Limits on the validity of the semiclassical theory

L. Sriramkumar†
IUCAA, Post Bag 4, Ganeshkhind,
Pune 411007, INDIA.

Abstract

For want of a more natural proposal, it is generally assumed that the back-reaction of a quantised matter field on a classical metric is given by the expectation value of its energy-momentum tensor, evaluated in a specified state. This semiclassical theory can be reliable only when the fluctuations in the energy-momentum tensor of the quantum field are negligible. Based on this condition, Kuo and Ford have constructed a dimensionless quantity, whose magnitude reflects the amount of fluctuations in the back-reaction term and hence on the validity of the semiclassical theory. In this paper we evaluate this quantity for the minisuperspace model of a quantised massless scalar field in a Friedmann universe. We conclude that the semiclassical theory for the model we consider here can be relied upon only if the scalar field is in states like coherent states. The implications of this investigation on the complete field theory are also discussed.

IUCAA – 15/95, April, 1995 : Submitted for publication

---

† sriram@iucaa.ernet.in

1. Introduction

There exists a domain during the evolution of the universe when the energies of the ongoing physical processes lie between Compton and Planck scales. In this domain, though it is sufficient to describe gravity by a classical metric, the quantum nature of any matter field present has to be taken into account. In general relativity, the theory which we assume to describe gravity adequately in the regime of our interest, it is the energy-momentum tensor of the matter field, $T_{\mu\nu}$, that is reponsible for the classical geometry. The energy-momentum tensor for a quantum field being an operator, a c-number ought to be constructed out of this operator before the effect of the quantum field on a classical metric can be studied. It has been suggested earlier in literature[1], that the transition element $\langle\, out \,|\, \hat{T}_{\mu\nu} \,|\, in \rangle$ (where $|\, in\,\rangle$ and $|\, out\,\rangle$ are the asymptotic states of the quantum field), obtained by the variation of the effective action, be considered as the backreaction term. This transition element is in general a complex quantity and may lead to a complex metric which will prove rather difficult to interpret unless the imaginary part happens to be negligible or is dropped in an ad hoc manner. A more natural and plausible proposal[2 - 6] would be to consider the expectation value of the energy-momentum operator of the quantum field as the term that induces the non-trivial geometry. Since the theory being considered here, by itself, is incapable of providing us with a preferred state for the quantum matter field, the expectation value $\langle\, \hat{T}_{\mu\nu} \,\rangle$ has to be evaluated in a state specified by hand that is consistent with the dynamics. So the analysis of the back-reaction of a quantum field, say a massless scalar field, on the classical background metric reduces to that of solving the Einstein's equations

$$G_{\mu\nu} \;=\; R_{\mu\nu} \;-\; \frac{1}{2}\, g_{\mu\nu}\, R \;=\; 8\,\pi\,\langle\, \hat{T}_{\mu\nu} \,\rangle \tag{1}$$

where $\langle\, \hat{T}_{\mu\nu} \,\rangle$ is the expectation value of the energy-momentum operator (in the specified state) of the scalar field and the Klein-Gordon equation

$$\nabla_\mu \nabla^\mu \hat{\Phi}(x) \;=\; 0 \tag{2}$$

where $\hat{\Phi}$ is the operator corresponding to the quantised scalar field, self-consistently. (We adopt the convention $\hbar = G = c = 1$ and a metric signature of (-2) in this paper.)

Apart from the fact that the energy scales involved should be far below the Planck scale for the semi-classical theory as proposed above to be valid, the fluctuations in the energy-momentum densities of the quantum field should not be too large either[7], *i.e* we must demand

$$\langle\, \hat{T}_{\alpha\beta}(x)\, \hat{T}_{\mu\nu}(y) \,\rangle \;\approx\; \langle\, \hat{T}_{\alpha\beta}(x) \,\rangle\, \langle\, \hat{T}_{\mu\nu}(y) \,\rangle. \tag{3}$$

So, equation (1) will prove to be inadequate to describe a situation when the fluctuations in the energy-momentum densities are large. The goal of this present paper is to check the validity of the semiclassical theory based on the equations (1) and (2) in time dependant background metrics like for instance, Friedmann models, for different states prescribed for the quantum field.

The calculations necessary for drawing the limits on the validity of the semiclassical theory, with aid of the condition (3), will involve evaluating expecation values of the operators $\hat{T}_{\mu\nu}$ and $\hat{T}_{\mu\nu}\,\hat{T}_{\alpha\beta}$. These calculations will involve divergences of quantum field theory, which arise because of the infinite degrees of freedom associated with the fields, and these infinities will have to be removed in a systematic manner. Since these issues will eventually sidetrack our main concern, we, in this paper, will study the back-reaction problem for a minisuperspace model of a Friedmann universe with a quantised massless scalar field when all but one mode of the scalar field are 'frozen'. In such a case, the divergences that may arise because of the infinite degrees of freedom are avoided.

