

NONLINEAR GRAVITATIONAL CLUSTERING: DREAMS OF A PARADIGM

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ABSTRACT

We discuss the *late-time* evolution of the gravitational clustering in an expanding universe, based on the nonlinear scaling relations (NSR) that connect the nonlinear and linear two-point correlation functions. The existence of critical indices for the NSR suggests that the evolution may proceed toward a universal profile that does not change its shape at late times. We begin by clarifying the relation between the density profiles of the individual halos and the slope of the correlation function, and we discuss the conditions under which the slopes of the correlation function at the extreme nonlinear end can be independent of the initial power spectrum. If the evolution should lead to a profile that preserves the shape at late times, then the correlation function should grow as a^2 (in a $\Omega = 1$ universe), even at nonlinear scales. We prove that such exact solutions do not exist; however, there exists a class of solutions (“pseudolinear profiles”) that evolve as a^2 to a good approximation. It turns out that pseudolinear profiles are the correlation functions that arise if the individual halos are assumed to be isothermal spheres. They are also configurations of mass in which the nonlinear effects of gravitational clustering are a minimum, and hence they can act as building blocks of the nonlinear universe. We discuss the implications of this result.

Subject headings: cosmology: theory — large-scale structure of universe

1. INTRODUCTION

The evolution of a large number of particles under their mutual gravitational influence is a well-defined mathematical problem. If such a system occupies a finite region of phase space at an initial instant and evolves via Newtonian gravity, then it does not reach any sensible “equilibrium” state. The core region of the system will keep on shrinking, eventually to be dominated by a few hard binaries. The rest of the particles will evaporate away to large distances, gaining kinetic energy from the shrinking core (for a discussion of such systems, see Padmanabhan 1990).

The situation is drastically different in the presence of an expanding background universe characterized by an expansion factor $a(t)$. First, the expansion tends to keep particles apart, thereby exerting a civilizing influence against Newtonian attraction. Second, it is now possible to consider an infinite region of space filled with particles. The average density of particles will contribute to the expansion of the background universe, and the deviations from the uniformity will lead to clustering. Particles evaporating from a local overdense cluster cannot escape to “large distances,” but will necessarily encounter other deep potential wells. Naively, one would expect the local overdense regions eventually to form gravitationally bound objects, with a hotter distribution of particles hovering uniformly all over. As the background expands, the velocity dispersion of the second component will keep decreasing, and they will be captured by the deeper potential wells. Meanwhile, the clustered component will also evolve dynamically and participate in, e.g., mergers. If the background expansion and the initial conditions have no length scale, then it is likely that the clustering will continue in a hierarchical manner ad infinitum.

Most practicing cosmologists will broadly agree with the above picture of gravitational clustering in an expanding universe. It is, however, not easy to translate these concepts into a well-defined mathematical formalism and to provide

a more quantitative description of the gravitational clustering. One of the key questions concerning this system that needs to be addressed is the following: Can one make any general statements about the very late-stage evolution of the clustering? For example, does the power spectrum at late times “remember” the initial power spectrum, or does it possess some universal characteristics that are reasonably independent of initial conditions? (This question is closely related to the issue of whether gravitational clustering leads to density profiles that are universal; Navarro, Frenk, & White 1996.)

We address some aspects of this issue in this paper and show that it is possible to provide at least partial answers to these questions, based on a simple paradigm. The key assumption that we shall make is the following: Let the ratio between mean relative pair velocity $v(a, x)$ and the negative Hubble velocity ($-\dot{a}x$) be denoted by $h(a, x)$, and let $\xi(a, x)$ be the mean correlation function, averaged over a sphere of radius x . *We shall assume that $h(a, x)$ depends on a and x only through $\xi(a, x)$; that is, $h(a, x) = h[\xi(a, x)]$.* With such a minimal assumption, we will be able to reach several conclusions regarding the evolution of the power spectrum in the universe. Such an assumption was originally introduced, in a different form, by Hamilton et al. (1991). The present form, as well as its theoretical implications, were discussed in Nityananda & Padmanabhan (1994), and a theoretical model for the scaling was attempted by Padmanabhan (1996a). It must be noted that simulations indicate a dependence of the relation $h(a, x) = h[\xi(a, x)]$ on the initial spectrum, and also on cosmological parameters (Peacock & Dodds 1994, 1996; Padmanabhan et al. 1996; Mo, Jain, & White 1995). Most of our discussion in this paper is independent of this fact or can be easily generalized to such cases. When we need to use an explicit form for h , we shall use the original form suggested by Hamilton because of its simplicity.

As this paper addresses several independent but related

questions, we provide here a brief summary of how it is organized. Section 2 presents a study of some aspects of nonlinear evolution, based on the assumption mentioned above. We begin by summarizing some previously known results in § 2.1, in order to set up the notation and to collect together in one place the formulae that we will later need. In § 2.2 we make a brief comment about the critical indices in gravitational dynamics so as to motivate later discussion. In § 3 we discuss the relation between density profiles of halos and correlation functions, and we derive the conditions under which one may expect universal density profiles in gravitational clustering. In § 4 we show that gravitational clustering does not admit self-similar evolution, except in a very special case. We also discuss the conditions necessary for approximate self-similarity to hold. In § 5 we discuss the question of whether one can expect to find power spectra that preserve their shape as they evolve, even in the nonlinear regime. We first show, based on the results of § 4, that such *exact* solutions cannot exist. We then discuss the conditions for the existence of some *approximate* solutions. We obtain one prototypical approximate solution and discuss its properties. The solution also allows us to understand the connection between the statistical mechanics of gravitating systems on the small scale and the evolution of correlation functions on the large scale. Finally, the results are discussed in § 6.

It may be worthwhile to emphasize the overall aim of the paper and its relevance. We are not attempting to provide a definitive approach that will tackle the difficult problem of nonlinear gravitational clustering in its totality. The aim is more limited; we are attempting to ask certain general questions and to provide a framework within which these questions can be answered. It is hoped that this approach, which is fairly different from the conventional approach to the subject, might prove fruitful in future. In particular, we explore alternatives to the stable clustering hypothesis in the nonlinear regime and reach conclusions regarding the possibility of identifying the units of the nonlinear universe. The paper should be viewed in the above spirit.

2. GENERAL FEATURES OF NONLINEAR EVOLUTION

Consider the evolution of a system starting from Gaussian initial fluctuations with an initial power spectrum, $P_{in}(k)$. The Fourier transform of the power spectrum defines the correlation function $\xi(a, x)$, where $a \propto t^{2/3}$ is the expansion factor in a universe with $\Omega = 1$. It is more convenient to work with the average correlation function inside a sphere of radius x , defined by

$$\bar{\xi}(a, x) \equiv \frac{3}{x^3} \int_0^x \xi(a, y) y^2 dy . \tag{1}$$

This quantity is related to the power spectrum $P(a, k)$ by

$$\bar{\xi}(x, a) = \frac{3}{2\pi^2 x^3} \int_0^\infty \frac{dk}{k} P(a, k) [\sin(kx) - kx \cos(kx)] , \tag{2}$$

with the inverse relation

$$P(a, k) = \frac{4\pi}{3k} \int_0^\infty dx x \bar{\xi}(a, x) [\sin(kx) - kx \cos(kx)] . \tag{3}$$

In the linear regime, we have $\bar{\xi}_L(a, x) \propto a^2 \bar{\xi}_{in}(a_i, x)$.

