Newman-Janis algorithm revisited I: Wang-Wu functions and rotating solutions in General Relativity

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June 27, 2003

Abstract

In this paper an application of Newman-Janis algorithm in spherical symmetric metrics with the mass function $M(u, r)$ and the charge $e(u, r)$ has been discussed. After the transformation of the metric via this algorithm, these two functions $M(u, r)$ and $e(u, r)$ might be of the three variables u, r, θ. With these functions of three variables, all the Newman-Penrose (NP) spin coefficients, the Ricci as well as the Weyl scalars have been calculated and presented here general *rotating* solutions in NP formalism. From these NP quantities for *rotating* spherically symmetric metric, a class of *rotating* solutions of Einstein's field equations could be generated. These so-generated solutions may include (a) the rotating Kerr-Newman (known), (b) the *rotating* Vaidya solution, (c) *rotating* Vaidya-Bonnor solution, (d) *rotating* Husain's solution, (e) *rotating* Wang-Wu solutions. It is found that the technique of Wang and Wu is so powerful to generate solutions, that the *rotating* Vaidya solution obtained here could be combined smoothly with the *rotating* Kerr-Newman solution to generate *rotating* Kerr-Newman-Vaidya solution, and similarly, *rotating* Kerr-Newman-Vaidya-Bonnor solution of the field equations. It has also shown that the embedded universes, for example, Kerr-Newman de Sitter, *rotating* Vaidya-Bonnor-de Sitter, *rotating* Kerr-Newman-Vaidya-de Sitter might be derived from the *rotating* solutions with Wang-Wu function. All *rotating* embedded solutions derived here could be written in Kerr-Schild forms on embedded backgrounds, showing the extension of Xanthopoulos's theorem.

PACS number : 0420, 0420J, 0430, 0440N

1 Introduction

In an earlier paper [1] it is shown that Hawking's radiation [2] could be expressed in classical spacetime metrics, by considering the charge e to be function of the radial coordinate r of non-rotating Reissner-Nordstrom as well as rotating Kerr-Newman black holes. Since these two black holes describe the 'stationary' metrics, it has intended to search for 'non-stationary' rotating metrics in order to incorporate relativistic aspect of Hawking's radiation in general relativity. The *non-rotating* Vaidya metric is a 'non-stationary' generalization of Schwarzschild vacuum solution, describing the gravitational field of a null radiating star. Many attempts have been made to generate 'non-stationary' rotating metrics which describe the rotating external gravitational field of radiating bodies. To mention with, Vaidya and Patel [3] obtained a non-stationary rotating metric with mass $M = -m\{1 + u(b/a^2) - (b^2/4a^2)\cos^2\theta\}^{3/2}$ having minus sign, where m, a, b are constant and u is the retarded time coordinate. They claimed that their metric recovers the Kerr metric when $b = 0$. However, it is well known that the Kerr metric has no mass with negative sign. Carmeli and Kaye [4] have shown, by considering the mass M of Kerr metric directly as function of coordinate u, that the Kerr metric can be made a 'non-stationary' rotating metric, called variable mass Kerr metric. However, Herrera and Martinez [5], Herrera et. al. [6] argued that the interpretation of variable mass Kerr metric of Carmeli and Kaye is not completely clear. Gonzalez et.al.[7] have presented a non-stationary generalization of the Kerr-Newman metric, by allowing the three parameters a, m and e to be functions of coordinate u and shown that the variable $a(u)$ does not represent the rotating electromagnetic field of the Einstein-Maxwell equations and concluded that to take the parameters of a metric as functions of u does not generalize the solutions enough. Mallett [8] applied Newman-Janis algorithm to the Reissner-Nordstrom-de Sitter 'seed' solution to derive a rotating Kerr-Newman-de Sitter solution and afterward he introduced the retarded time coordinate u to get 'non-stationary' charged radiating metric. However, Xu [9] has commented that Mallett's result does not satisfy the Einstein's equations. Jing and Wang [10] consider the mass $M(u)$ and the charge $e(u)$ unchanged after the application of Newman-Janis algorithm $r = r' - i\,a\cos\theta$, and $u = u' + i\,a\cos\theta$, to the *non-rotating* Vaidya-Bonnor 'seed' solution with mass $M(u)$ and charge $e(u)$. In fact, after the application of the algorithm, the mass $M(u)$ and charge $e(u)$ should be functions $M(u,\theta)$, $e(u,\theta)$ of two variables u and θ. This situation could be seen in equations (2.10) and (3.3) of their paper. Because of these comments and arguments on these known 'non-stationary' rotating metrics cited in [3,4,8,10], it is not suggestible to utilize these metrics directly unless one personally verifies these results.

For this purpose, we employ the Newman-Janis algorithm [11] to generate 'rotating non-stationary' metrics from the spherically symmetric 'seed' metric with mass $M(u,r)$ and the charge $e(u,r)$, where u and r are the coordinates of the spacetime geometry. Newman-Janis algorithm [11] is a *complex coordinate transformation,* which has been introduced by Newman and Janis to obtain Kerr metric, a rotating Schwarzschild *vacuum* solution of Einstein's field equations from the non-rotating Schwarzschild 'seed' solution. In another paper Newman et.al. [12] again applied the same transformation to the *non-rotating* charged Reissner-Nordstrom solution to get *rotating* charged Reissner-Nordstrom solution, which is now commonly known as Kerr-Newman black hole solution in General Relativity. So this complex coordinate transformation is a powerful technique to derive *rotating* solutions from the *non-rotating* 'seed' solutions of Einstein's equations with spherical symmetric metrics. Herrera and Jimenez [13] applied the same Newman-Janis algorithm to an interior *non-rotating* spherically symmetric seed metric and the resulting *rotating* interior was tried to match with the exterior Kerr metric on the boundary of the source. Drake and Turolla [14] have generated a class of metrics as possible sources for the Kerr metric by applying the same algorithm to any static spherically symmetric 'seed' metric. Drake and Szekeres [15] have shown the uniqueness of this algorithm in generating the Kerr-Newman metric and proved that the only electrovac Petrov type D space-time generated by the algorithm with a vanishing Ricci scalar Λ is the Kerr-Newman space-time. Yazadjiev [16] has also shown that Sen's rotating dilation-axiom black-hole solution [17] can be derived from the static spherically symmetric dilation black hole solution via this algorithm too.

