

## Angular momentum transport in quasi-Keplerian accretion disks

Prasad Subramanian,<sup>1</sup> \* B. S. Pujari,<sup>2</sup> †, Peter A. Becker,<sup>3</sup> ‡

<sup>1</sup>*Inter-University Center for Astronomy and Astrophysics, P.O Bag 4, Ganeshkhind, Pune - 411 007, India*

<sup>2</sup>*Dept. of Physics, University of Pune, Pune - 411 007, India*

<sup>3</sup>*School of Computational Sciences, George Mason University, Fairfax, VA 22030, USA*

2010 August 20

**Abstract.** We reexamine arguments advanced by Hayashi & Matsuda (2001), who claim that several simple, physically motivated derivations based on mean free path theory for calculating the viscous torque in a quasi-Keplerian accretion disk yield results that are inconsistent with the generally accepted model. If correct, the ideas proposed by Hayashi & Matsuda would radically alter our understanding of the nature of the angular momentum transport in the disk, which is a central feature of accretion disk theory. However, in this paper we point out several fallacies in their arguments and show that there indeed exists a simple derivation based on mean free path theory that yields an expression for the viscous torque that is proportional to the radial derivative of the angular velocity in the accretion disk, as expected. The derivation is based on the analysis of the epicyclic motion of gas parcels in adjacent eddies in the disk.

*Key words:* Accretion - viscosity, disks, angular momentum

### 1. Introduction

Accretion disks around a central compact object are ubiquitous in astrophysics. In such systems, the plasma accreting onto the central object typically takes the form of a swirling disk. The disk-like nature of accretion is due to the angular momentum originally carried by the accreting material, and also to the spin of the central object, which provides a preferred axis of rotation. At a given radius in a quasi-Keplerian accretion disk, the average motion of the gas parcels is an approximately

---

\*e-mail: psubrama@iucaa.ernet.in

†e-mail: bspujari@rediffmail.com

‡e-mail: pbecker@gmu.edu

circular Keplerian orbits, combined with a radial sinking motion as the parcel gradually moves deeper into the potential well of the central mass. The radial velocity that is far smaller than the Keplerian (azimuthal) velocity. A detailed description of quasi-Keplerian accretion disks can be found in Pringle (1981), among several other references.

The specific angular momentum (angular momentum per unit mass) carried by a parcel of gas in a quasi-Keplerian accretion disk is very close to the Keplerian value  $\sqrt{GM/R}$ , where  $R$  is the radius measured from the central object with mass  $M$ . It follows that a parcel of plasma has to lose angular momentum in order to sink from a larger orbit into a smaller one and eventually cross the event horizon. In the absence of strong winds or jets, the angular momentum must therefore flow *outwards* in such an accretion disk in order to enable accretion to proceed. In disks of turbulent fluid, the flow of angular momentum is due to the exchange of plasma parcels in neighboring annuli with different angular velocities. This fluid picture may describe certain astrophysical cases such as the rings of Saturn, but in many other cases of astrophysical interest, the coupling between adjacent annuli is provided not by “parcel interchange,” but rather by the magnetic field. In either the fluid or magnetohydrodynamical (MHD) scenarios, the angular momentum flow is expressed as a torque between neighboring annuli. Our focus here is on the fluid picture, and on the validity of the various attempts to analyze quantitatively the angular momentum transport associated with the interchange of parcels.