We now recall that the conservation of pairs of particles gives an exact equation that is satisfied by the correlation function (Peebles 1980):

$$\frac{\partial \xi}{\partial t} + \frac{1}{ax^2} \frac{\partial}{\partial x} [x^2(1 + \xi)v] = 0 , \tag{4}$$

where $v(a, x)$ denotes the mean relative velocity of pairs at separation x and epoch a . Using the mean correlation function $\bar{\xi}$ and a dimensionless pair velocity $h(a, x) \equiv -(v/\dot{a}x)$, equation (4) can be written as

$$\left(\frac{\partial}{\partial \ln a} - h \frac{\partial}{\partial \ln x} \right) (1 + \bar{\xi}) = 3h(1 + \bar{\xi}) . \tag{5}$$

This equation can be simplified by introducing the variables

$$A = \ln a , \quad X = \ln x , \quad \text{and} \quad D(X, A) = \ln(1 + \bar{\xi}) , \tag{6}$$

in terms of which we have (Nityananda & Padmanabhan 1994)

$$\frac{\partial D}{\partial A} - h(A, X) \frac{\partial D}{\partial X} = 3h(A, X) . \tag{7}$$

At this stage we shall introduce our key assumption, viz., that h depends on (A, X) only through $\bar{\xi}$ (or, equivalently, through D). Given this single assumption, several results follow that we shall now summarize.

At this juncture we would like to stress that all the results and discussion in this paper depend strongly on this assumption. It has been invoked in several papers in the past (Nityananda & Padmanabhan 1994; Mo et al. 1995; Padmanabhan 1996a; Hamilton et al. 1991; Munshi & Padmanabhan 1997) and seems to have certain *prima facie* validation in numerical experiments. If this assumption is not made, there is no hope of solving the above equation in closed form. The alternatives to this assumption involve handling a BBGKY hierarchy of equations using some truncation scheme. In such attempts (which are also explored in the literature) it is very difficult to ascertain the validity of the truncation schemes. Our assumption of $h = h(\bar{\xi})$ can be thought of as yet another way of truncating the BBGKY hierarchy, although it is again very difficult to estimate its accuracy. The numerical experiments mentioned above, however, do show that this approximation has a fair amount of validity. In this paper we shall take this assumption as a basic postulate, and we will not address the question of its limits of validity.

2.1. Formal Solution

Given that $h = h[\bar{\xi}(a, x)]$, one can easily integrate equation (5) to find the general solution (see Nityananda & Padmanabhan 1994). The characteristics of equation (5) satisfy the condition

$$x^3(1 + \bar{\xi}) = l^3 , \tag{8}$$

where l is another length scale. When the evolution is linear at all the relevant scales, $\bar{\xi} \ll 1$ and $l \approx x$. As clustering develops, $\bar{\xi}$ increases, and x becomes considerably smaller than l . The behavior of clustering at some scale x is then determined by the original *linear* power spectrum at the scale l through the “flow of information” along the characteristics. This suggests that *we can express the true correlation function $\bar{\xi}(a, x)$ in terms of the linear correlation function $\bar{\xi}_L(a, l)$ evaluated at a different point.* This is indeed

true, and the general solution can be expressed as a nonlinear scaling relation (NSR) between $\bar{\xi}_L(a, l)$ and $\bar{\xi}(a, x)$, with l and x being related by equation (8). To express this solution, we define two functions $\mathcal{V}(z)$ and $\mathcal{U}(z)$, where $\mathcal{V}(z)$ is related to the function $h(z)$ by

$$\mathcal{V}(z) = \exp \left[\frac{2}{3} \int^z \frac{dz}{h(z)(1+z)} \right], \quad (9)$$

and $\mathcal{U}(z)$ is the inverse function of $\mathcal{V}(z)$. Then, the solution to equation (5) can be written in either of two equivalent forms, as

$$\bar{\xi}(a, x) = \mathcal{U}[\bar{\xi}_L(a, l)], \quad \text{or} \quad \bar{\xi}_L(a, l) = \mathcal{V}[\bar{\xi}(a, x)], \quad (10)$$

where $l^3 = x^3(1 + \bar{\xi})$ (Nityananda & Padmanabhan 1994). Given the form of $h(\bar{\xi})$, this allows one to relate the nonlinear correlation function to the linear one.

From general theoretical considerations (see Padmanabhan 1996a), it can be shown that $\mathcal{V}(z)$ has the form

$$\mathcal{V}(z) = \begin{cases} z & (z \ll 1), \\ z^{1/3} & (1 \gtrsim z \gtrsim 200), \\ z^{2/3} & (200 \ll z). \end{cases} \quad (11)$$

In these three regions $h(z) \approx [(2z/3), 2, 1]$, respectively. We shall call these regimes linear, intermediate, and nonlinear, respectively. More exact-fitting functions to $\mathcal{V}(z)$ and $\mathcal{U}(z)$ have been suggested in the literature (see Hamilton et al. 1991; Mo et al. 1995; Peacock & Dodds 1994). When necessary in this paper, we shall use those given in Hamilton et al. 1991:

$$\mathcal{V}(z) = z \left(\frac{1 + 0.0158z^2 + 0.000115z^3}{1 + 0.926z^2 - 0.0743z^3 + 0.0156z^4} \right)^{1/3}, \quad (12)$$

and

$$\mathcal{U}(z) = \frac{z + 0.358z^3 + 0.0236z^6}{1 + 0.0134z^3 + 0.0020z^{9/2}}. \quad (13)$$

Equations (10), (12), and (13) implicitly determine $\bar{\xi}(a, x)$ in terms of $\bar{\xi}_L(a, x)$.

2.2. Critical Indices

These NSRs already allow one to reach some general conclusions regarding the evolution. To do this most effectively, let us define a local index for the rate of clustering by

$$n_a(a, x) \equiv \frac{\partial \ln \bar{\xi}(a, x)}{\partial \ln a}, \quad (14)$$

which measures how fast $\bar{\xi}(a, x)$ is growing. When $\bar{\xi}(a, x) \ll 1$, then $n_a = 2$, irrespective of the spatial variation of $\bar{\xi}(a, x)$, and the evolution preserves the shape of $\bar{\xi}(a, x)$. However, as clustering develops, the growth rate will depend on the spatial variation of $\bar{\xi}(a, x)$. Defining the effective spatial slope by

$$- [n_{\text{eff}}(a, x) + 3] \equiv \frac{\partial \ln \bar{\xi}(a, x)}{\partial \ln x}, \quad (15)$$

one can rewrite equation (5) as

$$n_a = h \left[\frac{3}{\bar{\xi}(a, x)} - n_{\text{eff}} \right]. \quad (16)$$

At any given scale of nonlinearity, decided by $\bar{\xi}(a, x)$, there exists a critical spatial slope n_c , such that $n_a > 2$ for $n_{\text{eff}} < n_c$ (implying that the rate of growth is faster than predicted by linear theory), and $n_a < 2$ for $n_{\text{eff}} > n_c$ (with the rate of growth being slower than predicted by linear theory). The critical index n_c is fixed by setting $n_a = 2$ in equation (16) at any instant. (This requirement is established from the physically motivated wish to have a form of the two-point correlation function that remains invariant under time evolution. Since the linear end of the two-point correlation function scales as a^2 , the required invariance of form constrains the index n_a to be 2 at all scales.) The fact that $n_a > 2$ for $n_{\text{eff}} < n_c$, and $n_a < 2$ for $n_{\text{eff}} > n_c$, will tend to “straighten out” correlation functions toward the critical slope. [We are assuming that $\bar{\xi}(a, x)$ has a slope that is decreasing with scale, which is true for any physically interesting case.] From the fitting function, it is easy to see that in the range $1 \gtrsim \bar{\xi} \gtrsim 200$, the critical index is $n_c \approx -1$, and for $200 \gtrsim \bar{\xi}$, the critical index is $n_c \approx -2$ (Bagla & Padmanabhan 1997). This clearly suggests that the local effect of evolution is to drive the correlation function toward a shape with $1/x$ behavior in the nonlinear regime and $1/x^2$ behavior in the intermediate regime. Such a correlation function will have $n_a \approx 2$ and hence will grow at a rate close to a^2 . We shall discuss this further in § 3.