The purposes of this paper are

1. to apply the Newman-Janis algorithm to the spherical symmetric 'seed' metric with the mass $M(u,r)$ and the charge $e(u,r)$ of two variables u, r,

2. to calculate all the Newman-Penrose (NP) spin coefficients, the Ricci as well as the Weyl scalars in NP formalism [18] in general,

3. to generate possible *rotating* solutions of Einstein's field equations from the *non-rotating* 'seed' solutions.

The spherically symmetric metric with the mass and charge functions of two variables u, r has been transformed via Newman-Janis algorithm [11] to get *rotating* spherical symmetric metric. After the transformation, the mass and the charge may be functions of three variables u, r, θ. Then we calculate all the Newman-Penrose (NP) spin coefficients, the Ricci as well as the Weyl scalars in general. Accordingly, the Einstein's tensors as well as the energy momentum tensors (EMT) of the *rotating* fields have been presented in terms of complex null tetrad vectors. From this EMT one may observe the description of having two fluids system in the field equations. To visualize the two fluid system we rewrite the Einstein's field equations in terms of one unit time-like and three unit space-like vectors constructed from the complex null vectors.

Thus, we could generate *rotating* solutions mentioned in the abstract above from the Ricci as well as the Weyl scalars of the transformed metric. Consequently, some of the results are cited for ready reference in the form of theorems based on *rotating* solutions discussed here.

Theorem 1. *If $g_{ab}^{\rm KN}$ is the Kerr-Newman solution of Einstein's field equations and ℓ_a is geodesic, shear free, rotating and expanding null vector and one of the double repeated principal null directions of the Weyl tensor of $g_{ab}^{\rm KN}$, then $g_{ab}^{\rm KNV} = g_{ab}^{\rm KN} + 2Q(u,r,\theta)\,\ell_a\,\ell_b$ would be a rotating Kerr-Newman-Vaidya solution with $Q(u,r,\theta) = -r\,f(u)\,R^{-2}$, where $f(u)$ is the mass function of rotating Vaidya solution.*

Theorem 2. *If $g_{ab}^{\rm V}$ is a rotating Vaidya solution of Einstein's field equations and ℓ_a is geodesic, shear free, rotating and expanding null vector of $g_{ab}^{\rm V}$, then $g_{ab}^{\rm KNV} = g_{ab}^{\rm V} + 2Q(r,\theta)\,\ell_a\,\ell_b$ would be a rotating Kerr-Newman-Vaidya solution with $Q(r,\theta) = -(r\,m - e^2/2)\,R^{-2}$, where m and e are constant and represent the mass and the charge of Kerr-Newman black hole.*

Theorem 3. *If $g_{ab}^{\rm dS}$ is the rotating de Sitter solution of Einstein's field equations and ℓ_a is geodesic, shear free, rotating and expanding null vector and one of the double repeated principal null directions of the Weyl tensor of $g_{ab}^{\rm dS}$, then $g_{ab}^{\rm KNdS} = g_{ab}^{\rm dS} + 2Q(r,\theta)\,\ell_a\,\ell_b$ would be a rotating Kerr-Newman-de Sitter solution with $Q(r,\theta) = -(r\,m - e^2/2)\,R^{-2}$, where m and e are constant and represent the mass and the charge of Kerr-Newman black hole.*

Theorem 4. *If $g_{ab}^{\rm KN}$ is a rotating Kerr-Newman solution of Einstein's field equations and ℓ_a is geodesic, shear free, rotating and expanding null vector of $g_{ab}^{\rm KN}$, then $g_{ab}^{\rm KNdS} = g_{ab}^{\rm KN} + 2Q(r,\theta)\,\ell_a\,\ell_b$ would be a rotating Kerr-Newman-de Sitter solution with $Q(r,\theta) = -(\Lambda^* r^4/6)R^{-2}$, where Λ^* is the de Sitter cosmological constant.*

Theorem 5. *All rotating stationary spherically symmetric solutions based on Newman-Janis algorithm are Petrov type D whose one of the repeated null vectors ℓ_a is geodesic, shear free, expanding as well as non-zero rotation.*

Theorem 6. *All rotating non-stationary spherically symmetric solutions, derivable from the application of Newman-Janis algorithm and possessing a geodesic, shear free, expanding and rotating null vector ℓ_a, are algebraically special in the Petrov classification.*

Theorem 2 is an alternative theorem of theorem 1. Theorem 2 may interpret as the Kerr-Newman black hole embedded into the rotating Vaidya null radiating background, describing Kerr-Newman-Vaidya black hole. Similarly, theorem 3 states that Kerr-Newman black hole may also be embedded into the de Sitter cosmological background, describing Kerr-Newman-de Sitter black hole. Its alternative form would be the theorem 4. The extension of these theorem 3 and 4 in *non-stationary* version might be stated in the case of *rotating* Vaidya-Bonnor-de Sitter solution. Proofs of these theorems may be found latter in the paper. Theorems 5 and 6 may follow from the *stationary* as well as *non-stationary* rotating solutions to be discussed in the next sections.

This paper is organized as follows: Section 2 presents a brief application of Newman-Janis algorithm to a spherically symmetric 'seed' metric with the mass $M(u,r)$ and the charge $e(u,r)$. A general expressions of NP quantities with $M(u,r,\theta)$ and $e(u,r,\theta)$ are cited for further use in section 3. The general properties of the *rotating* spherically symmetric metric is discussed after observing the nature of the NP quantities. These NP quantities can be used to generalize any known solutions written in NP formalism within the limitation of Newman-Janis algorithm. Section 4 discusses the general properties of energy momentum tensor admitted by the transformed metric. Section 5 proves the usefulness of NP quantities presented in section 3 by generating rotating 'non-stationary' solutions. In section 6 we introduce the Wang-Wu function in the rotating general solutions and it is shown that the general rotating solutions with Wang-Wu function are so powerful to generate rotating new solutions which could not generate by Newman-Janis algorithm alone. The conclusion of the paper is cited in section 7 with suggestions and remarks of the solutions discussed in the earlier sections.

The presentation of this paper is essentially based on the Newman-Penrose (NP) spin-coefficient formalism [18]. The NP quantities are calculated through the technique developed by McIntosh and Hickman [19] in (+,−,−,−) signature.

