact numerical value of the transition probability.

References

1. A. Friedmann, *Z. Phys.* **10** (1922), 377.
2. H. P. Robertson, *Astrophys. J.* **82** (1935), 248.
3. A. G. Walker, *Proc. London Math. Soc.* **42** (1936), 90.
4. A. K. Raychaudhuri, *Phy. Rev.* **98** (1955), 1123.
5. O. Heckmann and E. Schücking, *Z. Astrophys.* **38** (1955), 95.
6. S. W. Hawking and G. F. R. Ellis "The Large Scale Structure of Spacetime," Cambridge Univ. Press, Cambridge, 1973.
7. S. P. Boughn, D. M. Fram, and R. B. Partridge, *Astrophys. J.* **165** (1971), 439.
8. C. W. Misner, *Phy. Rev. Lett.* **22** (1969), 1071.
9. D. M. Chitre, Doctoral dissertation, University of Maryland, 1972.
10. R. H. Dicke and P. J. E. Peebles, *in* "General Relativity—An Einstein Centenary Survey" (S. W. Hawking and W. Israel, Eds.), pp. 504–517, Cambridge Univ. Press, Cambridge, 1979.
11. A. H. Guth, *Phy. Rev. D* **23** (1981), 347.
12. J. D. Barrow and M. S. Turner, *Nature* (London) **292** (1981), 337.
13. J. D. Barrow and M. S. Turner, *Nature* (London) **298** (1982), 801.
14. R. Penrose and M. A. H. MacCallum, *Phy. Rep. C* **6** (1973), 242.
15. B. S. DeWitt, *in* "General Relativity—An Einstein Centenary Survey" (S. W. Hawking and W. Israel, Eds.), pp. 680–745, Cambridge Univ. Press, Cambridge, 1979.
16. R. P. Feynman, *Acta Phys. Polon.* **24** (1963), 697.
17. J. V. Narlikar, *Gen. Rel. Grav.* **10** (1979), 883.
18. T. Padmanabhan and J. V. Narlikar, *Nature (London)* **295** (1982), 677.
19. T. Padmanabhan, *Phy. Rev. D* **26** (1982), 2162.
20. J. V. Narlikar, *Found. Phy.* **11** (1981), 473.
21. J. A. Wheeler, *in* "Battelle Rencontre" (C. M. DeWitt and J. A. Wheeler, Eds.), pp. 325–356 Benjamin, New York, 1968.
22. J. Isenberg and J. A. Wheeler, *in* "Relativity, Quanta and Cosmology" (M. Pantaleo and F. de Finis, Eds.), pp. 267–292, Johnson, New York, 1979.
23. R. P. Feynman and A. R. Hibbs "Quantum Mechanics and Path Integrals," McGraw-Hill, New York, 1965.

24. S. WEINBERG, *Phy. Scripta* **21** (1980), 773.
25. C. DEWITT, *Ann. Phys.* **97** (1976), 307.
26. H. BONDI, "Cosmology," Cambridge Univ. Press, Cambridge, 1961).
27. T. PADMANABHAN, "Quantum conformal fluctuations and stationary states," *Int. J. Theor. Phys.*, in press.
28. D. LINDLEY, *Nature (London)* **291** (1981), 392.
29. D. ATKATZ AND H. PAGELS, *Phy. Rev. D* **25** (1982), 2065.
30. R. BRONT, F. ENGLERT, J. M. FRERE, E. GUNZIG, P. NARADONE, AND C. TRUFFIN, *Nuc. Phys. B* **170** (1980), 228.
31. L. INFELD AND A. SCHILD, *Phys. Rev.* **68** (1945), 250.
32. G. W. GIBBONS, S. W. HAWKING, AND M. J. PERRY, *Nuc. Phy. B* **138** (1978), 14.
33. B. S. DEWITT, *Phy. Rev.* **160** (1967), 1113.
34. T. PADMANABHAN, *Gen. Rel. Grav.* **14** (1982), 549.
35. S. W. HAWKING AND G. W. GIBBONS, *Phys. Rev. D* **15** (1977), 2752.