3. CORRELATION FUNCTIONS, DENSITY PROFILES, AND STABLE CLUSTERING

Now that we have a NSR giving $\bar{\xi}(a, x)$ in terms of $\bar{\xi}_L(a, l)$, we can ask the following question: How does $\bar{\xi}(a, x)$ behave at highly nonlinear scales, or, equivalently, at any given scale at large a ?

To begin with, it is easy to see that we must have $v = -ax$ or $h = 1$ for sufficiently large $\bar{\xi}(a, x)$, if we assume that the evolution gets frozen in proper coordinates at highly nonlinear scales. Integrating equation (5) with $h = 1$, we obtain $\bar{\xi}(a, x) = a^3 F(ax)$; we shall call this phenomenon “stable clustering.” There are two points that need to be emphasized about stable clustering:

1. At present, there exists some evidence from simulations (Padmanabhan et al. 1996) that stable clustering does *not* occur in a $\Omega = 1$ model. In a *formal* sense, numerical simulations cannot disprove (or even prove, strictly speaking) the occurrence of stable clustering, because of the finite dynamic range of any simulation.

2. Theoretically speaking, the “naturalness” of stable clustering is often overstated. The usual argument is based on the assumption that at very small scales—corresponding to high nonlinearities—the structures are “expected to be” frozen at the proper coordinates. However, this argument does not take into account the fact that mergers are not negligible *at any scale* in an $\Omega = 1$ universe. In fact, stable clustering is more likely to be valid in models with $\Omega < 1$, a claim that seems again to be supported by simulations (Padmanabhan et al. 1996).

If stable clustering is valid, then the late-time behavior of $\bar{\xi}(a, x)$ cannot be independent of initial conditions. In other words, the two requirements—the validity of stable clustering at highly nonlinear scales and the independence of late-time behavior from initial conditions—are mutually exclusive. This is most easily seen for initial power spectra that are scale-free. If $P_{\text{in}}(k) \propto k^n$, so that $\bar{\xi}_L(a, x) \propto a^2 x^{-(n+3)}$, then it is easy to show that $\bar{\xi}(a, x)$, at

small scales, will vary as

$$\bar{\xi}(a, x) \propto a^{6/(n+5)} x^{-3(n+3)/(n+5)} \quad (\bar{\xi} \gg 200), \quad (17)$$

if stable clustering is true. Clearly, the power-law index in the nonlinear regime “remembers” the initial index. The same result holds for more general initial conditions. What does this result imply for the profiles of individual halos? To answer this question, let us start with the simple assumption that the density field $\rho(a, x)$ at late stages can be expressed as a superposition of several halos, each with some density profile; that is, we take

$$\rho(a, x) = \sum_i f(x - x_i, a), \quad (18)$$

where the i th halo is centered at x_i and contributes an amount $f(x - x_i, a)$ at the location x_i (We can easily generalize this equation to the situation in which there are halos with different properties, like core radius, mass, etc., by summing over the number density of objects with particular properties; we shall not bother to do this. At the other extreme, the exact description merely corresponds to taking the f s to be Dirac delta functions.) The power spectrum for the density contrast, $\delta(a, x) = (\rho/\rho_b - 1)$, corresponding to the $\rho(a, x)$ in equation (18), can be expressed as

$$P(\mathbf{k}, a) \propto [a^3 |f(\mathbf{k}, a)|]^2 \left| \sum_i \exp -i\mathbf{k} \cdot \mathbf{x}_i(a) \right|^2 \quad (19)$$

$$\propto [a^3 |f(\mathbf{k}, a)|]^2 P_{\text{cen}}(\mathbf{k}, a), \quad (20)$$

where $P_{\text{cen}}(\mathbf{k}, a)$ denotes the power spectrum of the distribution of centers of the halos.

If stable clustering is valid, then the density profiles of halos are frozen in proper coordinates, and we have $f(x - x_i, a) = f[a(x - x_i)]$. Hence, the Fourier transform will have the form $f(\mathbf{k}, a) = a^{-3} f(\mathbf{k}/a)$. On the other hand, the power spectrum at scales that participate in stable clustering must satisfy $P(\mathbf{k}, a) = P(\mathbf{k}/a)$. This is merely the requirement that $\bar{\xi}(a, x) = a^3 F(ax)$, reexpressed in Fourier space. From equation (20) it follows that we must have $P_{\text{cen}}(\mathbf{k}, a) = P_{\text{cen}}(\mathbf{k}/a)$. We can, however, take $P_{\text{cen}} = \text{constant}$ at sufficiently small scales. This is because we must necessarily have $P_{\text{cen}} \approx \text{constant}$ by definition, for length scales smaller than typical halo size, assuming that the halos do not overlap, when we are essentially probing the interior of a single halo at sufficiently small scales. We can relate the halo profile to the correlation function using equation (20). In particular, if the halo profile is a power law with $f \propto r^{-\epsilon}$, it follows that $\bar{\xi}(a, x)$ scales as $x^{-\gamma}$ (see also McClelland & Silk 1977; Sheth & Jain 1997), where

$$\gamma = 2\epsilon - 3. \quad (21)$$

Now if the correlation function scales as $x^{-3(n+3)/(n+5)}$, we see that the halo density profiles should be related to the initial power-law index through the relation

$$\epsilon = \frac{3(n+4)}{n+5}. \quad (22)$$

Clearly, the halos of highly virialized systems still “remember” the initial power spectrum.

Alternatively, without the help of the stable clustering hypothesis, one can try to “reason out” the profiles of the individual halos and use them to obtain the scaling relation for correlation functions. One of the favorite arguments used by cosmologists to obtain such a “reasonable” halo profile is based on spherical scale-invariant collapse. It

turns out that one can provide a series of arguments, based on spherical collapse, to show that, under certain circumstances, the *density profiles* at the nonlinear end scale as $x^{-3(n+3)/(n+5)}$. The simplest variant of this argument runs as follows: If we start with an initial density profile that is $r^{-\alpha}$, then scale-invariant spherical collapse will lead to a profile that goes as $r^{-\beta}$ with $\beta = 3\alpha/(1 + \alpha)$ (see, e.g., Padmanabhan 1996a, 1996b, and references therein). Taking the initial slope as $\alpha = (n + 3)/2$ will immediately give $\beta = 3(n + 3)/(n + 5)$. Our definition of stable clustering in § 2 is based on the scaling of the correlation function, and it gave the slope of $-3(n + 3)/(n + 5)$ for the *correlation function*. The spherical collapse gives the same slope for *halo profiles*. In this case, when the halos have a slope of $\epsilon = 3(n + 3)/(n + 5)$, the correlation function should have a slope

$$\gamma = \frac{3(n+1)}{n+5}. \quad (23)$$

Once again, the final state “remembers” the initial index n .

Is this conclusion true? Unfortunately, simulations do not have sufficient dynamic range to provide a clear answer, but there are some claims (see Navarro et al. 1996) that the halo profiles are “universal” and independent of initial conditions. The theoretical arguments given above are also far from rigorous (in spite of the popularity they seem to enjoy!). The argument for the correlation function’s scaling as $-3(n + 3)/(n + 5)$ is based on the assumption of $h = 1$ asymptotically, which may not be true. The argument, leading to density profiles scaling as $x^{-3(n+3)/(n+5)}$, is based on scale-invariant spherical collapse that does not do justice to nonradial motions. To illustrate the situations in which one may obtain final configurations that are independent of the initial index n , we shall discuss two possibilities.