There are several previous papers in which derivations of the viscous torque in fluid dynamical situations are developed based on simple, physically motivated arguments (e.g., Frank, King, & Raine 1985, 1992, 2002; Hartmann 1998). However, some of these do not yield an expression for the viscous torque that is proportional to the radial derivative of the angular velocity. This is a problem because the viscosity is fundamentally due to the “rubbing” of matter in adjacent radial annuli in the disk, and consequently the viscosity should *vanish* in the case of solid body rotation with  $\Omega(R) = \text{constant}$ , where  $\Omega(R)$  is the angular velocity in the disk at radius  $R$ . Hayashi & Matsuda (2001; hereafter HM) recognized this point, and attempted to clear up some of the confusion by carefully examining the previously published derivations of the viscosity that were based on the mean free path approach. They concluded that the mean free path approach inevitably leads to an inward rather than outward flow of angular momentum in the disk, which is unphysical. However, we argue that the reasoning of HM was flawed because they did not consider the epicyclic nature of the parcel trajectories. We present a simple derivation based on mean free path theory, combined with the actual epicyclic motion of the gas parcels, that in fact yields a physically reasonable expression for the viscous torque between neighboring annuli in a quasi-Keplerian accretion disk.

We discuss the standard fluid dynamical formulation of viscous torques in an accretion disk in § 2. We then discuss the problems with the various previous derivations that attempted to utilize a mean free path approach to compute the viscous torque in § 3. In § 4 we present a simple physical derivation based on analysis of the epicyclic (ballistic) motion of gas parcels in adjacent eddies, and we demon-

strate that this approach yields the expected form for the torque in terms of the gradient of the angular velocity. We present our final conclusions in § 5.

## 2. Standard fluid dynamics treatment of viscous torque

The equation of motion for a viscous, incompressible fluid can be written as (Landau & Lifshitz 1987)

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{v} = \frac{\vec{F}}{\rho}, \quad (1)$$

where the force density  $\vec{F}$  is defined via the viscous stress tensor  $\sigma_{ik}$  as follows:

$$F_i = \frac{\partial \sigma_{ik}}{\partial x_k},$$

$$\sigma_{ik} = -p \delta_{ik} + \eta \left( \frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} \right). \quad (2)$$

The fluid velocity is denoted by  $\vec{v}$  and its density and pressure by  $\rho$  and  $p$ , respectively. The coefficient of dynamic viscosity ( $\text{g cm}^{-1} \text{s}^{-1}$  in cgs units) is denoted by  $\eta$  and we have used the Einstein summation convention in equation (2). For the special case of a thin, azimuthally symmetric accretion disk, the only non-negligible component of the viscous stress tensor in cylindrical coordinates  $(R, \phi, z)$  is

$$\sigma_{R\phi} = -\eta R \frac{\partial \Omega}{\partial R}. \quad (3)$$

The viscous stress (force per unit area) is thus directly proportional to the radial derivative of the angular velocity. This is by far the cleanest and most rigorous way to derive the azimuthal equation of motion for a quasi-Keplerian accretion disk (see, for example, chapter 1 of Subramanian 1997). It is implicitly assumed that the dynamic viscosity  $\eta$  arises out of local effects; i.e., due to momentum exchange between neighboring annuli of the accretion disk, as with molecular viscosity. It is well known (Shakura & Sunyaev 1973; Pringle 1981) that molecular viscosity is far too small to account for angular momentum transport in accretion disks around active galactic nuclei. Identifying suitable candidates for the microphysical viscosity mechanism operative in such disks is a subject of intensive research.

## 3. Simplified treatments of viscous torque in accretion disks

Pringle (1981) derived the azimuthal component of the equation of motion in accretion disks starting from first principles in a simple, physically motivated manner. The equation reads as follows:

$$\frac{\partial (\Sigma R^3 \Omega)}{\partial t} + \frac{\partial (\Sigma v_R R^3 \Omega)}{\partial R} = -\frac{1}{2\pi} \frac{\partial \mathcal{G}}{\partial R}, \quad (4)$$

where  $\Sigma$  is the surface density of plasma in the disk,  $v_R < 0$  is the radial accretion velocity, and  $\mathcal{G} > 0$  is the torque exerted by the material inside radius  $R$  on the material outside that radius. Throughout the remainder of the paper, we shall assume that the disk has a steady-state (time-independent) structure, although this is not essential for our results. In the standard approach introduced by Shakura & Sunyaev (1973) and adopted by Pringle (1981), the torque  $\mathcal{G}$  is related to  $\Omega$  via