This paper is organised as follows. In section **2**, we discuss the minisuperspace model we intend to study and in section **3** we extend the criterion suggested by Kuo and Ford to draw the limits on the validity of the semiclassical theory to our model. In section **4** we utilise this criterion to study the reliability of the semiclassical theory for our model when the quantum state of the scalar field mode is in a (i) vacuum, (ii) n-particle or a (iii) coherent state. In section **5**, we discuss the implications of our analysis on field theory and we close with section **6**, where we draw the possible conclusions from our analysis.

2. Friedmann universe with a massless scalar field: Minisuperspace model

The action for a massless scalar field coupled to gravity is

$$\mathcal{A} = \int d^4x \sqrt{-g} \left(\frac{1}{16\pi} R + \frac{1}{2} \partial_\mu \Phi \, \partial^\mu \Phi \right). \tag{4}$$

Consider a homogenous and and isotropic spacetime described by the line element

$$ds^2 = N^2(t)\, dt^2 - a^2(t) \left(dx^2 + dy^2 + dz^2 \right). \tag{5}$$

In such a spacetime, the scalar field can be decomposed into its Fourier modes. For the case of the metric (5), the action, after the $\ddot{a}$ terms have been integrated away by parts and the scalar field has been decomposed into its Fourier modes, will be

$$\mathcal{A} = \int dt\, a^3 \left(-\frac{3V}{8\pi N} \left\{ \frac{\dot{a}^2}{a^2} \right\} + \sum_{\mathbf{k}} \frac{1}{2} \left\{ \frac{1}{N} |\, q_k |^2 - N\omega^2 \,|\, \dot{q}_k |^2 \right\} \right), \tag{6}$$

where $q_{\mathbf{k}}(t)$ are the spatial Fourier transforms of the scalar field, $\omega(t) = (|\,\mathbf{k}\,|/a)$ and V is the volume of the universe. As mentioned earlier, when the scalar field is quantised, because of the infinite degrees of freedom associated with the scalar field divergences will arise in the expectation values. To avoid these divergences, we will consider the evolution of just a single mode $\mathbf{k}$ of the scalar field. That is, we will carry out our anlaysis for a system, which has only a finite number of degrees of freedom, described by the action

$$\mathcal{A} = \int dt\, a^3 \left(-\frac{3V}{8\pi N} \left\{ \frac{\dot{a}^2}{a^2} \right\} + \frac{1}{2} \left\{ \frac{1}{N} \dot{q}^2 - N\omega^2 q^2 \right\} \right). \tag{7}$$

Varying the above action with respect to N and setting $N = 1$ after the variation yields

$$\dot{a}^2\, a = \left(\frac{8\pi}{3V} \right) \left\{ \frac{a^3}{2} \left(\dot{q}^2 + \omega^2 q^2 \right) \right\}, \tag{8}$$

which is the Friedmann equation we will be interested in.

In the semiclassical domain, when the single mode q of the scalar field is quantised it satisfies the following Heisenberg equation of motion

$$\frac{d^2\hat{q}}{dt^2} + 3 \left(\frac{\dot{a}}{a} \right) \frac{d\hat{q}}{dt} + \omega^2 \hat{q} = 0. \tag{9}$$

Let $\hat{q} = (\hat{A}\, Q + \hat{A}^\dagger\, Q^*)$, where Q satisfies the same differential eqaution as $\hat{q}$ and $\hat{A}$ is an operator independent of time. If we also assume that Q and $\dot{Q}$ are given by[8]

$$Q = \left(\alpha(t)\, f(t) + \beta(t)\, f^*(t) \right) \quad ; \quad \dot{Q} = -i\omega \left(\alpha(t)\, f(t) + \beta(t)\, f^*(t) \right), \tag{10}$$

where

$$f = \left(\frac{1}{\sqrt{2\omega a^3}} \right) \exp -i \left\{ \int_{t_0}^{t} dt'\, \omega(t') \right\} \tag{11}$$

and t_0 is an *early* time when the initial conditions for the differential equation (9) have to be specified, then we find that α and β satisfy the following set of coupled differential equations

$$\dot{\alpha} = \left(\frac{\dot{a}}{a} \right) \beta\, f^{*2} \quad ; \quad \dot{\beta} = \left(\frac{\dot{a}}{a} \right) \alpha\, f^2. \tag{12}$$

If the initial conditions for Q are chosen such that $\alpha(t_0) = 1$ and $\beta(t_0) = 0$, then the wronskian condition corresponding to the differential equation (9) is

$$|\,\alpha\,|^2 - |\,\beta\,|^2 = 1. \tag{13}$$

So, $\hat{q} = (a(t)\, f \,+\, \hat{a}^{\dagger}(t)\, f^{*})$, where $\hat{a} = (\alpha\, \hat{A} \,+\, \beta^{*}\, \hat{A}^{\dagger})$ and $\hat{a}(t_0) = \hat{A}$. This solution for $\hat{q}$ corresponds to an instantaneous diaganolisation of the scalar field Hamiltonian, which at any time $t \,\geq\, t_0$ is given by

$$\hat{H}(t) \;=\; \left(\hat{a}^{\dagger}\, \hat{a} \,+\, (1/2)\right)\, \omega. \tag{14}$$