As a first example, we will try to see when the slope of the correlation function is universal and obtain the slope of halos in the nonlinear limit using equation (21). Such an interesting situation can develop if we assume that h reaches a constant value asymptotically that is not necessarily unity. In that case we can integrate equation (5) to get $\bar{\xi}(a, x) = a^{3h} F(a^h x)$, where h now denotes the constant asymptotic value of the function. For an initial spectrum that is a scale-free power law with index n , this result translates to

$$\bar{\xi}(a, x) \propto a^{2\gamma/(n+3)} x^{-\gamma}, \quad (24)$$

where γ is given by

$$\gamma = \frac{3h(n+3)}{2+h(n+3)}. \quad (25)$$

We now notice that one can obtain a γ that is independent of the initial power-law index, provided that h satisfies the condition $h(n+3) = c$, a constant. In this case the nonlinear correlation function will be given by

$$\bar{\xi}(a, x) \propto a^{6c/[(2+c)(n+3)]} x^{-3c/(2+c)}. \quad (26)$$

The halo index will be independent of n and will be given by

$$\epsilon = 3 \left(\frac{c+1}{c+2} \right). \quad (27)$$

Note that we are now demanding that the asymptotic value of h depend explicitly on the initial conditions, although the *spatial* dependence of $\bar{\xi}(a, x)$ does not. In other words, the velocity distribution—which is related to h —still

“remembers” the initial conditions. This is indirectly reflected by the fact that the growth of $\bar{\xi}(a, x)$, represented by $a^{6c/[2+c(n+3)]}$, does depend on the index n .

We emphasize the fact that the velocity distribution remembers the initial condition, because it is usual (in published literature) to ignore the memory in velocity and to concentrate entirely on the correlation function. It is not clear to us (or, we suppose, to anyone else) whether it is possible to come up with a clustering scenario in which no physical feature remembers the initial conditions. This could probably occur when virialization has run its full course, but even then, it is not clear whether the particles that evaporate from a given potential well (and form a uniform hot component) will forget all the initial conditions.

As an example of the power of such a seemingly simple analysis, note the following: since $c \geq 0$, it follows that $\epsilon > 3/2$; invariant profiles with shallower indices (e.g., with $\epsilon = 1$) are not consistent with the evolution described above.

As our second example, we shall make an *Ansatz* for the halo profile and use it to determine the correlation function. We assume, based on small-scale dynamics, that the density profiles of individual halos should resemble that of isothermal spheres, with $\epsilon = 2$, irrespective of initial conditions. Converting this halo profile to the correlation function in the nonlinear regime is straightforward and is based on equation (21): if $\epsilon = 2$, we must have $\gamma = 2\epsilon - 3 = 1$ at small scales; that is, $\bar{\xi}(a, x) \propto x^{-1}$ in the nonlinear regime. Note that this corresponds to the critical index at the nonlinear end, $n_{\text{eff}} = n_c = -2$, for which the growth rate is a^2 , as in linear theory. [This a^2 growth, however, is possible for initial power-law spectra only if $\epsilon = 2$, i.e., $h(n+3) = 1$ at very nonlinear scales. Testing the conjecture that $h(n+3)$ is a constant is probably a little easier than looking for invariant profiles in the simulations, but the results are still uncertain.]

The corresponding analysis for the intermediate regime, with $1 \gtrsim \bar{\xi}(a, x) \gtrsim 200$, is more involved. This is clearly seen in equation (20), which shows that the power spectrum (and hence the correlation function) depends both on the Fourier transform of the halo profiles and on the power spectrum of the distribution of halo centers. In general, both quantities will evolve with time, and we cannot ignore the effect of $P_{\text{cen}}(k, a)$ and relate $P(k, a)$ to $f(k, a)$. The density profile around a *local maximum* will scale approximately as $\rho \propto \xi$, while the density profile around a *randomly chosen point* will scale as $\rho \propto \xi^{1/2}$. (The relation $\gamma = 2\epsilon - 3$ expresses the latter scaling of $\xi \propto \rho^2$.) There is, however, reason to believe that the intermediate regime (with $1 \gtrsim \bar{\xi} \gtrsim 200$) is dominated by the collapse of high peaks (Padmanabhan 1996a). In this case we expect the correlation function and the density profile to have the same slope in the intermediate regime, with $\bar{\xi}(a, x) \propto (1/x^2)$. Remarkably enough, this corresponds to the critical index $n_{\text{eff}} = n_c = -1$ for the intermediate regime, for which the growth is proportional to a^2 .

We thus see that if (i) the individual halos are isothermal spheres with a $1/x^2$ profile, and (ii) $\xi \propto \rho$ in the intermediate regime, and $\xi \propto \rho^2$ in the nonlinear regime, we end up with a correlation function that grows as a^2 at all scales. Such an evolution, of course, preserves the shape and is a good candidate for the late-stage evolution of the clustering.

While the above arguments are suggestive, they are far from conclusive. It is, however, clear from the above

analysis that it is not easy to provide unique theoretical reasoning regarding the shapes of the halos. The situation gets more complicated if we include the fact that all halos will not have the same mass, core radius, etc., and we have to modify our equations by integrating over the abundance of halos with given values of the latter. This brings in more ambiguities that depend on the assumptions that we make for each of these components (e.g., abundance for halos of a particular mass could be based on Press-Schechter or Peaks formalism), and the final results have no real significance. It is, therefore, better (and probably easier) to attack the question based upon the evolution equation for the correlation function, rather than from “physical” arguments for density profiles. This is what we shall do in the next section.

4. SELF-SIMILAR EVOLUTION

Since the above discussion motivates us to look for correlation functions of the form $\bar{\xi}(a, x) = a^2 L(x)$, we will start by asking a more general question: Does equation (5) possess self-similar solutions of the form

$$\bar{\xi}(a, x) = a^\beta F\left(\frac{x}{a^\alpha}\right) = a^\beta F(q), \quad (28)$$

where $q \equiv xa^{-\alpha}$? Defining $Q = \ln q = X - \alpha A$ and changing independent variables from (A, X) to (A, Q) , we can transform equation (5) to the form

$$\left(\frac{\partial \bar{\xi}}{\partial A}\right)_Q - (h + \alpha) \left(\frac{\partial \bar{\xi}}{\partial Q}\right)_A = 3(1 + \bar{\xi})h(\bar{\xi}). \quad (29)$$

Using the relations $(\partial \bar{\xi} / \partial A)_Q = \beta \bar{\xi}$ and $(\partial \bar{\xi} / \partial Q)_A = (\bar{\xi}/F)(dF/dQ)$, we can rewrite this equation as

$$\frac{\beta \bar{\xi} - 3(1 + \bar{\xi})h(\bar{\xi})}{[\alpha + h(\bar{\xi})]\bar{\xi}} = \frac{1}{F} \frac{dF}{dQ} \equiv K(Q). \quad (30)$$

The right-hand side of this equation depends only on Q and hence will vanish if differentiated with respect to A at constant Q . Imposing this condition on the left-hand side of equation (30), and noticing that it is a function of $\bar{\xi}(a, x)$, we obtain

$$\left(\frac{\partial \bar{\xi}}{\partial A}\right)_Q \frac{d}{d\bar{\xi}} = 0. \quad (31)$$

To satisfy this condition, we need either (i) $(\partial \bar{\xi} / \partial A)_Q = \beta \bar{\xi} = 0$, implying that $\beta = 0$, or (ii) the left-hand side of equation (30) to be a constant. Let us consider the two cases separately.