$$\mathcal{G} = 4\pi R^2 H \sigma_{R\phi} = -2\pi R^3 \Sigma \nu \frac{d\Omega}{dR}, \quad (5)$$

where  $H(R)$  is the half-thickness of the disk at radius  $R$  and  $\nu = \eta/\rho$  is the kinematic viscosity coefficient. The stress and torque are therefore proportional to  $d\Omega/dR$ , in agreement with equation (3). This prescription has been applied in many disk structure calculations. In particular, it has been shown recently (Becker & Le 2003) that fully relativistic and self-consistent models for hot, advection-dominated accretion disks can be constructed by applying the Shakura-Sunyaev viscosity prescription throughout the entire disk, including the region close to the event horizon.

A number of authors have attempted to confirm the general form of equation (5) by using simple physical arguments. However, several of these derivations have errors in them, and they do not always result in an expression for  $\mathcal{G}$  that is proportional to  $d\Omega/dR$ , as pointed out by HM. We briefly review the relevant derivations below, and we also point out errors in the approach adopted by HM. We then present a new, heuristic derivation of equation (5) that is based on a careful analysis of the ballistic motion of two parcels as they exchange radii. This derivation leads to the expected conclusion that  $\mathcal{G} \propto d\Omega/dR$ .

Although the argument given by HM is rather indirect, their main point can be understood through a simple examination of the angular momentum transport resulting from the interchange of fluid elements in a disk that is rotating as a solid body, i.e., with  $\Omega(R) = \Omega_0 = \text{constant}$ . In this case, the angular momentum per unit mass, denoted by  $J \equiv R^2 \Omega(R)$ , is given by  $J = R^2 \Omega_0$ , and this quantity increases rather strongly as a function of the radius  $R$ . Hence if two parcels of fluid on opposite sides of radius  $R$  were exchanged due to some turbulent or convective process, then clearly angular momentum would be transported in the *inward* direction, since the blob that was originally outside the annulus will have more angular momentum than the interior blob. However, this result is unphysical, because in the case of solid body rotation, there is no “rubbing” between adjacent fluid annuli, and therefore there should be no torque and no angular momentum transport. Any successful microphysical model for the angular momentum transport in the disk based on mean free path theory must somehow resolve this apparent paradox. In the following sections, we provide a detailed consideration of the reasoning employed in the previous published derivations, including that of HM, and we conclude that when properly carried out, the mean free path approach can yield a result for the viscous torque that correctly vanishes in the case of solid body rotation.

### 3.1 Derivation of $\mathcal{G}$ by Hartmann (1998)

The specific angular momentum  $J_{\text{in}}$  carried by material originating at a radius  $R - \lambda/2$  in a quasi-Keplerian accretion disk is given by Hartmann (1998) as  $J_{\text{in}} = (R - \lambda/2) \Omega(R - \lambda/2)$ , where  $\lambda$  is the mean free path over which parcels of plasma exchange angular momentum. As pointed out by HM, this expression is incorrect, and the correct expression should read as follows:

$$J_{\text{in}} = \left(R - \frac{\lambda}{2}\right)^2 \Omega\left(R - \frac{\lambda}{2}\right). \quad (6)$$

One can write an analogous expression for  $J_{\text{out}}$ , the specific angular momentum carried by material originating at a radius  $R + \lambda/2$ , by reversing the sign of  $\lambda$ . As shown by HM, if one expands  $\Omega(R - \lambda/2)$  to first order in  $\lambda$  as  $\Omega(R - \lambda/2) \sim \Omega(R) - (\lambda/2)(d\Omega/dR)$ , the result obtained for the difference between the specific angular momenta is

$$J_{\text{in}} - J_{\text{out}} = -\lambda \frac{d}{dR} (R^2 \Omega). \quad (7)$$