In the semiclassical domain, when the single mode q of the scalar field has been quantised as discussed above, the semiclassical equation corresponding to (1) for our minisuperspace model is

$$\dot{a}^2\, a \;=\; \langle\, \psi \,|\, \hat{H} \,|\, \psi \rangle \tag{15}$$

where $|\, \psi \,\rangle$ is the state of the scalar field mode and $\hat{H}$ is given by (14). The quantum state $|\, \psi \,\rangle$ is independent of time in the Heisenberg picture and it can be defined at the time t_0 when the initial conditions for the differential equation (9) have been specified. The three quantum states of the scalar field mode we will be interested in *viz* the (i) vacuum($|\, 0 \,\rangle$), (ii) n-particle ($|\, n \,\rangle$) and (iii) coherent ($|\, \lambda \,\rangle$) states can then be defined as follows:

$$\hat{A}\,|\, 0 \,\rangle \,=\, 0 \qquad ; \qquad \hat{A}^{\dagger}\, \hat{A}\,|\, n \,\rangle \,=\, n\,|\, n \,\rangle \qquad ; \qquad \hat{A}\,|\, \lambda \,\rangle \,=\, \lambda\,|\, \lambda \,\rangle. \tag{16}$$

3. Criterion for drawing the limits on the validity of the semiclassical theory

The semiclassical theory as described by the equations (1) and (2) does not account for the fluctuations in the energy-momentum densities of the quantum field. So, as discussed in the introduction, this theory can be relied upon only when the fluctuations in the energy-momentum densities of the quantum field are small when compared to their expectation values.

Motivated by this fact, Kuo and Ford[9] have suggested that the dimensionless quantity

$$\Delta_{\alpha\beta\mu\nu}(x,\, y) \;\equiv\; \left|\, \frac{\langle\, : \hat{T}_{\alpha\beta}(x)\, \hat{T}_{\mu\nu}(y) :\, \rangle \,-\, \langle\, : \hat{T}_{\alpha\beta}(x) :\, \rangle \langle\, : \hat{T}_{\mu\nu}(x) :\, \rangle}{\langle\, : \hat{T}_{\alpha\beta(x)}\, \hat{T}_{\mu\nu}(y) :\, \rangle} \,\right| \tag{17}$$

(where the colons represent normal ordering) be considered as a measure of the fluctuations in the energy-momentum densities of the quantum field. When the fluctuations in the energy-momentum densities are negligible, this quantity will be far less than unity and the semiclassical theory as described by equations (1) and (2) will prove to be quite sound. But when the fluctuations are large the above quantity is expected to be of order unity reflecting a breakdown of the theory.

The numerous components and the dependance on the two spacetime points make the quantity $\Delta_{\alpha\beta\mu\nu}(x,\, y)$ an extremely cumbersome object to handle. For the sake of simplicty, as Kuo and Ford themselves suggest, we can confine our attention to either the evaluation of the purely temporal component of this quantity in the coincidence limit (*i.e* when $x \,\rightarrow\, y$)

$$\Delta_{KF2}(x) \;=\; \left|\, \frac{\langle\, : \hat{T}_{00}^2(x) :\, \rangle \,-\, \langle\, : \hat{T}_{00}(x) :\, \rangle^2}{\langle\, : \hat{T}_{00}^2(x) :\, \rangle} \,\right| \tag{18}$$

(subscript KF standing for Kuo and Ford) or the quantity

$$\Delta_{KF1}(x) \;=\; \left|\, \frac{\langle\, : \hat{T}_{00}^2(x) :\, \rangle \,-\, \langle\, : \hat{T}_{00}(x) :\, \rangle^2}{\langle\, : \hat{T}_{00}(x) :\, \rangle^2} \,\right|. \tag{19}$$

The quantities Δ_{KF1} and Δ_{KF2} are related to each other by the equation

$$\Delta_{KF2} \;=\; \left(\frac{\Delta_{KF1}}{\Delta_{KF1} \,+\, 1}\right). \tag{20}$$

In (15), the semiclassical equation for our minisuperspace model, the back-reaction term is the expectation value of the hamiltonian operator of the scalar field mode. The validity of equation (15) will then depend on the magnitude of fluctuations in $\langle \hat{H} \rangle$. Since the minisuperspace model we are considering here, has only a finite number of degrees of freedom, no divergences occur in the expectation values. So no normal ordering needs to be carried out. Then, the quantity that we will have to concentrate on, to draw the limits on the validity of equation (15) is either

$$\Delta_{SC1}(t) \equiv \left| \frac{\langle \hat{H}^2 \rangle - \langle \hat{H} \rangle^2}{\langle \hat{H} \rangle^2} \right|, \tag{21}$$

(subscript SC stands for semiclassical) or

$$\Delta_{SC2}(t) \equiv \left| \frac{\langle \hat{H}^2 \rangle - \langle \hat{H} \rangle^2}{\langle \hat{H}^2 \rangle} \right|. \tag{22}$$

The two quantities Δ_{SC1} and Δ_{SC2} are related to each other by the equation

$$\Delta_{SC2} = \left(\frac{\Delta_{SC1}}{\Delta_{SC1} + 1} \right). \tag{23}$$

(Δ_{SC1} and Δ_{SC2} are expected to yield equivalent results.)