The simpler case (i) corresponds to $\beta = 0$, which implies that $\bar{\xi}(a, x) = F(Q)$. Setting $\beta = 0$ in equation (30), we get

$$\frac{d\bar{\xi}}{dQ} = -\frac{3(1 + \bar{\xi})h(\bar{\xi})}{\alpha + h(\bar{\xi})}, \quad (32)$$

which can be integrated in a straightforward manner to give a relation between $q = \exp Q$ and $\bar{\xi}$:

$$\begin{aligned} q &= q_0(1 + \bar{\xi})^{-1/3} \exp \left[-\frac{\alpha}{3} \int \frac{d\bar{\xi}}{(1 + \bar{\xi})h(\bar{\xi})} \right] \\ &= q_0(1 + \bar{\xi})^{-1/3} \gamma(\bar{\xi})^{-\alpha/2}. \end{aligned} \quad (33)$$

Given the form of $h[\bar{\xi}(a, x)]$, this equation can, in principle, be inverted to give $\bar{\xi}$ as a function of $q = xa^{-\alpha}$.

To understand when such a solution will exist, we should look at the limit of $\bar{\xi} \ll 1$. In this limit, when linear theory is

valid, we know that $h \approx \frac{2}{3}\bar{\xi}$ (see Peebles 1980). Using the latter in equation (33), we obtain the solution $\ln \bar{\xi} = -(2/\alpha) \ln q$, or

$$\bar{\xi} \propto q^{-2/\alpha} \propto x^{-2/\alpha} a^2 \propto a^2 x^{-(n+3)}, \tag{34}$$

with the definition $\alpha \equiv 2/(n + 3)$. This clearly shows that our solution is valid, *if and only if* the linear correlation function is a scale-free power law. In this case, of course, it is well known that solutions of the type $\bar{\xi}(a, x) = F(q)$, with $q = xa^{-2/(n+3)}$, exist. [Equation (33) gives the explicit form of the function $F(q)$.] This result shows that this is the only possibility. It should be noted that even though we have no explicit length scale in the problem, the function $\bar{\xi}(q)$, determined from the above equation, does exhibit different behavior at different scales of nonlinearity. Roughly speaking, the three regimes in equation (11) translate into nonlinear density contrasts in the ranges $\delta < 1$, $1 < \delta < 200$, and $\delta > 200$, and the function $\bar{\xi}(q)$ has different characteristics in these three regimes. This shows that gravity can intrinsically select out a density contrast of $\delta \approx 200$, which, of course, is well known from the study of spherical top-hat collapse.

Let us next consider the second possibility, viz., that the left-hand side of equation (30) is a constant. If the constant is denoted by μ , we obtain $F = F_0 q^\mu$ and

$$\beta \bar{\xi} - 3(1 + \bar{\xi})h(\bar{\xi}) = \mu \alpha \bar{\xi} + \mu h \bar{\xi}, \tag{35}$$

which can be rearranged to give

$$h = \frac{(\beta - \alpha\mu)\bar{\xi}}{3 + (\mu + 3)\bar{\xi}}. \tag{36}$$

This relation shows that solutions of the form $\bar{\xi}(a, x) = a^\beta F(x/a^\alpha)$ with $\beta \neq 0$ are possible only if $h[\bar{\xi}(a, x)]$ has the very specific form given by equation (36). In this form h is a monotonically increasing function of $\bar{\xi}(a, x)$. There is, however, firm theoretical and numerical evidence (Hamilton et al. 1991; Padmanabhan 1996a) to suggest that h increases with $\bar{\xi}(a, x)$ first, reaches a maximum, and then decreases. In other words, the h for actual gravitational clustering is not in the form that is suggested by equation (36). *We therefore conclude that solutions of the form of equation (28) with $\beta \neq 0$ cannot exist in gravitational clustering.*

[In stating the above conclusion, we are relying on the fact that numerical evidence is fairly strong against a monotonically increasing functional form for a two-point correlation function. This may not be a mathematical proof, but it will be very difficult to arrive at a monotonically increasing $h(\bar{\xi})$ in a realistic clustering scenario. Systems that turn around and collapse will tend to overshoot in any sensible gravitational potential. This suggests that the form of $h(\bar{\xi})$ will have at least one maximum value.]

With a similar analysis, we can prove a stronger result: there are no solutions of the form $\bar{\xi}(a, x) = \bar{\xi}[x/F(a)]$, except when $F(a) \propto a^2$. So, self-similar evolution in clustering is a very special situation.

This result, incidentally, has an important implication. It shows that power-law initial conditions are very special in gravitational clustering and may not represent generic behavior. This is because for power laws, we have a strong constraint that the correlations, etc., can only depend on $q = xa^{-2/(n+3)}$. For more realistic—non-power law—initial conditions, the shape can be distorted in a generic way during evolution.

The entire discussion thus far has been related to finding exact scaling solutions. It is, however, possible to find approximate scaling solutions that are of practical interest. Note that we normally expect constants like α, β, μ , etc., to be of order unity, while $\bar{\xi}(a, x)$ can take arbitrarily large values. If $\bar{\xi}(a, x) \gg 1$, then equation (36) shows that h is approximately a constant, with $h = (\beta - \alpha\mu)/(\mu + 3)$. In this case

$$\bar{\xi}(a, x) = a^\beta F(q) \propto a^\beta q^\mu \propto a^{(\beta - \alpha\mu)x^\mu} \propto a^{h(\mu+3)x^\mu}, \tag{37}$$

which has the form $\bar{\xi}(a, x) = a^{3h} F(a^h x)$, obtained earlier by directly integrating equation (5) with constant h . We shall say more about such approximate solutions in the next section.

5. UNITS OF THE NONLINEAR UNIVERSE

Having reached the conclusion that *exact* solutions of the form $\bar{\xi}(a, x) = a^2 G(x)$ are not possible, we will ask the question: Are there such *approximate* solutions? If so, what do they look like? We will see that such profiles, which we shall call “pseudolinear profiles,” and which evolve very close to the the above form, do indeed exist. In order to obtain such a solution and to check its validity, it is better to use the results of § 2.1 and to proceed as follows.

We are trying to find an approximate solution of the form $\bar{\xi}(a, x) = a^2 G(x)$ for equation (5). Since the linear correlation function $\bar{\xi}_L(a, x)$ does grow as a^2 at fixed x , continuity demands that $\bar{\xi}(a, x) = \bar{\xi}_L(a, x)$ for all a and x . [This can be proved more formally as follows: Let $\bar{\xi} = a^2 G(x)$ and $\bar{\xi}_L = a^2 G_1(x)$ for some range $x_1 < x < x_2$. Consider a sufficiently early epoch $a = a_i$ at which all the scales in the range (x_1, x_2) are described by linear theory, so that $\bar{\xi}(a_i, x) = \bar{\xi}_L(a_i, x)$. It follows that $G_1(x) = G(x)$ for all $x_1 < x < x_2$. Hence $\bar{\xi}(a, x) = \bar{\xi}_L(a, x)$ for all a in $x_1 < x < x_2$. By choosing sufficiently small a_i , we can cover any range (x_1, x_2) . So, $\bar{\xi} = \bar{\xi}_L$ for any arbitrary range. Q.E.D.] Since we have a formal relation (eq. [10]) between nonlinear and linear correlation functions, we should be able to determine the form of $G(x)$.