The characteristic time for the interchange of the matter between the two radii is  $\Delta t = \lambda/w$ , where  $w$  is the turbulent velocity of the fluid parcels. The total mass of fluid involved in the interchange is  $\Delta M = 2\pi R \lambda \Sigma$ , and it follows that the net rate of flow of angular momentum from the inner ring at  $R - \lambda/2$  towards the outer ring at  $R + \lambda/2$  (or equivalently, the viscous torque exerted by the ring at  $R - \lambda/2$  on the ring at  $R + \lambda/2$ ) is

$$\mathcal{G} = \frac{\Delta M}{\Delta t} (J_{\text{in}} - J_{\text{out}}) = 2\pi R \Sigma w (J_{\text{in}} - J_{\text{out}}), \quad (8)$$

or

$$\mathcal{G} = -2\pi R \Sigma \beta \nu \frac{d}{dR} (R^2 \Omega), \quad (9)$$

where we have set  $\nu = w\lambda/\beta$ , with  $\beta$  denoting a constant of order unity. The viscous torque in a quasi-Keplerian accretion disk should tend to smooth out gradients in the angular velocity so as to attain solid body rotation ( $d\Omega/dR = 0$ ). However, the expression for the viscous torque in equation (9) is such that it tends to attain a flow with *constant angular momentum*, i.e., the ‘‘equilibrium’’ condition is  $d(R^2 \Omega)/dR = 0$ . As HM point out, this expression is therefore unphysical.

### 3.2 Derivation of $\mathcal{G}$ by Frank, King, & Raine (1992)

We next turn our attention to another derivation of the viscous torque given by Frank, King, & Raine (1992). At the heart of their derivation is the claim that the linear velocity of material at radius  $R - \lambda/2$  as seen by an observer situated at radius  $R$  is given by

$$v_{\text{rel}}\left(R - \frac{\lambda}{2}\right) = \left(R - \frac{\lambda}{2}\right) \Omega\left(R - \frac{\lambda}{2}\right) + \Omega(R) \frac{\lambda}{2}. \quad (10)$$

Based on this, one can write an expression for  $L_{\text{in}}$ , the rate at which angular momentum crosses an annulus at radius  $R$  in the direction of increasing  $R$ , as

$$\begin{aligned} L_{\text{in}} &= 2\pi R \Sigma w \left( R - \frac{\lambda}{2} \right) v_{\text{rel}} \left( R - \frac{\lambda}{2} \right) \\ &= 2\pi R \Sigma w \left( R - \frac{\lambda}{2} \right) \left[ \left( R - \frac{\lambda}{2} \right) \Omega \left( R - \frac{\lambda}{2} \right) + \Omega(R) \frac{\lambda}{2} \right] \\ &\simeq 2\pi R \Sigma w \left( R - \frac{\lambda}{2} \right) \left[ R \Omega(R) - R \frac{\lambda}{2} \frac{d\Omega}{dR} \right], \end{aligned} \quad (11)$$

where we have expanded  $\Omega(R - \lambda/2)$  to first order in  $\lambda$  to arrive at the final expression. We can write an analogous expression for  $L_{\text{out}}$ , the rate at which angular momentum crosses radius  $R$  in the inward direction, by reversing the sign of  $\lambda$ . To first order in  $\lambda$ , the torque exerted on the plasma outside radius  $R$  by the plasma inside that radius is then

$$\mathcal{G} = L_{\text{in}} - L_{\text{out}} = -2\pi R^2 \beta \nu \Sigma \frac{d}{dR} (R \Omega). \quad (12)$$

Since the right-hand side is proportional to  $d(R\Omega)/dR$ , in this case the torque will lead to a uniform *linear velocity*, and not to a uniform *angular velocity* as we require on physical grounds. As HM point out, this expression for  $\mathcal{G}$  is therefore also incorrect.