In the adiabatic limit, *i.e* when the background metric is evolving very slowly, the ground state energy of each mode of the quantum field just shifts and no excitation of these modes takes place. Or, in other words, no particle creation takes place. In this limit the semiclassical equation (1) proves to be quite reliable[10]. On the other hand, when the metric is evolving very rapidly, a large number of particles get created, with the result that the expectation value of the energy-momentum density of the quantum field, ceases to account for the backreaction adequately. The adiabatic limit for our minisuperspace model corresponds to the case when the scale factor a of the Friedmann universe is a slowly varying function of time, *i.e* when $(\dot{a} / a) \to 0$. In this limit, for the initial conditions we have chosen *viz* $\alpha(t_0) = 1$ and $\beta(t_0) = 0$, equation (12) implies that $\beta \to 0$. So, when $\beta \to 0$, we expect Δ_{SC1} and Δ_{SC2} to vanish thus suggesting a perfect validity of equation (15). And, when $\beta \to \infty$, *i.e* when $(\dot{a} / a)$ is large, we expect Δ_{SC1} and Δ_{SC2} to be of order unity implying that (15) does not describe the backreaction problem adequately.

4. Δ_{SC} for different quantum states of the scalar field mode

In the following three sub-sections we evaluate Δ_{SC1} and Δ_{SC2} for the (i) vacuum, (ii) n-particle and (iii) coherent states of the scalar field mode q.

(i). For a vacuum state

If the quantum state of q is specified to be a vacuum state at $t = t_0$, then the expectation values of the operators $\hat{H}$ and $\hat{H}^2$ are

$$\langle \hat{H} \rangle = \langle 0 | (\hat{a}^\dagger \hat{a} + (1/2)) | 0 \rangle \omega = \left(|\beta|^2 + (1/2) \right) \omega \tag{24}$$

and

$$\begin{aligned} \langle \hat{H}^2 \rangle &= \langle 0 | (\hat{a}^\dagger \hat{a} + (1/2)) (\hat{a}^\dagger \hat{a} + (1/2)) | 0 \rangle \omega^2 \\ &= \left(3|\beta|^2 + 3|\beta|^4 + (1/4) \right) \omega^2. \end{aligned} \tag{25}$$

And, the quantities Δ_{SC1} and Δ_{SC2} are then given by

$$\Delta_{SC1} = \left(\frac{2|\beta|^2 + 2|\beta|^4}{|\beta|^2 + |\beta|^4 + (1/4)} \right) \quad ; \quad \Delta_{SC2} = \left(\frac{2|\beta|^2 + 2|\beta|^4}{3|\beta|^2 + 3|\beta|^4 + (1/4)} \right). \tag{26}$$

(ii). For a n-particle state

If the quantum state of q at $t = t_0$ is a n-particle state then the expectation values of the operators $\hat{H}$ and $\hat{H}^2$ are

$$\langle \hat{H} \rangle = \langle n | (\hat{a}^\dagger \hat{a} + (1/2)) | n \rangle \omega = \left(|\beta|^2 (2n + 1) + n + (1/2) \right) \omega \tag{27}$$

and

$$\begin{aligned} \langle \hat{H}^2 \rangle &= \langle n | (\hat{a}^\dagger \hat{a} + (1/2)) (\hat{a}^\dagger \hat{a} + (1/2)) | n \rangle \omega^2 \\ &= \left\{ (n^2 + n) \left(1 + 6|\beta|^2 + 6|\beta|^4 \right) + \left(3|\beta|^2 + 3|\beta|^4 + (1/4) \right) \right\} \omega^2. \end{aligned} \tag{28}$$

The quantities Δ_{SC1} and Δ_{SC2} are then given by the expressions

$$\Delta_{SC1} = \left\{ \left(\frac{2|\beta|^2 + 2|\beta|^4}{1 + 4|\beta|^2 + 4|\beta|^4} \right) \left(\frac{n^2 + n + 1}{n^2 + n + (1/4)} \right) \right\} \tag{29}$$

and

$$\Delta_{SC2} = \left\{ \left(\frac{2|\beta|^2 + 2|\beta|^4}{1 + 6|\beta|^2 + 6|\beta|^4} \right) \left(\frac{n^2 + n + 1}{n^2 + n + (1/2)} \right) \right\}. \tag{30}$$