2 Newman-Janis algorithm

For application of Newman-Janis algorithm, we start with a spherical symmetric 'seed' metric written in the form

$$ds^2 = e^{2\phi}\, du^2 + 2du\, dr - r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{2.1}$$

where $e^{2\phi} = 1 - 2M(u,r)/r + e^2(u,r)/r^2$ and the coordinate chosen are $\{x^1, x^2, x^3, x^4\}$ $=\{u, r, \theta, \phi\}$. The u-coordinate is related to the retarded time in flat space-time. So u-constant surfaces are null cones open to the future. The r-constant is null coordinate. The θ and ϕ are usual angle coordinates. The retarded time coordinate are used to evaluate the radiating (or outgoing) energy momentum tensor around the astronomical body [9]. Here the mass M and the charge e are the functions of the retarded time coordinate u and the radial coordinate r. Initially, when M, e are constant, this metric provides the non-rotating Reissner-Nordstrom solution and also when both M, e are functions of u, it becomes the non-rotating Vaidya-Bonnor solution. The contravariant components of the metric (2.1) are

$$g^{ab} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -e^{2\phi}/r^2 & 0 & 0 \\ 0 & 0 & -1/r^2 & 0 \\ 0 & 0 & 0 & -1/r^2\sin^2\theta \end{pmatrix}, \quad a, b = 1, 2, 3, 4. \tag{2.2}$$

These metric components may be expressed in terms of complex null tetrad [18]

$$\ell^a = \delta^a_2,$$

$$n^a = \delta^a_1 - \frac{1}{2}(1 - \frac{2M}{r} + \frac{e^2}{r^2})\,\delta^a_2, \tag{2.3}$$

$$m^a = \frac{1}{\sqrt{2}r}\,(\delta^a_3 + \frac{i}{\sin\theta}\,\delta^a_4)$$

$$\overline{m}^a = \frac{1}{\sqrt{2}r}\,(\delta^a_3 - \frac{i}{\sin\theta}\,\delta^a_4).$$

Then the metric tensor g^{ab} of the line element (2.1) is expressed in these null tetrad vectors as

$$g^{ab} = \ell^a\,n^b + n^a\,\ell^b - m^a\,\overline{m}^b - \overline{m}^a\,m^b. \tag{2.4}$$

Here the null vectors ℓ^a and n^a are real, and m^a and $\overline{m}^a$ are complex conjugates of each other. According to Newman and Janis [11], one may complexify the coordinate r and u by the following transformation

$$r = r' - i\,a\cos\theta,\ u = u' + i\,a\cos\theta,\ \theta = \theta',\ \phi = \phi'. \tag{2.5}$$

This complexification may only be done by considering r' and u' real. During the transformation, we have used the substitution $d\theta = -i\sin\theta\,d\phi$ when required. Then the covariant complex null tetrad vectors take the forms

$$\ell_a = \delta^1_a - a\sin^2\theta\,\delta^4_a,$$

$$n_a = \frac{1}{2}\,H(u,r,\theta)\,\delta^1_a + \delta^2_a - \frac{1}{2}\,H(u,r,\theta)\,a\sin^2\theta\,\delta^4_a,$$

$$m_a = -\frac{1}{\sqrt{2}R}\,\big\{ - ia\sin\theta\,\delta^1_a + R^2\,\delta^3_a + i(r^2+a^2)\sin\theta\,\delta^4_a\big\}, \tag{2.6}$$

$$\overline{m}_a = -\frac{1}{\sqrt{2}\overline{R}}\,\big\{ia\sin\theta\,\delta^1_a + R^2\,\delta^3_a - i(r^2+a^2)\sin\theta\,\delta^4_a\big\}.$$

where $R = r + ia\cos\theta$, $\overline{R} = r - ia\cos\theta$ and $R^2 = r^2 + a^2\cos^2\theta$. The null tetrad vectors chosen here are *different* from those chosen in [12], but are similar to those given in Chandrasekhar [20]. Now, after the transformation (2.5), the mass M and the charge e of the body must be of the three variables u, r, θ, however the old ones had explicitly u and r dependence. That is,

$$H(u,r,\theta) = 1 - \frac{2rM(u,r,\theta)}{R^2} + \frac{e^2(u,r,\theta)}{R^2} + \frac{a^2\sin^2\theta}{R^2}. \tag{2.7}$$

All the primes are being dropped for convenience of notation. The line element of the *rotating* spherically symmetric metric would be of the form

$$\begin{aligned} ds^2 &= e^{2\phi}\, du^2 + 2du\, dr + 2a\sin^2\theta(1-e^{2\phi})\, du\, d\phi \\ &\quad -2a\sin^2\theta\, dr\, d\phi - R^2 d\theta^2 - \{R^2 - a^2\sin^2\theta\,(e^{2\phi}-2)\}\sin^2\theta\, d\phi^2, \end{aligned} \tag{2.8}$$

where $e^{2\phi} = 1 - 2rM(u,r,\theta)/R^2 + e^2(u,r,\theta)/R^2$.

Then the *rotating* spherically symmetric line element has the covariant components of the metric tensor g_{ab}:

$$g_{ab} = \begin{pmatrix} e^{2\phi} & 1 & 0 & a\sin^2\theta(1-e^{2\phi}) \\ 1 & 0 & 0 & -a\sin^2\theta \\ 0 & 0 & -R^2 & 0 \\ a\sin^2\theta\,(1-e^{2\phi}) & -a\sin^2\theta & 0 & -\{R^2 - a^2\sin^2\theta(e^{2\phi}-2)\}\sin^2\theta \end{pmatrix} \tag{2.9}$$

$a, b = 1,2,3,4$. This completes the application of Newman-Janis algorithm to the spherically symmetric 'seed' metric (2.1). The usefulness of this transformed metric (2.8) would be discussed in the following sections.

3 NP quantities for the rotating metric

In this section we could derive the general NP spin coefficients, the Ricci scalars and the Weyl scalars for the spherically symmetric metric (2.8) and present the general properties of the metric after observing the conditions of these NP quantities. First, the basis one-form of the tetrad vectors (2.6) are given below:

$$\begin{aligned} \theta^1 &\equiv n_a\, dx^a = \frac{1}{2}H\, du + dr - \frac{1}{2}\, a\, H\sin^2\theta\, d\phi, \\ \theta^2 &\equiv \ell_a\, dx^a = du - a\sin^2\theta\, d\phi, \\ \theta^3 &\equiv -\overline{m}_a\, dx^a = \frac{1}{\sqrt{2}\overline{R}}\Big\{ia\sin\theta\, du + R^2\, d\theta - i(r^2+a^2)\, d\phi\Big\}, \\ \theta^4 &\equiv -m_a\, dx^a = \frac{1}{\sqrt{2}\, R}\Big\{-ia\sin\theta\, du + R^2\, d\theta + i(r^2+a^2)\, d\phi\Big\}. \end{aligned} \tag{3.1}$$