### 3.3 Correction proposed by HM

HM claim that this is because Frank, King, & Raine (1992) have used an incorrect expression for  $v_{\text{rel}}$  (i.e., eq. [10]). They assert that the linear velocity of the plasma at  $(R - \lambda/2)$  as viewed by an observer at radius  $R$  should instead be given by

$$v_{\text{rel}} \left( R - \frac{\lambda}{2} \right) = \left( R - \frac{\lambda}{2} \right) \Omega \left( R - \frac{\lambda}{2} \right) - R \Omega(R) + \Omega(R) \frac{\lambda}{2}. \quad (13)$$

By employing this expression for  $v_{\text{rel}}$  and following the same procedure used to obtain equation (12), they find that to first order in  $\lambda$ ,

$$\mathcal{G} = L_{\text{in}} - L_{\text{out}} = -2\pi R^3 \beta \nu \Sigma \frac{d\Omega}{dR}. \quad (14)$$

This expression for  $\mathcal{G}$  does indeed have the correct dependence on  $d\Omega/dR$ , but nonetheless we claim that equation (13) for the relative velocity used by HM is incorrect. The correct expressions for the relative velocities are in fact (see, e.g., Mihalas & Binney 1981)

$$\begin{aligned} v_{\text{rel}} \left( R - \frac{\lambda}{2} \right) &= \left( R - \frac{\lambda}{2} \right) \Omega \left( R - \frac{\lambda}{2} \right) - R \Omega(R) \\ v_{\text{rel}} \left( R + \frac{\lambda}{2} \right) &= \left( R + \frac{\lambda}{2} \right) \Omega \left( R + \frac{\lambda}{2} \right) - R \Omega(R), \end{aligned} \quad (15)$$

where  $v_{\text{rel}}(R - \lambda/2)$  denotes the velocity of a plasma parcel at  $R - \lambda/2$  as seen by an observer at  $R$ , and  $v_{\text{rel}}(R + \lambda/2)$  denotes the velocity of a plasma parcel at  $R + \lambda/2$  as seen by an observer at  $R$ . In a quasi-Keplerian accretion disk,  $\Omega \propto R^{-3/2}$ , and therefore the first velocity is positive and the second is negative. These expressions for the relative velocities assume that the plasma parcels at  $R$ ,  $R - \lambda/2$ , and  $R + \lambda/2$  all lie on the same radial line through the central object. If this is not true, then the expressions will contain additional terms, as Mihalas & Binney (1981) show in the context of the relative velocity between the sun and stars in our galaxy. Since we are dealing with material transport over lengths of the order of  $\lambda$  that are very small in comparison with  $R$ , this assumption is quite valid in our accretion disk application. Using equation (15) for the relative velocities, we now obtain

$$\begin{aligned} L_{\text{in}} &= 2\pi R \Sigma w \left(R - \frac{\lambda}{2}\right) \left[ \left(R - \frac{\lambda}{2}\right) \Omega \left(R - \frac{\lambda}{2}\right) - R \Omega(R) \right] \\ &\simeq -2\pi R^2 \Sigma w \frac{\lambda}{2} \left[ R \frac{d\Omega}{dR} + \Omega(R) \right] \\ L_{\text{out}} &= 2\pi R \Sigma w \left(R + \frac{\lambda}{2}\right) \left[ \left(R + \frac{\lambda}{2}\right) \Omega \left(R + \frac{\lambda}{2}\right) - R \Omega(R) \right] \\ &\simeq 2\pi R^2 \Sigma w \frac{\lambda}{2} \left[ R \frac{d\Omega}{dR} + \Omega(R) \right], \end{aligned} \quad (16)$$

where the final expressions for  $L_{\text{in}}$  and  $L_{\text{out}}$  are correct to first order in  $\lambda$ . We thus obtain for the viscous torque