To do this we shall invert the form of the linear correlation function $\bar{\xi}_L(a, l) = a^2 G(l)$ and write $l = G^{-1}(a^{-2}\bar{\xi}_L) \equiv F(a^{-2}\bar{\xi}_L)$, where F is the inverse function of G . We also know that the linear correlation function $\bar{\xi}_L(a, l)$ at scale l can be expressed as $\mathcal{V}[\bar{\xi}(a, x)]$, in terms of the true correlation function $\bar{\xi}(a, x)$ at scale x , where

$$l = x[1 + \bar{\xi}(a, x)]^{1/3}. \tag{38}$$

So we can write

$$l = F\left[\frac{\bar{\xi}_L(a, l)}{a^2}\right] = F\left\{\frac{\mathcal{V}[\bar{\xi}(a, x)]}{a^2}\right\}. \tag{39}$$

But x can be expressed as $x = F[\bar{\xi}_L(a, x)/a^2]$; substituting this in equation (38), we have

$$l = F\left[\frac{\bar{\xi}_L(a, x)}{a^2}\right](1 + \bar{\xi})^{1/3}. \tag{40}$$

From our assumption, $\bar{\xi}_L(a, x) = \bar{\xi}(a, x)$; therefore, this relation can also be written as

$$l = F\left[\frac{\bar{\xi}(a, x)}{a^2}\right](1 + \bar{\xi})^{1/3}. \tag{41}$$

Equating the expressions for l in equations (39) and (41), we get an implicit functional equation for F :

$$F\left[\frac{\mathcal{V}(\bar{\xi})}{a^2}\right] = F\left(\frac{\bar{\xi}}{a^2}\right), \tag{42}$$

which can be rewritten as

$$\frac{F[\mathcal{V}(\bar{\xi})/a^2]}{F(\bar{\xi}/a^2)} = (1 + \bar{\xi})^{1/3}. \tag{43}$$

This equation should be satisfied by the function F , if we need to maintain the relation $\xi(a, x) = \bar{\xi}_L(a, x)$.

To see what this implies, note that the left-hand side should not vary with a at fixed $\bar{\xi}$. This is possible only if F is a power law:

$$F(\bar{\xi}) = A\bar{\xi}^m, \tag{44}$$

which in turn constrains the form of $\mathcal{V}(\bar{\xi})$ to be

$$\mathcal{V}(\bar{\xi}) = \bar{\xi}(1 + \bar{\xi})^{1/3m}. \tag{45}$$

Knowing the particular form for \mathcal{V} , we can compute the corresponding $h(\bar{\xi})$ from the relation

$$\frac{d \ln \mathcal{V}}{d \bar{\xi}} = \frac{2}{3} \frac{1}{(1 + \bar{\xi})h(\bar{\xi})}. \tag{46}$$

For the $\mathcal{V}(\bar{\xi})$ considered in equation (45), we obtain

$$h = \frac{2\bar{\xi}}{3 + (3 + 1/m)\bar{\xi}}, \tag{47}$$

which is the same result as obtained by letting $\beta = 2$ and $\alpha = 0$ in equation (28). We thus recover our old result—that *exact* solutions of the form $\bar{\xi}(a, x) = \bar{\xi}_L(a, x) = a^2 G(x)$ are *not* possible, because the correct $\mathcal{V}(\bar{\xi})$ and $h(\bar{\xi})$ do not have the forms in equations (45) and (47), respectively. But, as in the previous section, we can look for approximate solutions.

We note from equation (45) that for $\bar{\xi} \gg 1$, we have

$$\mathcal{V}(\bar{\xi}) = \bar{\xi}^{(1+1/3m)}, \quad F(\bar{\xi}) \propto \bar{\xi}^m, \quad \text{and} \quad G(\bar{\xi}) \propto \bar{\xi}^{1/m}. \tag{48}$$

This can be rewritten as

$$\mathcal{V}(\bar{\xi}) = \bar{\xi}^\nu, \quad F(\bar{\xi}) \propto \bar{\xi}^{1/3(\nu-1)}, \quad \text{and} \quad G(\bar{\xi}) \propto \bar{\xi}^{3/(\nu-1)}. \tag{49}$$

In other words, if $\mathcal{V}(\bar{\xi})$ can be approximated as $\bar{\xi}^\nu$, we have an approximate solution of the form

$$\bar{\xi}(a, x) = a^2 G(x) = a^2 x^{3(\nu-1)}. \tag{50}$$

Since \mathcal{V} in equation (12) is well approximated by the power laws in equation (11), so that

$$\mathcal{V}(\bar{\xi}) \propto \bar{\xi}^{1/3} \quad (1 \gtrsim \bar{\xi} \gtrsim 200), \tag{51}$$

$$\mathcal{V}(\bar{\xi}) \propto \bar{\xi}^{2/3} \quad (200 \gtrsim \bar{\xi}), \tag{52}$$

we can take $\nu = \frac{1}{3}$ in the intermediate regime and $\nu = \frac{2}{3}$ in the nonlinear regime. It follows from equations (49) that the approximate solution should have the form

$$F(\bar{\xi}) \propto \frac{1}{\sqrt{\bar{\xi}}} \quad (1 \gtrsim \bar{\xi} \gtrsim 200), \tag{53}$$

$$F(\bar{\xi}) \propto \frac{1}{\bar{\xi}} \quad (200 \gtrsim \bar{\xi}), \tag{54}$$

This gives the approximate form of a pseudolinear profile that will grow as a^2 at all scales.

There is another way of looking at this solution that is probably more physical and throws light on the scalings of pseudolinear profiles. We recall that in the study of finite gravitating systems made up of point particles and interacting via Newtonian gravity, isothermal spheres play an important role. They can be shown to be the local maxima of entropy (see Padmanabhan 1990), and hence, dynamical evolution drives the system toward a $1/x^2$ profile. Since one expects similar considerations to hold at small scales, during the late stages of the evolution of the universe, we may hope that isothermal spheres with a $1/x^2$ profile may still play a role in the late stages of evolution of clustering in an expanding background. However, while converting the profile to correlation, we have to take note of the issues discussed in § 2. In the intermediate regime, dominated by scale-invariant radial collapse (Padmanabhan 1996a), the density will scale as the correlation function, and we will have $\bar{\xi} \propto 1/x^2$. On the other hand, at the nonlinear end, we have the relation $\gamma = 2\epsilon - 3$ (see eq. [21]), which gives $\bar{\xi} \propto 1/x$ for $\epsilon = 2$. Thus, if isothermal spheres are the generic contributors, then we expect the correlation function to vary as $1/x$ at nonlinear scales, steepening to $1/x^2$ at intermediate scales. Further, since isothermal spheres are local maxima of entropy, a configuration like this should remain undistorted for a long duration. This argument suggests that a $\bar{\xi}$ that varies as $1/x$ at small scales and as $1/x^2$ at intermediate scales is likely to be a candidate for a pseudolinear profile. And we found that this is indeed the case.