(iii). For a coherent state

When the quantum state for q is specified to be a coherent state, the expectation values of $\hat{H}$ and $\hat{H}^2$ are

$$\begin{aligned} \langle \hat{H} \rangle &= \langle \lambda | (\hat{a}^\dagger \hat{a} + (1/2)) | \lambda \rangle \omega \\ &= \left\{ |\lambda|^2 \left(1 + 2|\beta|^2 \right) + \lambda^2 \alpha \beta + \lambda^{*2} \alpha^* \beta^* + |\beta|^2 + (1/2) \right\} \omega \end{aligned} \tag{31}$$

and

$$
\begin{aligned}
\langle \hat{H}^2 \rangle &= \langle \lambda | \left(\hat{a}^\dagger \hat{a} + (1/2) \right) \left(\hat{a}^\dagger \hat{a} + (1/2) \right\} | \lambda \rangle \omega^2 \\
&= \Big\{ \left(|\lambda|^4 + 2|\lambda|^2 \right) \left(1 + 6|\beta|^2 + 6|\beta|^4 \right) \\
&+ \left(2|\lambda|^2 + 3 \right) \left(\lambda^2 \alpha \beta + \lambda^{*2} \alpha^* \beta^* \right) \left(1 + 2|\beta|^2 \right) \\
&+ \left(\lambda^4 \alpha^2 \beta^2 + \lambda^{*4} \alpha^{*2} \beta^{*2} \right) + \left(3|\beta|^2 + 3|\beta|^4 \right) + (1/4) \Big\} \omega^2 .
\end{aligned} \tag{32}
$$

The expressions for the Δ_{SC1} and Δ_{SC2} for the coherent state prove to be rather lenghthy. Due to this reason, we do not write them down here explicitly. Their values in the different limits of interest will be quoted in the tables below.

Δ_{SC1} and Δ_{SC2} for the three quantum states specified for q in the limits $\beta \rightarrow 0$ and $\beta \rightarrow \infty$ are tabulated below.

Table I ($\beta \rightarrow 0$)

	Vacuum	nth excited	Coherent
Δ_{SC1}	0	0	$\left(\frac{\lvert\lambda\rvert^2}{\lvert\lambda\rvert^4 + \lvert\lambda\rvert^2 + (1/4)} \right)$
Δ_{SC2}	0	0	$\left(\frac{\lvert\lambda\rvert^2}{\lvert\lambda\rvert^4 + 2\lvert\lambda\rvert^2 + (1/4)} \right)$

Table II ($\beta \rightarrow \infty$)

	Vacuum	nth excited	Coherent
Δ_{SC1}	2	$\left(\frac{n^2 + n + 1}{2n^2 + 2n + (1/2)} \right)$	$\left(\frac{\lvert\lambda\rvert^2 (8 + 4c_1) + 2}{\left(\lvert\lambda\rvert^2 (2 + c_1) + 1 \right)^2} \right)$
Δ_{SC2}	$\left(\frac{2}{3} \right)$	$\left(\frac{n^2 + n + 1}{3n^2 + 3n + \frac{3}{2}} \right)$	$\left(\frac{\lvert\lambda\rvert^2 (8 + 4c_1) + 2}{\lvert\lambda\rvert^4 (6 + 4c_1 + c_2) + \lvert\lambda\rvert^2 (12 + 6c_1) + 3} \right)$

The quantities c_1 and c_2 in the table II are

$$
c_1 = 2\cos(a + b + 2l) \quad ; \quad c_1 = 2\cos(2a + 2b + 4l), \tag{33}
$$

where a, b and l are the arguments of the complex quantities α, β and λ respectively.

The results tabulated above show that in the adiabatic limit, *i.e* when $\beta \rightarrow 0$, Δ_{SC1} and Δ_{SC2} identically vanish for the vacuum and n-particle states whereas they die down as $|\lambda|^{-2}$ (for a large λ) for coherent states. And in the limit when the Friedmann metric is evolving rapidly, *i.e* when $\beta \rightarrow \infty$ we find that Δ_{SC1} and Δ_{SC2} are of order unity for vacuum and n-particle (even for a large n) implying a breakdown of the semiclassical theory. For coherent states with a large λ they still go as $|\lambda|^{-2}$. These results imply that the semiclassical theory for our minisuperspace model as described by equation (15) is valid, during all stages of evolution, only if the scalar field mode is specified to be in coherent like states.

5. Δ_{KF} for different quantum states of the scalar field mode

Had we been dealing with the complete field theory instead of a minisuperspace model we would have encountered divergences when evaluating the expectation values of the operators involving quantum fields. These infinities would have had to be systematically removed. In particular it would have been necessary to normal order the operators.