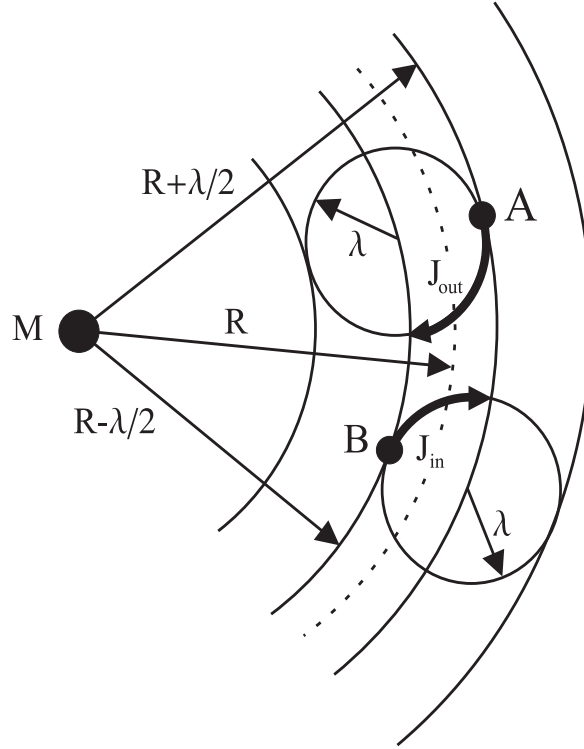
$$\mathcal{G} = L_{\text{in}} - L_{\text{out}} = -2\pi R^2 \Sigma \beta \nu \frac{d}{dR} (R \Omega). \quad (17)$$

This result is identical to equation (12), and therefore it too is incorrect.

#### 4. Derivation based on epicyclic parcel motion

We have demonstrated in § 3 that due to various errors, the previous derivations of the viscous torque based on mean free path theory do not yield results consistent with the classical theory of viscous transport. We shall now focus on a heuristic analysis of the angular momentum transport that combines the mean free path approach with a proper treatment of the epicyclic motion of parcels of gas in adjacent eddies in the accretion disk. The physical picture is presented in Fig. 1. Let us suppose that  $\lambda$  represents the mean distance a parcel travels freely before having its motion disrupted by interaction with the surrounding material in the disk. The disruption in this case involves hydrodynamical interaction, and therefore  $\lambda$  is the damping length for the turbulence as well as the mean free path for the parcel motion. It follows that the turbulence in the disk can be characterized by whirls with a mean radius  $\lambda$ .

Next we focus on the interchange of two gas parcels of unit mass, A and B,



**Figure 1.** Angular momentum transport in the disk involves the interchange of two parcels, A and B, initially located at radii  $R + \lambda/2$  and  $R - \lambda/2$ , respectively, around central mass  $M$ . They each participate in ballistic, epicyclic motion in eddies of radius  $\lambda$ , with A moving inward and B moving outward. See the discussion in the text.

which are initially located at radii  $R + \lambda/2$  and  $R - \lambda/2$ , respectively (see Fig. 1). The two parcels each “ride” turbulent eddies with radius  $\lambda$ , and they start out at the outer (parcel A) or inner (parcel B) edge of their respective eddies. Hence each begins its motion with zero radial velocity, at a turning point in its orbit. Parcel A moves in the inward direction, and therefore its angular momentum is sub-Keplerian compared with the average disk at radius  $R + \lambda/2$ . Similarly, parcel B is super-Keplerian compared with the mean disk at radius  $R - \lambda/2$  and therefore it moves in the outward direction. The motion of the parcels is epicyclic as viewed from the reference frame of an observer who remains at the starting radius and travels along with the Keplerian angular velocity. Since  $\lambda$  is the mean damping length for the turbulence, each of the parcels will experience ballistic motion over a radial length scale comparable to  $\lambda$ , after which the ballistic motion (and the parcel itself) will be absorbed by the surrounding gas. It follows that, on average, parcel A will deposit its angular momentum ( $J_{out}$ ) at radius  $R - \lambda/2$ , and parcel B

will deposit its angular momentum ( $J_{\text{in}}$ ) at radius  $R + \lambda/2$ . Hence the two parcels exchange radii during the process. We shall proceed to compute the net angular momentum exchange,  $J_{\text{in}} - J_{\text{out}}$ , by considering the motion of these parcels in detail.