To go from the scalings in the two limits given by equation (53) to an actual profile, we can use some fitting function. By making the fitting function sufficiently complicated, we can make the pseudolinear profile more exact. We shall, however, choose the simplest interpolation between the two limits and try the *Ansatz*

$$F(z) = \frac{A}{\sqrt{z}(\sqrt{z} + B)}, \tag{55}$$

where A and B are constants. Using the original definition, $l = F(\bar{\xi}_L/a^2)$, and the condition that $\bar{\xi} = \bar{\xi}_L$, we obtain

$$\frac{A}{\sqrt{\bar{\xi}/a^2}(\sqrt{\bar{\xi}/a^2} + B)} = l. \tag{56}$$

This relation implicitly fixes our pseudolinear profile. Solving for $\bar{\xi}$, we obtain

$$\bar{\xi}(a, x) = \left[\frac{Ba}{2} \left(\sqrt{1 + \frac{L}{x}} - 1 \right) \right]^2, \tag{57}$$

with $L = 4A/B^2$. Since this profile is not a pure power law, it will satisfy the equation (43) only approximately. We choose B such that the relation

$$F\left[\frac{\mathcal{V}(\bar{\xi})}{a^2}\right] = F\left(\frac{\bar{\xi}}{a^2}\right)(1 + \bar{\xi})^{1/3} \tag{58}$$

is satisfied to greatest accuracy at $a = 1$.

This approximate profile works reasonably well. Figures 1 and 2 show this result. In Figure 1 we have plotted the

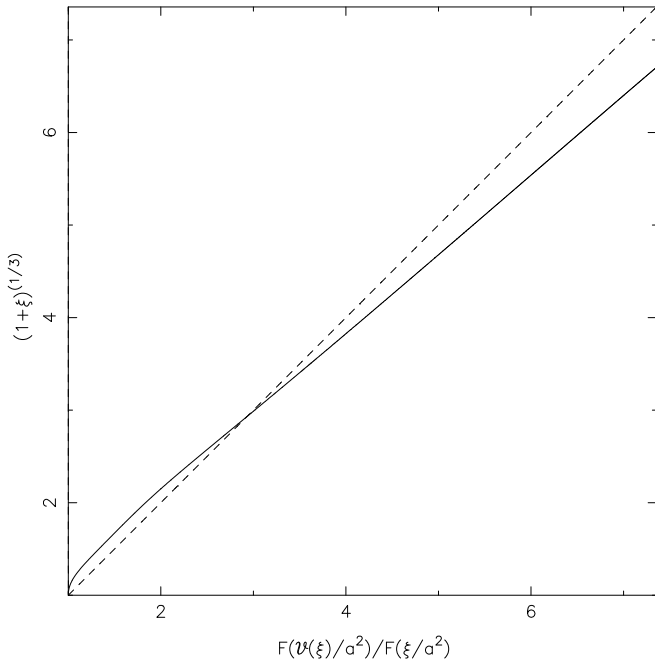


FIG. 1.—Approximate solution to the functional equation determining the pseudolinear profile. See text for discussion.

ratio $F[\mathcal{V}(\bar{\xi})/a^2]/F(\bar{\xi}/a^2)$ on the x -axis and the function $(1 + \bar{\xi})^{1/3}$ on the y -axis. If the function in equation (57) satisfied equation (43) exactly, we would get a 45° line in the figure, which is shown by a dashed line. The fact that our

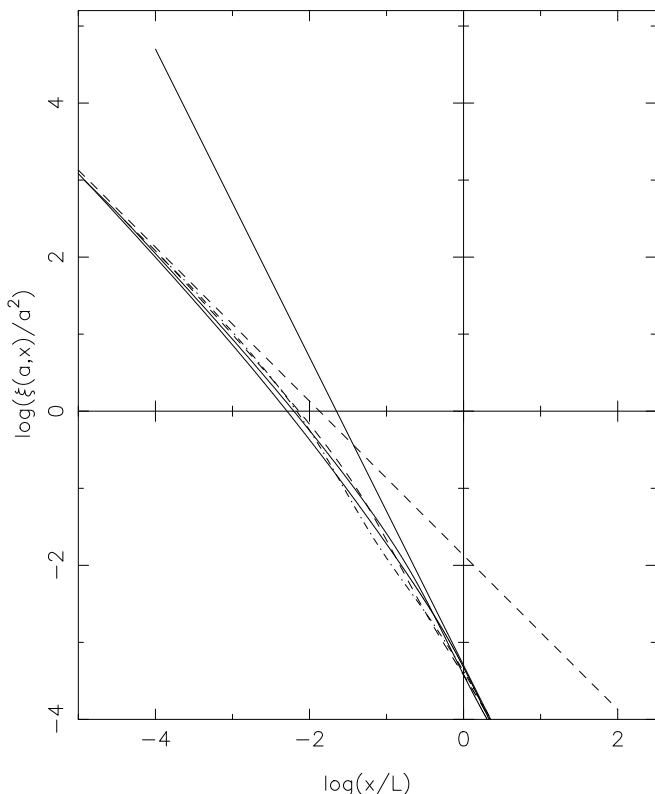


FIG. 2.—Dot-dashed, dashed, and two solid curves (upper curve, $a^2 = 100$; lower curve, $a^2 = 900$) for $a^2 = 1, 9, 100,$ and $900,$ respectively. The dashed straight line is of slope -1 and the solid straight line is of slope $-2,$ showing both the $1/x$ and $1/x^2$ regions of the profile.

curve is pretty close to this line shows that the *Ansatz* in equation (57) satisfies equation (43) fairly well. The optimum value of B chosen for this figure is $B = 38.6$. When a is varied from 1 to $10^3,$ the percentage of error between the 45° line and our curve is less than about 20% in the worst case. It is clear that our profile in equation (57) satisfies equation (58) quite well for a dynamic range of 10^6 in a^2 .

Figure 2 shows this result more directly. We evolve the pseudolinear profile from $a^2 = 1$ to 1000 using the NSR, and we plot $\bar{\xi}(a, x)/a^2$ against x . The dot-dashed, dashed, and two solid curves (upper curve, $a^2 = 100$; lower curve, $a^2 = 900$) are for $a^2 = 1, 9, 100,$ and $900,$ respectively. The overlap of the curves shows that the profile does grow approximately as a^2 . Also shown are lines of slope -1 (dotted) and -2 (solid); clearly, $\bar{\xi} \propto x^{-1}$ for small $x,$ and $\bar{\xi} \propto x^{-2}$ in the intermediate regime.

We emphasize the fact that we have chosen, in equation (57), the simplest kind of *Ansatz* combining the two regimes, and we have used only the two parameters A and B . It is quite possible to come up with more elaborate fitting functions that will solve our functional equation far more accurately, but we have not done so for two reasons. First, the fitting function in equation (11) for $\mathcal{V}(z)$ itself is approximate and is probably accurate only at the 10%–20% level. There have also been repeated claims in the literature that these functions have a weaker dependence on $n,$ which we have ignored for simplicity in this paper. Second, one must remember that only those $\bar{\xi}$ that correspond to positive definite $P(k)$ are physically meaningful. This happens to be the case for our choice (which can be verified by explicit numerical integration with a cutoff at large x), but it may not be true for arbitrarily complicated fitting functions.

If a fitting with greater accuracy is required, one can obtain it more directly from equation (16). Setting $n_a = 2$ in the equation predicts the instantaneous spatial slope of $\bar{\xi}(a, x)$ to be

$$\frac{\partial \ln \bar{\xi}(a, x)}{\partial \ln x} = \frac{2}{h[\bar{\xi}(a, x)]} - 3 \left[1 + \frac{1}{\bar{\xi}(a, x)} \right], \quad (59)$$

which can be integrated to give

$$\ln \frac{x}{L} = \int_{\bar{\xi}(L)}^{\bar{\xi}(x)} \frac{h d\bar{\xi}}{\bar{\xi}^2(2 - 3h) - 3h} \quad (60)$$

at $a = 1,$ with L being an arbitrary integration constant. Numerical integration of this equation will give a profile that varies as $1/x$ at small scales, goes to $1/x^2,$ and then to $1/x^3, 1/x^4, \dots,$ etc., with an asymptotic logarithmic dependence. In the regime $\bar{\xi}(a, x) > 1,$ this will give results that are reasonably close to our fitting function.