In this section, we will evaluate the quantities that correspond to Δ_{KF1} and Δ_{KF2} for our model. These quantities would be

$$\Delta_{KF1}(t) = \left| \frac{\langle :\hat{H}^2: \rangle - \langle :\hat{H}: \rangle^2}{\langle :\hat{H}: \rangle^2} \right|, \tag{34}$$

and

$$\Delta_{KF2}(t) = \left| \frac{\langle :\hat{H}^2: \rangle - \langle :\hat{H}: \rangle^2}{\langle :\hat{H}^2: \rangle} \right|, \tag{35}$$

where the colons denote normal ordering. For our model the operators have to be normal ordered with respect to $\hat{a}$. This has to be so, because, if the expression for $\langle 0|\hat{H}|0\rangle$ is normal ordered with respect to $\hat{A}$ instead of $\hat{a}$ it will kill the $|\beta|^2$ term in (24) which otherwise will contribute to the back-reaction. Alternatively one can try to regularise the expectation values by subtracting the vacuum contribution, *i.e* the $\langle 0|(\hat{A}^\dagger \hat{A} + (1/2))|0\rangle\omega = (\omega/2)$ and $\langle 0|(\hat{A}^\dagger \hat{A} + (1/2))(\hat{A}^\dagger \hat{A} + (1/2))|0\rangle\omega^2 = (\omega^2/4)$ terms can be removed from $\langle \hat{H}\rangle$ and $\langle \hat{H}^2\rangle$ respectively. The goal of this section is to point out a drawback when the magnitude of Δ_{KF1} or Δ_{KF2} is used to decide the validity of the semiclassical theory in the adiabatic limit.

(i). For a vacuum state

When the state of the scalar field mode q is defined to be a vacuum state and the operators are normal ordered with respect to $\hat{a}$, we get

$$\langle :\hat{H}: \rangle_{NO} = \langle 0|\hat{a}^\dagger \hat{a}|0\rangle\omega = |\beta|^2\omega \tag{36}$$

and

$$\langle :\hat{H}^2: \rangle_{NO} = \langle 0|a^\dagger \hat{a}^\dagger \hat{a}\hat{a}|0\rangle = \left(|\beta|^2 + 3|\beta|^4\right)\omega^2. \tag{37}$$

When the vacuum terms are subtracted, *i.e*

$$\langle :\hat{H}: \rangle_{VS} = \langle 0|(\hat{a}^\dagger \hat{a} + (1/2))\,\omega|0\rangle\omega - (\omega/2) \tag{38}$$

and

$$\langle :\hat{H}^2: \rangle_{(VS)} = \langle 0|(\hat{a}^\dagger \hat{a} + (1/2))\,(\hat{a}^\dagger \hat{a} + (1/2))\,|0\rangle\omega^2 - (\omega^2/4), \tag{39}$$

the expressions for $\langle :\hat{H}:\rangle_{VS}$ and $\langle :\hat{H}^2:\rangle_{VS}$ are the same as the quantities $\langle \hat{H}\rangle$ and $\langle \hat{H}^2\rangle$ in equations (24) and (25) but without the $(\omega/2)$ and $(\omega^2/4)$ terms respectively. Substituting these in equations (34) and (35), we obtain that

$$\Delta_{KF1(NO)} = \left(\frac{1 + 2|\beta|^2}{|\beta|^2}\right) \quad ; \quad \Delta_{KF2(NO)} = \left(\frac{1 + 2|\beta|^2}{1 + 3|\beta|^2}\right) \tag{40}$$

and

$$\Delta_{KF1(VS)} = \left(\frac{3 + 2|\beta|^2}{|\beta|^2}\right) \quad ; \quad \Delta_{KF2(VS)} = \left(\frac{3 + 2|\beta|^2}{3 + 3|\beta|^2}\right), \tag{41}$$

where the subscripts NO and VS represent regularisation by normal ordering and vacuum subtraction respectively.

(ii). For a n particle state

For the case when the quantum state of the mode q of the scalar field is specified to be a n-particle state, the expectation values, when the operators are normal ordered, are given by the expressions

$$\left\langle :\hat{H}: \right\rangle_{(NO)} = \left\langle n \,|\, (\hat{a}^\dagger \hat{a})\, \omega | n \right\rangle = \Big(|\beta|^2 (2n+1) + n \Big)\, \omega \tag{42}$$

and

$$\begin{aligned}\left\langle :\hat{H}^2: \right\rangle_{(NO)} &= \left\langle n \,|\, (\hat{a}^\dagger \hat{a}^\dagger \hat{a} \hat{a})\, | n \right\rangle \omega^2 \\ &= \Big\{ n^2 \left(1 + 6|\beta|^2 + 6|\beta|^4\right) + n\left(-1 + 2|\beta|^2 + 6|\beta|^4\right) \\ &\qquad + \left(|\beta|^2 + 3|\beta|^4\right) \Big\}\, \omega^2.\end{aligned} \tag{43}$$