Gas parcel A participates in the motion of turbulent eddy A, which has its center located at radius  $R - \lambda/2$  and possesses turning points at radii  $R + \lambda/2$  and  $R - 3\lambda/2$ . Likewise, parcel B experiences the motion of eddy B, with its center located at radius  $R + \lambda/2$ , and with turning points a radii  $R + 3\lambda/2$  and  $R - \lambda/2$ . The specific energy  $E$  and the specific angular momentum  $J$  of each parcel is conserved during the ballistic phase of its motion, until it travels a mean distance  $\lambda$  and is damped by the surrounding material. For a particle moving ballistically in the Newtonian gravitational field of a central mass  $M$ , the quantities  $E$  and  $J$  are related by the classical energy equation

$$E = \frac{1}{2} v_R^2 + \frac{1}{2} \frac{J^2}{R^2} - \frac{GM}{R}, \quad (18)$$

where  $v_R$  is the radial component of the velocity. Turning points in the radial motion occur where  $v_R$  vanishes, so that we can write

$$E = \frac{1}{2} \frac{J^2}{R_1^2} - \frac{GM}{R_1} = \frac{1}{2} \frac{J^2}{R_2^2} - \frac{GM}{R_2}, \quad (19)$$

where  $R_1$  and  $R_2$  denote the two turning point radii. This equation can be easily solved for the angular momentum  $J$  as a function of  $R_1$  and  $R_2$ . The result obtained is

$$J = \left( \frac{2GM R_1 R_2}{R_1 + R_2} \right)^{1/2}. \quad (20)$$

We can use equation (20) to conclude that the angular momentum of parcel A is equal to

$$J_{\text{out}} = \sqrt{GMR} \left( 1 + \frac{1}{2} \frac{\lambda}{R} \right) \left( 1 - \frac{3}{2} \frac{\lambda}{R} \right)^{1/2} \left( 1 - \frac{1}{4} \frac{\lambda^2}{R^2} \right)^{-1/2}. \quad (21)$$

Likewise, the angular momentum of parcel B is given by

$$J_{\text{in}} = \sqrt{GMR} \left( 1 - \frac{1}{2} \frac{\lambda}{R} \right) \left( 1 + \frac{3}{2} \frac{\lambda}{R} \right)^{1/2} \left( 1 - \frac{1}{4} \frac{\lambda^2}{R^2} \right)^{-1/2}. \quad (22)$$

In the spirit of the mean free path approach, we are interested in computing the value of the net angular momentum transport,  $J_{\text{in}} - J_{\text{out}}$ , to first order in the small parameter  $\lambda/R$ . The corresponding results obtained for  $J_{\text{in}}$  and  $J_{\text{out}}$  are

$$J_{\text{out}} = \sqrt{GMR} \left( 1 - \frac{1}{4} \frac{\lambda}{R} \right) + \text{O} \left[ \frac{\lambda^2}{R^2} \right], \quad (23)$$

$$J_{\text{in}} = \sqrt{GMR} \left( 1 + \frac{1}{4} \frac{\lambda}{R} \right) + \mathcal{O} \left[ \frac{\lambda^2}{R^2} \right]. \quad (24)$$

The angular velocity  $\Omega(R)$  in a quasi-Keplerian accretion disk is very close to the Keplerian value, and therefore we can write

$$\Omega(R) = \sqrt{\frac{GM}{R^3}}. \quad (25)$$

It follows that

$$R^2 \Omega \left( R + \frac{\lambda}{6} \right) = \sqrt{GMR} \left( 1 + \frac{1}{6} \frac{\lambda}{R} \right)^{-3/2}, \quad (26)$$

or, to first order in  $\lambda/R$ ,

$$R^2 \Omega \left( R + \frac{\lambda}{6} \right) = \left( 1 - \frac{1}{4} \frac{\lambda}{R} \right) + \mathcal{O} \left[ \frac{\lambda^2}{R^2} \right]. \quad (27)$$