The intrinsic derivative operators for the metric (2.8) take the following forms:

$$D \equiv \ell^a\, \partial_a = \partial_r,$$

$$\Delta \equiv n^a\, \partial_a = \frac{r^2+a^2}{R^2}\, \partial_u - \frac{H}{2}\, \partial_r + \frac{a}{R^2}\partial_\phi,$$

$$\delta \equiv m^a\,\partial_a = \frac{1}{\sqrt{2}\,R}\Big\{ia\sin\theta\,\partial_u + \partial_\theta + \frac{i}{\sin\theta}\,\partial_\phi\Big\}, \tag{3.2}$$

$$\overline{\delta} \equiv \overline{m}^a\,\partial_a = \frac{1}{\sqrt{2}\,\overline{R}}\Big\{-ia\sin\theta\,\partial_u + \partial_\theta - \frac{i}{\sin\theta}\,\partial_\phi\Big\}.$$

By taking the exterior derivative of basis one-forms (3.1), one may obtained the spin coefficients from the Cartan's equations of structure written in Newman-Penrose *spin coefficients* [19]:

$$\kappa = \sigma = \lambda = \epsilon = 0,$$

$$\rho = -\frac{1}{\overline{R}}, \qquad \mu = -\frac{H(u,r,\theta)}{2\overline{R}},$$

$$\alpha = \frac{(2ai - R\cos\theta)}{2\sqrt{2}\overline{R}\,R\sin\theta}, \qquad \beta = \frac{\cot\theta}{2\sqrt{2}R},$$

$$\pi = \frac{i\,a\sin\theta}{\sqrt{2}\overline{R}\,\overline{R}}, \qquad \tau = -\frac{i\,a\sin\theta}{\sqrt{2}R^2}, \tag{3.3}$$

$$\gamma = \frac{1}{\sqrt{2}\,\overline{R}\,R^2}\Big[(r - M - r\,M_{,r} + e\,e_{,r})\,\overline{R} - \Delta^*\Big],$$

$$\nu = \frac{1}{\sqrt{2}\,\overline{R}\,R^2}\Big[i\,a\sin\theta(r\,M_{,u} - e\,e_{,u}) - (r\,M_{,\theta} - e\,e_{,\theta})\Big],$$

where $\Delta^* = r^2 - 2rM(u,r,\theta) + a^2 + e^2(u,r,\theta)$ and the function $H(u,r,\theta)$ is given in (2.7). From these NP spin coefficients we could conclude that the transformed metric (2.8) with the mass and the charge of three variables u, r, θ possesses, in general, a geodesic ($\kappa = \epsilon = 0$), shear free ($\sigma = 0$), expanding ($\theta \neq 0$) and rotating ($\omega^{*2} \neq 0$) null vector ℓ_a [20] where

$$\theta \equiv -\frac{1}{2}(\rho + \overline{\rho}) = \frac{r}{R^2}, \tag{3.4}$$

$$\omega^{*2} \equiv -\frac{1}{4}(\rho - \overline{\rho})^2 = -\frac{a^2\cos^2\theta}{R^2\,R^2}. \tag{3.5}$$

Further we calculate the Weyl scalars:

$$\psi_0 = \psi_1 = 0,$$

$$\psi_2 = \frac{1}{\overline{R}\overline{R}\,R^2}\Big[(-RM + e^2) + \overline{R}\,(rM_{,r} - ee_{,r}) + \frac{1}{6}\overline{R}\overline{R}\Big(-2M_{,r} - rM_{,rr} + e^2_{,r} + ee_{,rr}\Big)\Big],$$

$$\psi_3 = \frac{-1}{2\sqrt{2}\overline{R}R\,R^2}\Big[4\big\{i\,a\sin\theta\,(rM_{,u}-ee_{,u})-(rM_{,\theta}-ee_{,\theta})\big\}$$
$$+\overline{R}\big\{i\,a\sin\theta(r\,M_{,u}-e\,e_{,u})_{,r}-(r\,M_{,\theta}-e\,e_{,\theta})_{,r}\big\}\Big], \qquad (3.6)$$

$$\psi_4 = \frac{1}{2\overline{R}\,\overline{R}\,R^2}\Big[a^2\sin^2\theta\,(rM_{,u}-ee_{,u})_{,u}+2ia\sin\theta\,(rM_{,u}-ee_{,u})_{,\theta}-(rM_{,\theta}-ee_{,\theta})_{,\theta}\Big]$$
$$-\frac{r\,a^2\sin^2\theta}{\overline{R}\,\overline{R}\,R^2\,R^2}(rM_{,u}-ee_{,u})-\frac{2\,r\,a\sin^2\theta-R^2\cos\theta}{2\overline{R}\,\overline{R}\,R^2\,R^2\sin\theta}(r\,M_{,\theta}-e\,e_{,\theta}).$$

The non-vanishing of the Weyl scalars ($\psi_2 \neq \psi_3 \neq \psi_4 \neq 0$) means that the metric (2.8) is an 'algebraically special' in the Petrov classification. In the expression of ψ_2 it is found in general that there is no differential terms of $M(u,r,\theta)$ and $e(u,r,\theta)$ with respect to u and θ. This leads that for a static 'rotating' metric with the mass $M(r)$ and the charge $e(r)$, the spacetime metric may be a Petrov type D ($\psi_2 \neq 0, \psi_3 = \psi_4 = 0$), whose one of the repeated principal null vectors would be a geodesic, shear free, expanding (3.4) and rotating (3.5) vector ℓ_a.