When the vacuum terms have been subtracted from the expectation values, *i.e*

$$\left\langle :\hat{H}: \right\rangle_{(VS)} = \left\langle n \,|\, (\hat{a}^\dagger \hat{a} + (1/2))\, | n \right\rangle - (\omega/2) \tag{44}$$

and

$$\left\langle :\hat{H}^2: \right\rangle_{(VS)} = \left\langle n \,|\, (\hat{a}^\dagger \hat{a} + (1/2))\, (\hat{a}^\dagger \hat{a} + (1/2))\, | n \right\rangle - (\omega^2/4), \tag{45}$$

the expressions for $\langle :\hat{H}: \rangle_{VS}$ and $\langle :\hat{H}^2: \rangle_{VS}$ are the same as the quantities $\langle \hat{H} \rangle$ and $\langle \hat{H}^2 \rangle$ in equations (27) and (28) but without the $(\omega/2)$ and $(\omega^2/4)$ terms respectively. Substituting the quantities evaluated above in the equations (34) and (35), we obtain that

$$\Delta_{KF1(NO)} = \left(\left| \frac{|\beta|^4 (2n^2 + 2n + 2) + |\beta|^2 (2n+1) - n}{|\beta|^4 (4n^2 + 4n + 1) + |\beta|^2 (4n^2 + 2n) + n^2} \right| \right), \tag{46}$$

$$\Delta_{KF2(NO)} = \left(\left| \frac{|\beta|^4 (2n^2 + 2n + 2) + |\beta|^2 (2n+1) - n}{|\beta|^4 (6n^2 + 6n + 3) + |\beta|^2 (6n^2 + 2n + 1) + (n^2 - n)} \right| \right), \tag{47}$$

$$\Delta_{KF1(VS)} = \left(\frac{|\beta|^4 (2n^2 + 2n + 2) + |\beta|^2 (2n^2 + 4n + 3) + n}{|\beta|^4 (4n^2 + 4n + 1) + |\beta|^2 (4n^2 + 2n) + n^2} \right), \tag{48}$$

and

$$\Delta_{KF2(VS)} = \left(\frac{|\beta|^4 (2n^2 + 2n + 2) + |\beta|^2 (2n^2 + 4n + 3) + n}{\left(|\beta|^4 + |\beta|^2\right)(6n^2 + 6n + 3) + (n^2 + n)} \right). \tag{49}$$

(iii). For a coherent state

When the quantum state for q is specified to be a coherent state, the expectation values when the operators are normal ordered are

$$\begin{aligned}\left\langle :\hat{H}: \right\rangle_{(NO)} &= \left\langle \lambda \,|\, \hat{a}^\dagger \hat{a} \,|\, \lambda \right\rangle \omega \\ &= \left(|\lambda|^2 \left(1 + 2|\beta|^2\right) + \lambda^2 \alpha \beta + \lambda^{*2} \alpha^* \beta^* + |\beta|^2 \right) \omega\end{aligned} \tag{50}$$

and

$$
\begin{aligned}
\langle : \hat{H}^2 : \rangle_{(NO)} &= \langle \lambda | \hat{a}^\dagger \hat{a}^\dagger \hat{a} \hat{a} | \lambda \rangle \omega^2 \\
&= \Big\{ |\lambda|^4 \left(1 + 6|\beta|^2 + 6|\beta|^4\right) + |\lambda|^2 \left(8|\beta|^2 + 12|\beta|^4\right) \\
&+ \left(\lambda^2 \alpha \beta + \lambda^{*2} \alpha^* \beta^*\right) \left\{1 + 6|\beta|^2 + |\lambda|^2 \left(2 + 4|\beta|^2\right)\right\} \\
&+ \left(\lambda^4 \alpha^2 \beta^2 + \lambda^{*4} \alpha^{*2} \beta^{*2}\right) + 3|\beta|^4 + |\beta|^2 \Big\} \omega^2.
\end{aligned} \tag{51}
$$

For the case, when the vacuum terms are subtracted, *i.e*

$$
\langle : \hat{H} : \rangle_{(VS)} = \langle \lambda | (\hat{a}^\dagger \hat{a} + (1/2)) | \lambda \rangle \omega - (\omega/2) \tag{52}
$$

and

$$
\langle : \hat{H}^2 : \rangle_{(VS)} = \langle \lambda | (\hat{a}^\dagger \hat{a} + (1/2)) (\hat{a}^\dagger \hat{a} + (1/2)) | \lambda \rangle \omega^2 - (\omega^2/4), \tag{53}
$$

the expectation values, $\langle : \hat{H} : \rangle_{VS}$ and $\langle : \hat{H}^2 : \rangle_{VS}$ are the same as the quantities $\langle \hat{H} \rangle$ and $\langle \hat{H}^2 \rangle$ in equations (31) and (32) but without the $(\omega/2)$ and $(\omega^2/4)$ terms respectively. For the coherent state, the expressions for the Δ_{KF1} and Δ_{KF2} prove to be rather lenghthy. So, we do not write them down here explicitly but just quote their values in the different limits of interest in the tables below.

The expressions for the different Δ_{KF1} and Δ_{KF2} in the two limits of interest, *viz* $\beta \longrightarrow 0$ and $\beta \longrightarrow \infty$, are summarised below in tables III and IV.