This also implies that

$$R^2 \Omega \left( R - \frac{\lambda}{6} \right) = \left( 1 + \frac{1}{4} \frac{\lambda}{R} \right) + \mathcal{O} \left[ \frac{\lambda^2}{R^2} \right]. \quad (28)$$

Comparing equations (23) and (24) with equations (27) and (28), we find that to first order in  $\lambda/R$ , the net angular momentum transfer is given by

$$J_{\text{in}} - J_{\text{out}} = R^2 \Omega \left( R - \frac{\lambda}{6} \right) - R^2 \Omega \left( R + \frac{\lambda}{6} \right) \simeq -\frac{\lambda}{3} R^2 \frac{d\Omega}{dR}. \quad (29)$$

By following the same steps leading to equation (8), we now obtain

$$\mathcal{G} = 2\pi R \Sigma w (J_{\text{in}} - J_{\text{out}}) \simeq -2\pi R^3 \nu \Sigma \frac{d\Omega}{dR}, \quad (30)$$

where the final expression is correct to first order in  $\lambda$  and we have set  $\nu = w\lambda/3$  so that  $\beta = 3$ . Note that this result agrees very well with the Shakura-Sunyaev form (eq. [5]). Hence we have demonstrated using a simple heuristic derivation that the viscous torque is indeed proportional to the gradient of the angular velocity in an accretion disk within the context of a mean free path, parcel-exchange picture.

## 5. Conclusion

The derivation presented in § 4 clearly employs mean free path theory, since we assumed that the fluid parcels travel an average radial distance  $\lambda$  before being damped by the surrounding gas. We have thus shown that there does exist a simple derivation, based on mean free path theory, that yields an expression for the viscous

torque  $\mathcal{G}$  (eq. [30]) that is directly proportional to the radial derivative of the angular velocity,  $d\Omega/dR$ , in agreement with our physical expectation. This expression for  $\mathcal{G}$  can be used in equation (4) to proceed further in deriving the structure of the accretion disk, as in Pringle (1981). Our results provide a simple but important unification of the “parcel interchange” viscosity model with the MHD viscosity model, in which the coupling that transports the angular momentum is provided by the magnetic field rather than by fluid turbulence (Frank, King, & Raine 2002). In the MHD model, the torque is found to be proportional to the gradient of the angular velocity,  $d\Omega/dR$ , in agreement with the results we have obtained here by applying the mean free path approach to the case of fluid turbulence.

## References

- Becker, P. A., Le, T. 2003, *Astrophys. J.*, **588**, 408
- Frank, J., King, A., Raine, D. 1985, *Accretion Power in Astrophysics, 1st edition*, Cambridge University Press
- Frank, J., King, A., Raine, D. 1992, *Accretion Power in Astrophysics, 2nd edition*, Cambridge University Press
- Frank, J., King, A., Raine, D. 2002, *Accretion Power in Astrophysics, 3rd edition*, Cambridge University Press
- Hartmann, L. 1998, *Accretion Processes in Star Formation*, Springer
- Hayashi, E., Matsuda, T. 2001, *Prog. Theor. Phys.*, **105**, 531 (HM)
- Landau, L. D., Lifshitz, E. M. 1987, *Fluid Mechanics, 2nd edition*, Pergamon Press
- Mihalas, D., Binney, J. 1981, *Galactic Astronomy, 2nd edition*, W. H. Freeman & Co., p. 468
- Pringle, J. 1981, *Ann. Rev. Astron. Astrophys.*, **19**, 137
- Shakura, N. I., Sunyaev, R. I. 1973, *Astron. Astrophys.*, **24**, 337
- Subramanian, P. 1997, *Ph.D. Thesis, George Mason University*