The Ricci scalars of the metric (2.8) are obtained as follows:

$$\phi_{00} = \phi_{01} = \phi_{10} = \phi_{20} = \phi_{02} = 0,$$

$$\phi_{11} = \frac{1}{4\,R^2\,R^2}\Big[2\,e^2+4r(r\,M_{,r}-ee_{,r})+R^2\Big(-2M_{,r}-r\,M_{,r\,r}+e_{,r}^2+ee_{,r\,r}\Big)\Big],$$

$$\phi_{12} = \frac{1}{2\sqrt{2}\,R^2\,R^2}\Big[i\,a\sin\theta\big\{(R\,M_{,u}-2\,e\,e_{,u})-(r\,M_{,r}-e\,e_{,r})_{,u}\overline{R}\big\}$$
$$+\big\{(R\,M_{,\theta}-2\,e\,e_{,\theta})-(r\,M_{,r}-e\,e_{,r})_{,\theta}\overline{R}\big\}\Big],$$

$$\phi_{21} = -\frac{1}{2\sqrt{2}\,R^2\,R^2}\Big[i\,a\sin\theta\big\{(\overline{R}\,M_{,u}-2\,e\,e_{,u})-(r\,M_{,r}-e\,e_{,r})_{,u}\,R\big\}$$
$$+\big\{(\overline{R}\,M_{,\theta}-2\,e\,e_{,\theta})-(r\,M_{,r}-e\,e_{,r})_{,\theta}\,R\big\}\Big], \qquad (3.7)$$

$$\phi_{22} = -\frac{1}{2\,R^2\,R^2}\Big[2r(r\,M_{,u}-e\,e_{,u})-\cot\theta(rM_{,\theta}-e\,e_{,\theta})$$
$$+a^2\sin^2\theta(r\,M_{,u}-e\,e_{,u})_{,u}-(rM_{,\theta}-e\,e_{,\theta})_{,\theta}\Big],$$

$$\Lambda = \frac{1}{12\,R^2}\Big(2M_{,r}+r\,M_{,r\,r}-e_{,r}^2-ee_{,r\,r}\Big).$$

Here we observe that the general expressions of ϕ_{11} and Λ do not involve any differential terms with respect to u and θ, although the mass and the charge of the rotating spherically symmetric stars (2.8) are functions of three variables u, r, θ. The vanishing of ϕ_{00} suggests the possibility that the transformed metric (2.8) may not include the *perfect fluid* $T_{ab} = (\rho^* + p) u_a u_b - p\, g_{ab}$ as $\phi_{00} = 2\phi_{11} = \phi_{22} = -K(\rho^* + p)/4$, $\Lambda = K(3p - \rho^*)/24$ with a time-like vector $u^a = (\ell^a + n^a)/\sqrt{2}$ [21,22]. It is also worth mentioning that for a static rotating metric with $M(r)$ and the charge $e(r)$, the Ricci scalars ϕ_{12} and ϕ_{22} will be vanished.

Then the Einstein's tensor is computed from these Ricci scalars (3.7) as follows

$$\begin{aligned} G_{ab} &= -2\,\phi_{22}\,\ell_a\,\ell_b - 4\,\phi_{11}\,\{\ell_{(a}\,n_{b)} + m_{(a}\overline{m}_{b)}\} - 6\,\Lambda\,g_{ab} \\ &\quad +4\,\phi_{12}\,\ell_{(a}\,\overline{m}_{b)} + 4\,\phi_{21}\,\ell_{(a}\,m_{b)}, \end{aligned} \tag{3.8}$$

where $2\ell_{(a}\,n_{b)} = \ell_a\,n_b + n_a\,\ell_b$. For *non-rotating fields* $(a = 0)$, the Ricci scalars ϕ_{12}, ϕ_{21} would vanish and this Einstein's tensor might reduce to that presented by Glass and Krisch [23]. From the Einstein's equations:

$$G_{ab} \equiv R_{ab} - \frac{1}{2} R\, g_{ab} = -K\, T_{ab} \tag{3.9}$$

we obtain the null density μ^*, the matter density ρ^*, the pressure p as well as the rotation function ω as

$$K\,\mu^* = 2\,\phi_{22}, \quad K\,\omega = -2\,\phi_{12}, \quad K\,\rho^* = 2\,\phi_{11} + 6\,\Lambda, \quad K\,p = 2\,\phi_{11} - 6\,\Lambda \tag{3.10}$$

where the Ricci scalars ϕ_{11}, ϕ_{12}, ϕ_{22}, Λ are given in (3.7).

To have the two-rotating fluid description we might introduce a time-like unit vector u^a and three unit space-like vectors v^a, w^a, z^a such that

$$u_a = \frac{1}{\sqrt{2}}(\ell_a + n_a), \quad v_a = \frac{1}{\sqrt{2}}(\ell_a - n_a), \tag{3.11}$$

$$w_a = \frac{1}{\sqrt{2}}(m_a + \overline{m}_a), \quad z_a = -\frac{i}{\sqrt{2}}(m_a - \overline{m}_a) \tag{3.12}$$

with the normalization conditions $u_a\,u^a = 1$, $v_a\,v^a = w_a\,w^a = z_a\,z^a = -1$. Then the explicit form of these unit vectors may be written as follows

$$u_a = \frac{1}{\sqrt{2}}\Big\{(1 + \frac{1}{2}H)\,\delta^1_a + \delta^2_a - (1 + \frac{1}{2}H)\,a\sin^2\theta\,\delta^4_a\Big\},$$

$$v_a = \frac{1}{\sqrt{2}}\Big\{(1 - \frac{1}{2}H)\,\delta^1_a - \delta^2_a - (1 - \frac{1}{2}H)\,a\sin^2\theta\,\delta^4_a\Big\},$$

$$w_a = -\frac{1}{R^2}\Big\{-a^2\sin\theta\cos\theta\,\delta^1_a + r\,R^2\delta^3_a + a\,(r^2 + a^2)\sin\theta\cos\theta\,\delta^4_a\Big\},$$

$$z_a = \frac{1}{R^2}\Big\{a\,r\sin\theta\,\delta^1_a + a\cos\theta\,R^2\delta^3_a - r\,(r^2 + a^2)\sin\theta\,\delta^4_a\Big\},$$

The metric tensor g_{ab} may be expressed in these unit vectors

$$g_{ab} = u_a\, u_b - v_a\, v_b - w_a\, w_b - z_a\, z_b. \tag{3.13}$$

Thus the Einstein's equations are written in *two-fluid* system

$$\begin{aligned} G_{ab} &= -K[\mu^*\,\ell_a\,\ell_b + \rho^*\,(u_a\,u_b - v_a\,v_b) + p\,(w_a\,w_b + z_a\,z_b) \\ &\quad +(\omega+\overline{\omega})\,\{u_{(a}\,w_{b)} + v_{(a}\,w_{b)}\} - i(\omega-\overline{\omega})\,\{u_{(a}\,z_{b)} + v_{(a}\,z_{b)}\}], \end{aligned} \tag{3.14}$$

where μ^*, ρ^*, p and ω are related with the Ricci scalars given in (3.7) as:

$$\begin{aligned} K\,\mu^* &= -\frac{1}{R^2\,R^2}\Big\{2r(r\,M_{,u} - e\,e_{,u}) - \cot\theta(r M_{,\theta} - e\,e_{,\theta}) \\ &\quad +a^2\sin^2\theta(r\,M_{,u} - e\,e_{,u})_{,u} - (r M_{,\theta} - e\,e_{,\theta})_{,\theta}\Big\} \end{aligned}$$

$$K\,\rho^* = \frac{1}{R^2\,R^2}\Big\{e^2 + 2\,r(r\,M_{,r} - e\,e_{,r})\Big\}$$

$$K\,p = \frac{1}{R^2\,R^2}\Big\{e^2 + 2\,r(r\,M_{,r} - e\,e_{,r}) - R^2(2M_{,r} + r\,M_{,r\,r} - e_{,r}^2 - e\,e_{,r\,r})\Big\}, \tag{3.15}$$

$$\begin{aligned} K\,\omega &= -\frac{1}{\sqrt{2}\,R^2\,R^2}\Big[i\,a\sin\theta\Big\{(R\,M_{,u} - 2\,e\,e_{,u}) - (r\,M_{,r} - e\,e_{,r})_{,u}\overline{R}\Big\} \\ &\quad +\Big\{(R\,M_{,\theta} - 2\,e\,e_{,\theta}) - (r\,M_{,r} - e\,e_{,r})_{,\theta}\overline{R}\Big\}\Big] \end{aligned}$$

The expression of null radiation density μ^* involves the derivative of the mass $M(u, r, \theta)$ and the charge $e(u, r, \theta)$ with respect to u and θ. Those of ρ^* and p are with respect to r only. However, the expression of the rotation function ω is involved the derivative of mass and the charge with respect to three variables u, r, θ. From the above equations, it is observed that the Einstein's tensor (3.14) of the *rotating* string fluid would reduce to those of Glass and Krisch [23] of *non-rotating* string fluid when $a = e = 0$ and $M = M(u, r)$. In the non-rotating Vaidya-type radiation null fluid, the null density μ^* takes the form $\mu^* = -2M_{,u}/Kr^2$. This shows that μ^* is always negative, since $\partial M/\partial u$ is positive [24].

4 Stress-energy tensor and energy conditions

From the Einstein tensor (3.9) and the relations (3.10) of Ricci scalars with μ^*, ρ^*, p, ω we could introduce the total energy momentum tensor (EMT) for a *rotating string fluid* as follows:

$$\begin{aligned} T_{ab} &= T^{(\mathrm{n})}_{ab} + T^{(\mathrm{m})}_{ab} \\ &= \mu^*\,\ell_a\,\ell_b + 2\,\rho^*\,\ell_{(a}\,n_{b)} + 2\,p\,m_{(a}\overline{m}_{b)} \\ &\quad +2\,\omega\,\ell_{(a}\,\overline{m}_{b)} + 2\,\overline{\omega}\,\ell_{(a}\,m_{b)} \end{aligned} \tag{4.1}$$

where the EMTs for the *rotating null fluid* as well as that of the *rotating matter* are given respectively below:

$$T^{(\mathrm{n})}_{ab} = \mu^{*}\,\ell_a\,\ell_b + \omega\,\ell_{(a}\,\overline{m}_{b)} + \overline{\omega}\,\ell_{(a}\,m_{b)} \tag{4.2}$$

$$T^{(\mathrm{m})}_{ab} = 2\,(\rho^{*} + p)\,\ell_{(a}\,n_{b)} - p\,g_{ab} + \omega\,\ell_{(a}\,\overline{m}_{b)} + \overline{\omega}\,\ell_{(a}\,m_{b)}, \tag{4.3}$$

where $\overline{\omega}$ is the complex conjugate of ω. When $\omega = 0$ initially, these EMTs may be similar to those introduced by Husain [25] in the case of non-rotating fluid.

In General Relativity the stress-energy tensor represents the matter that describes the gravitation in the space-time geometry through Einstein's field equations. From the conditions of Ricci scalar $\phi_{00} = 0$, obtained above for the space-time geometry (2.8), it may conclude that the stress-energy tensor given above does not, in general describe a perfect fluid, as for a perfect fluid the Ricci scalar $\phi_{00} = -K(\rho^{*}+p)/4$ must not vanish. Hence, it may be interesting to study the nature of the energy conditions for rotating non-perfect fluid given in (4.1). When the rotation factor ω vanishes, this fluid may be thought of null radiation fluid of non-rotating Vaidya space-time. So we refer to this rotating ($\omega \neq 0$) null radiation fluid as rotating Vaidya type radiating fluid, shortly *rotating Vaidya fluid* (4.2).

As the T_{ab} does not include the perfect fluid, it seems that the stress-energy tensor may represent the interaction of rotating Vaidya fluid with rotating non-perfect fluid (i.e. electromagnetic field, string fluid etc). Since there is a coupling term of the rotation scalar a with $\partial M/\partial u$ in the expression of ω appearing in T_{ab}, the energy condition of this T_{ab} satisfying Einstein's field equations would be a new area to discuss in the classical General Relativity. For this purpose, we may write the matter part $T^{(\mathrm{m})}_{ab}$ of T_{ab} in terms of time-like as well as space-like vectors as

$$\begin{aligned} T_{ab} \;&=\; \mu^{*}\,\ell_a\,\ell_b + (\rho^{*} + p)\,(u_a\,u_b - v_a\,v_b) - p\,g_{ab} \\ &\quad +(\omega + \overline{\omega})\,\{u_{(a}\,w_{b)} + v_{(a}\,w_{b)}\} - i(\omega - \overline{\omega})\,\{u_{(a}\,z_{b)} + v_{(a}\,z_{b)}\}, \end{aligned} \tag{4.4}$$

and its trace is $T \equiv T_{ab}g^{ab} = 2(\rho^{*} - p)$, which is different from that of a perfect fluid. This trace will be vanished when $\rho* = p$. This means that the matter part $T^{(\mathrm{m})}_{ab}$ of the stress-energy tensor may be that of electromagnetic field whose trace is zero. [The stress-energy tensor for a non-rotating perfect fluid is $T^{(\mathrm{pf})}_{ab} = (\rho^{*}+p)u_a\,u_b - p\,g_{ab}$ with unit time-like vector u_a and trace $T^{(\mathrm{pf})} = \rho^{*} - 3p$, which will be zero when $\rho* = 3p$]. The energy flux $J_a = T_{ab}u^b$ of the stress-energy tensor (4.4) is

$$J_a = \frac{1}{\sqrt{2}}\,\mu^{*}\,\ell_a + \rho^{*}u_a + \frac{1}{2}\,(\omega + \overline{\omega})\,w_a - \frac{i}{2}(\omega - \overline{\omega})\,z_a, \tag{4.5}$$

with $J_aJ^a = (\mu^{*} + \rho^{*})\rho^{*} - \omega\,\overline{\omega}$. Clearly, the energy flux vector J_a may be *time-like* or *null* when $(\mu^{*} + \rho^{*})\rho^{*} \geq \omega\,\overline{\omega}$, otherwise it is *space-like*, when $(\mu^{*} + \rho^{*})\rho^{*} < \omega\,\overline{\omega}$. Ultimately, we have in general the following energy conditions:

(i) *Weak energy condition*: The energy momentum tensor obeys the inequality $T_{ab}u^a u^b \geq 0$ for any timelike vector u^a i,e., $T_{ab}u^a u^b \geq 0$ implies that $2\rho^* \geq -\mu^*$, since μ^* is always negative as $M_{,u}$ is positive, mentioned above.

(ii) *Strong energy condition*: The Ricci tensor for T_{ab} (4.4) satisfies the inequality $R_{ab}u^a u^b \geq 0$ for $u^a u_a = 1$, i.e. $T_{ab}u^a u^b \geq \frac{1}{2}T$, which implies that $2p \geq -\mu^*$ as $-\mu^*$ is always positive.

(iii) *Energy dominant condition*: Since for every time-like vector u^a, $T_{ab}u^b$ is a non-space like vector, the energy momentum tensor satisfies the energy dominant condition, i.e., it obeys the above inequality condition $(\mu^* + \rho^*)\rho^* \geq \omega\,\overline{\omega}$.

It is observed that the rotation function ω is involved only in the energy dominant condition (iii). The density ρ^* of the matter part is eigen-value function of the outgoing null ray ℓ^a as $T^a_b \ell^b = \rho^* \ell^a$. For a incoming null vector n^a, we have $T^a_b\, n^b = \mu^* \ell^a + \rho^* n^a$ which may be *time-like* or *null* when $\mu^* \rho^* \geq 0$.

5 Rotating solutions recovered from the general solutions

In the above section we present the full expressions of NP spin coefficients (3.3) the Weyl scalars (3.6) and the Ricci scalars (3.7) with arbitrary mass and charge functions of three coordinate variables u, r, θ. These NP quantities are so transparent that these would be able to explain the nature of any solution, known or unknown, of Einstein's equations. For example, the NP spin coefficients (3.3) easily explain that there is a null vector ℓ^a which is geodesic, shear free, expanding (3.4) as well as rotating (3.5), In this section we would discuss the utilities of the NP quantities presented above by generating rotating, known and unknown solutions. We also try to present briefly the nature of the spacetime metrics so generated here.

5.1 *Rotating Kerr-Newman solution: $e = M$ = constant, $a \neq 0$*

When $e = M$ = constant, $a \neq 0$, the equation (3.10) reduces to the Kerr-Newman solution

$$\mu^* = \omega = 0$$

$$\rho^* = p = \frac{e^2}{K\, R^2 \overline{R}^2}, \tag{5.1}$$

and the only existing Weyl scalar is

$$\psi_2 = \frac{1}{R\,\overline{R}\,\overline{R}^2}(e^2 - R\,M).$$

Then the total energy momentum tensor takes the form

$$T_{ab} \;=\; \rho^*\,(\ell_a\, n_b + n_a\, \ell_b) + p\,(m_a \overline{m}_b + \overline{m}_a\, m_b)$$

$$= \quad (e^2/KR^2R^2)\{(\ell_a n_b + n_a \ell_b) + (m_a \overline{m}_b + \overline{m}_a m_b)\}, \tag{5.2}$$

which is the EMT for *non-null electromagnetic field* with Maxwell scalar

$$\phi_1 \equiv \frac{1}{2} F_{ab}(\ell^a n^b + \overline{m}^a m^b) = \frac{e}{\surd(2K)\overline{R}\overline{R}} \tag{5.3}$$

for rotating Kerr-Newman solution. Here is the birth place of Kerr-Newman solution, originally applied the Newman-Janis algorithm by Newman *et. al.* [12] to generate this well known *rotating* solution from the *non-rotating* Reissner-Nordstrom 'seed' solution. The line element is

$$\begin{aligned} ds^2 \quad = \quad & \{1-(2rM-e^2)R^{-2}\}\, du^2 + 2du\, dr \\ & + 2aR^{-2}(2rM-e^2)\sin^2\theta\, du\, d\phi - 2a\sin^2\theta\, dr\, d\phi \\ & - R^2 d\theta^2 - \{(r^2+a^2)^2 - \Delta^* a^2 \sin^2\theta\, \}R^{-2}\sin^2\theta\, d\phi^2, \end{aligned} \tag{5.4}$$

where $\Delta^* = r^2 - 2rM + a^2 + e^2$. The charged Kerr-Newman black hole has an *external event horizon* at $r_+ = M + \surd(M^2 - a^2 - e^2)$ and an *internal Cauchy horizon* at $r_- = M - \surd(M^2 - a^2 - e^2)$. The *stationary limit* surface $g_{uu} > 0$ of the rotating black hole i. e. $r = r_e(\theta) = M + \surd(M^2 - a^2\cos^2\theta - e^2)$ does not coincide with the event horizon at r_+ thereby producing the *ergosphere*. This stationary limit coincides with the event horizon at the poles $\theta = 0$ and $\theta = \pi$ [20]. Naturally, this solution includes Kerr ($e = 0$), Reissner-Nordstrom ($a = 0,\ e \neq 0$) as well as Schwarzschild ($a = e = 0$) solutions.