Table III ($\beta \longrightarrow 0$)

	Vacuum	nth excited	Coherent
$\Delta_{KF1(NO)}$	∞	$\left(\frac{1}{n}\right)$	0
$\Delta_{KF2(NO)}$	1	$\left(\left\lvert\frac{1}{n-1}\right\rvert\right)$	0
$\Delta_{KF1(VS)}$	∞	$\left(\frac{1}{n}\right)$	$\left(\frac{2}{\lvert\lambda\rvert^2}\right)$
$\Delta_{KF2(VS)}$	1	$\left(\frac{1}{n-1}\right)$	$\left(\frac{2}{2+\lvert\lambda\rvert^2}\right)$

Table IV ($\beta \longrightarrow \infty$)

	Vacuum	nth excited	Coherent
$\Delta_{KF1(NO)}$	2	$\left(\frac{n^2+n+1}{2n^2+2n+(1/2)}\right)$	$\left(\frac{\lvert\lambda\rvert^2(8+4c_1)+2}{(\lvert\lambda\rvert^2(2+c_1)+1)^2}\right)$
$\Delta_{KF2(NO)}$	$\left(\frac{2}{3}\right)$	$\left(\frac{n^2+n+1}{3n^2+3n+\frac{3}{2}}\right)$	$\left(\frac{\lvert\lambda\rvert^2(8+4c_1)+2}{\lvert\lambda\rvert^4(6+4c_1+c_2)+\lvert\lambda\rvert^2(12+6c_1)+3}\right)$
$\Delta_{KF1(VS)}$	2	$\left(\frac{n^2+n+1}{2n^2+2n+(1/2)}\right)$	$\left(\frac{\lvert\lambda\rvert^2(8+4c_1)+2}{(\lvert\lambda\rvert^2(2+c_1)+1)^2}\right)$
$\Delta_{KF2(VS)}$	$\left(\frac{2}{3}\right)$	$\left(\frac{n^2+n+1}{3n^2+3n+\frac{3}{2}}\right)$	$\left(\frac{\lvert\lambda\rvert^2(8+4c_1)+2}{\lvert\lambda\rvert^4(6+4c_1+c_2)+\lvert\lambda\rvert^2(12+6c_1)+3}\right)$

The quantities c_1 and c_2 in the table IV above are the same as those defined after table II.

From table III it is clear that Δ_{KF1} and Δ_{KF2}, for our model do not vanish in the adiabatic limit, but, in fact are of order unity thus suggesting a breakdown of the semiclassical theory in this limit. In field theory, when Δ_{KF1} and Δ_{KF2} are calculated with regularised expectation values the same is bound to happen. But we do know that the semiclassical theory *is* perfectly valid in the adiabatic limit[10]. So, in field theory, where the expectation values *have* to be regularised, in the adiabatic limit it would be advisable to concentrate just on the magnitude of the fluctuations rather than on Δ_{KF1} or Δ_{KF2} to draw the limits on the validity of the semiclassical theory. Whereas, when $\beta \rightarrow \infty$, both Δ_{SC} and Δ_{KF} give identical results for our model and in field theoretic calculations where only Δ_{KF1} or Δ_{KF2} can be evaluated, we can expect these quantities to give reliable results to help draw the limits on the validity of the semiclassical theory.

6. Conclusions

The results of the section 4 quite clearly prove that the semiclassical theory that was considered for our model, can be relied upon, during all stages of the evolution, only if the quantum system, *viz* the scalar field mode is specified to be in states like coherent states. It is plausible that the results of our minisuperspace model will hold good even for quantum fields in curved spaces. In that case, if the backreaction problem *has* to be studied in those states for the quantum field, which do not possess a 'coherent' nature, the semiclassical theory based on equations (1) and (2) is bound to prove rather inadequate and the fluctuations will have to be accounted for in the backreaction. When done so, the back-reaction problem can possibly be expected to be described by an equation similar in form to the Langevin equation[11].

Acknowledgements

The author is being supported by the Senior Research Fellowship of the Council of Scientific and Industrial Research, India. This work was done under the guidance of T. Padmanabhan.

References

[1] See for *e.g*, B. S. DeWitt, Phys. Rep. C **19**, 295 (1975)

[2] V. G. Lapchinsky and V. A. Rubakov, Acta Phys. Polon. B **10**, 1041 (1975)

[3] J. B. Hartle, in *Gravitation in Astrophysics* (J. B. Hartle and B. Carter, Eds.), Plenum, New York (1986).

[4] R. D. Jordan, Phys. Rev. D **33**, 444 (1986)

[5] E. Calzetta and B. L. Hu, Phys. Rev. D **35**, 495 (1987)

[6] T. Padmanabhan and T. P. Singh, Ann. Phys. (N.Y.) **221**, 217 (1993).

[7] L. H. Ford, Ann. Phys. (N.Y.) **144**, 238 (1982)

[8] Ya. B. Zeldovich and A. A. Starobinskii, ZhETF **61**, 2161 (1971).

[9] Chung-I Kuo and L. H. Ford, Phys. Rev. D **47**, 4510 (1993).

[10] T. Padmanabhan and T. P. Singh, Class. Quantum Gravity **7**, 411 (1990)

[11] B. L. Hu and A. Matacz, Phys. Rev. D **51**, 1577 (1995)