It should be noted that equation (43) reduces to an identity for any F in the limit $\bar{\xi} \rightarrow 0,$ since in this limit, $\mathcal{V}(z) \approx z$. This shows that we are free to modify our pseudolinear profile at large scales into any other form (essentially determined by the input linear power spectrum) without affecting any of our conclusions.

In order to avoid possible misunderstanding, we would like to emphasize the following fact. Equation (60) is obtained by demanding that the correlation function grow as a^2 at all scales at a given instant. It is, of course, possible to make very many other demands on the partial differential equation and to obtain different kinds of approximate solutions (similar to eq. [60]). Which of these solutions are physically relevant has to be determined by certain well-

motivated physical arguments. In our case we have made the demand that the correlation function grow as a^2 based on the following observation: for any realistic system, with the correlation function decreasing monotonically with scale, the large-scale part will be in the linear regime and will be growing as a^2 . If the correlation function has to preserve its shape, then it has to grow as a^2 at all scales. It is the behavior of the correlation function at the linear end, added to the requirement of shape invariance, that gives a special status to a solution like this.

Finally, we will discuss a different way of thinking about pseudolinear profiles that may be useful.

In studying the evolution of the density contrast $\delta(a, x)$, it is conventional to expand it in term of the plane wave modes as

$$\delta(a, x) = \sum_{\mathbf{k}} \delta(a, \mathbf{k}) \exp(i\mathbf{k} \cdot \mathbf{x}). \quad (61)$$

In that case the *exact* equation governing the evolution of $\delta(a, \mathbf{k})$ is given by (Peebles 1980):

$$\frac{d^2 \delta_{\mathbf{k}}}{da^2} + \frac{3}{2a} \frac{d\delta_{\mathbf{k}}}{da} - \frac{3}{2a^2} \delta_{\mathbf{k}} = \mathcal{A}, \quad (62)$$

where \mathcal{A} denotes the terms responsible for the nonlinear coupling between different modes. The expansion in equation (61) is, of course, motivated by the fact that in the linear regime, we can ignore \mathcal{A} , and each of the modes evolves independently. For the same reason, this expansion is not of much value in the highly nonlinear regime.

This prompts one to ask the question: Is it possible to choose some other set of basis functions $Q(\alpha, x)$, instead of $\exp i\mathbf{k} \cdot \mathbf{x}$, and to expand $\delta(a, x)$ in the form

$$\delta(a, x) = \sum_{\alpha} \delta_{\alpha}(a) Q(\alpha, x), \quad (63)$$

so that the nonlinear effects are minimized? Here α stands for a set of parameters describing the basis functions. This question is extremely difficult to answer, partly because it is ill posed. To make any progress, we first have to give meaning to the concept of “minimizing the effects of nonlinearity.” One possible approach that we would like to suggest is the following: We know that when $\delta(a, x) \ll 1$, then $\delta(a, x) \propto aF(x)$ for any arbitrary $F(x)$; that is, all power spectra grow as a^2 in the linear regime. In the intermediate and nonlinear regimes, no such general statement can be made. But it is conceivable that there exist certain *special* power spectra for which $P(\mathbf{k}, a)$ grows (at least approximately) as a^2 even in the nonlinear regime. For such a spectrum, the left-hand side of equation (62) vanishes (approximately); hence, the right-hand side should also vanish. *Clearly, such power spectra are affected least by nonlinear effects.* Instead of looking for such a special $P(\mathbf{k}, a)$, we can, equivalently, look for a particular form of $\xi(a, x)$ that evolves as closely to the linear theory as possible. Such correlation functions and corresponding power spectra (which are the pseudolinear profiles) must be capable of capturing most of the essence of nonlinear dynamics. In this sense we can think of our pseudolinear profiles as the basic building blocks of the nonlinear universe. The fact that the correlation function is closely related to isothermal spheres indicates a connection between local gravitational dynamics and large-scale gravitational clustering.

The analysis of the two-point correlation function does not, of course, imply anything regarding the higher order

correlation functions directly. It has been shown in Munshi & Padmanabhan (1997), however, that nonlinear scaling relations exist even for higher order correlation functions. In that case there is a faint hope that one may actually be able to check whether a BBGKY hierarchy admits simple scaling solutions that respect the pseudolinear profiles. Hopefully, we will be able to address this question in a future publication.

6. CONCLUSIONS

It seems reasonable to hope that the late-stage evolution of collisionless point particles, interacting via Newtonian gravity in an expanding background, should be understandable in terms of a simple paradigm. This paper (as the title implies) tries to realize this dream within some well-defined framework. It should be viewed as a tentative first step in a new direction that seems promising.

There are three key points that emerge from this analysis. The first is the fact that we have been able to find approximate correlation functions that evolve while preserving their shapes. We achieved this by looking at the structure of an exact equation that obeys certain nonlinear scaling relations. As we emphasized before, the existence of such a special class of solutions to the equations of gravitational dynamics is an important feature.

Second, we should take note of the role played by the “isothermal” profile $1/x^2$ in our solution. Such a profile can lead to correlation functions that vary as $1/x$ at small scales and as $1/x^2$ in the intermediate scales. If this profile is indeed “special,” then one expects it to lead to a pseudolinear profile for the correlation function. Our analysis shows that there is indeed good evidence for this feature. If one accepts this evidence, then the next level of enquiry would be to ask why $1/x^2$ profiles are “special.” In the statistical mechanics of gravitating systems, one can show that these profiles arise as end stages of violent relaxation that operates at dynamical timescales. Whether a similar reasoning holds in an expanding background, independent of the index for power spectrum, is open to question. This is an important issue, and we hope to address it fully in a future work. We emphasize that our equations, along with NSR, naturally lead to a pseudolinear profile, which can be interpreted and understood in terms of isothermal density profiles for halos. We did not have to assume anything a priori regarding the halo profiles.

In a more pragmatic way, one can understand the pseudolinear profile from the dependence of the rate of growth of the correlation function on the local slope. The NSRs suggest that ξ grows (approximately) as $a^{6/(n_{\text{eff}}+4)}$ in the intermediate regime and as $a^{6/(n_{\text{eff}}+5)}$ in the nonlinear regime. This scaling shows that $n_{\text{eff}} = -1$ grows as a^2 in the intermediate regime, and that $n_{\text{eff}} = -2$ grows as a^2 in the nonlinear regime. This is precisely the form that our pseudolinear profile has. Also, in the intermediate regime, the correlation grows faster than a^2 if $n_{\text{eff}} < -1$ and slower than a^2 if $n_{\text{eff}} > -1$. The net effect is, of course, to straighten out a curved correlation and to drive it to $n = -1$. A similar effect drives the correlations to $n = -2$ in the nonlinear regime (see Bagla & Padmanabhan 1997 for a more detailed discussion of this aspect in the intermediate regime). Of course, one still needs to understand the dependence of the growth rate on n_{eff} from more physical considerations in order to get the complete picture. We have not addressed here the question of the timescale over which

clustering can lead to the pseudolinear profile, even granting that it does so. This requires further study.

The last aspect concerns what one can achieve by using the pseudolinear profiles. In principle, one would like to build the nonlinear density field through a superposition of pseudolinear profiles, but this is a mathematically complex problem. As a first step, one should understand why the nonlinear term in equation (62) is subdominant for such a profile. This itself is complicated, since we have only fixed the power spectrum—but not the phases of the density

modes—while the nonlinear terms do depend on the phase. Again, we hope to investigate this issue further in a future work.

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