5.2 *Rotating Vaidya solution:* $M = M(u)$, $a \neq 0$. $e = 0$

In this case the energy momentum tensor (4.2) would take

$$T_{ab} = \mu^* \,\ell_a\, \ell_b + \omega\, \ell_{(a}\, \overline{m}_{b)} + \overline{\omega}\, \ell_{(a}\, m_{b)} \tag{5.5}$$

where the null density μ^* and the rotation function ω in (3.15) become

$$\begin{aligned} K\,\mu^* &= -\frac{1}{R^2\,R^2}\{2r^2 M_{,u} + a^2 r\sin^2\theta\, M_{,uu}\}, \\ K\,\omega &= -\frac{1}{\surd 2\,\overline{R}\,R^2}\, i\, a \sin\theta\, M_{,u}, \end{aligned} \tag{5.6}$$

and the Weyl scalars are

$$\begin{aligned} \psi_2 &= -\frac{M}{R\overline{R}\overline{R}}, \\ \psi_3 &= -\frac{i\, a\sin\theta}{2\surd 2 \overline{R}\,\overline{R}\,R^2}\{4\, rM_{,u} + \overline{R}\, M_{,u}\}, \end{aligned} \tag{5.7}$$

$$\psi_4 = \frac{a^2\, r\sin^2\theta}{2\overline{R}\,\overline{R}\,R^2\,R^2}\,\{R^2\, M_{,uu} - 2r\, M_{,u}\}. \tag{5.8}$$

From the above we observe that ω, ψ_3, ψ_4 would vanish when $a = 0$, and the EMT might be that of the original *non-rotational* radiating Vaidya metric [26] with $\mu^* = -2\,M_{,u}/K\,r^2$. The line element of this *rotating* metric is

$$
\begin{aligned}
ds^2 &= \{1 - 2rM(u)R^{-2}\}\,du^2 + 2du\,dr \\
&\quad +4arM(u)\sin^2\theta\,R^{-2}\,du\,d\phi - 2a\sin^2\theta\,dr\,d\phi \\
&\quad -R^2d\theta^2 - \{(r^2+a^2)^2 - \Delta^*\,a^2\sin^2\theta\}R^{-2}\sin^2\theta\,d\phi^2,
\end{aligned} \tag{5.9}
$$

where $\Delta^* = r^2 - 2rM(u) + a^2$. This metric represents a non-stationary *rotating* solution of Einstein's equations possessing an energy-momentum tensor (5.5) for a *rotating* null radiating fluid, and would describe a non-stationary *rotating* black hole if $M(u) > a$. The involvement of ω in the energy-momentum tensor (5.5) indicates that the null fluid is a *rotating* Vaidya null fluid. When $M(u)$ = constant initially, this metric would reduce to rotating vacuum Kerr solution with vanishing μ^* and ω in (5.6).

Carmeli and Kaye [4] studied the metric (5.9) after considering the mass M of the Kerr solution as a function of coordinate u. That is why, they referred to the metric (5.9) as the variable-mass Kerr solution (see also in [27,28]) and discussed the properties of the metric using the NP quantities. Carmeli [27] referred to these ω terms as residues of the black hole. However, we would refer to the metric (5.9) as *rotating* Vaidya solution. In the next section this metric would be combining smoothly with the usual Kerr-Newman solution as *rotating* Kerr-Newman-Vaidya black hole. So the name – a *rotating* Vaidya solution might be suitable rather then the variable-mass Kerr solution.

5.3 *Rotating Vaidya-Bonnor solution:* $M = M(u)$, $a \neq 0$. $e = e(u)$

In this case the energy momentum tensor takes

$$
T_{ab} = \mu^*\,\ell_a\ell_b + 2\,\rho^*\{\ell_{(a}\,n_{b)} + m_{(a}\overline{m}_{b)}\} + 2\,\omega\,\ell_{(a}\,\overline{m}_{b)} + 2\,\overline{\omega}\,\ell_{(a}\,m_{b)} \tag{5.10}
$$

where

$$
\begin{aligned}
\mu^* &= -\frac{1}{K\,R^2\,\overline{R}^2}\Big\{2\,r\,(r\,M_{,u} - e\,e_{,u}) + a^2\sin^2\theta\,(r\,M_{,u} - e\,e_{,u})_{,u}\Big\}, \\
\rho^* &= p = \frac{e^2(u)}{K\,R^2\,\overline{R}^2}, \\
\omega &= \frac{-ia\sin\theta}{\sqrt{2}\,K\,R^2\,\overline{R}^2}\Big\{R\,M_{,u} - 2e\,e_{,u}\Big\},
\end{aligned} \tag{5.11}
$$

and the Weyl scalars are

$$
\psi_2 = \frac{1}{R\,\overline{R}\,\overline{R}^2}\big(e^2 - R\,M\big)
$$

$$\psi_3 = \frac{-i\,a\sin\theta}{2\sqrt{2}R\,\overline{R}\,R^2}\big\{4\,(rM_{,u} - e\,e_{,u}) + \overline{R}\,M_{,u}\big\}, \tag{5.12}$$

$$\psi_4 = \frac{a^2\sin^2\theta}{2R\,\overline{R}\,R^2\,R^2}\big\{R^2\,(r\,M_{,u} - e\,e_{,u})_{,u} - 2r\,(r\,M_{,u} - e\,e_{,u})\big\}.$$

The line element would be in the form

$$\begin{aligned} ds^2 \;=\;& [1 - \{2rM(u) - e^2(u)\}R^{-2}]\,du^2 + 2du\,dr \\ &+2aR^{-2}\{2rM(u) - e^2(u)\}\sin^2\theta\,du\,d\phi - 2a\sin^2\theta\,dr\,d\phi \\ &-R^2d\theta^2 - \{(r^2+a^2)^2 - \Delta^* a^2\sin^2\theta\}R^{-2}\sin^2\theta\,d\phi^2, \end{aligned} \tag{5.13}$$

where $\Delta^* = r^2 - 2rM(u) + a^2 + e^2(u)$. This solution would describe a black hole when $M(u) > a^2 + e^2(u)$ and has $r_\pm = M(u)^* \pm \sqrt{\{M^2(u) - a^2 - e^2(u)\}}$ as the roots of the equation $\Delta^* = 0$. So the rotating Vaidya-Bonnor solution has an *external event horizon* at $r_+ = M(u) + \sqrt{\{M^2(u) - a^2 - e^2(u)\}}$ and an *internal Cauchy horizon* at $r_- = M(u) - \sqrt{\{M^2(u) - a^2 - e^2(u)\}}$. The non-stationary limit surface $g_{uu} > 0$ of the rotating black hole *i.e.* $r \equiv r_e(u,\theta) = M(u) + \sqrt{\{M^2(u) - a^2\cos^2\theta - e^2(u)\}}$ does not coincide with the event horizon at r_+, thereby producing the *ergosphere*. The *rotating* Vaidya-Bonnor metric (5.13) could be written in Kerr-Schild form on the *rotating* Vaidya null radiating background as

$$g_{ab}^{\rm VB} = g_{ab}^{\rm V} + 2Q(u,r,\theta)\ell_a\ell_b \tag{5.14}$$

where

$$Q(u,r,\theta) = \frac{e^2(u)}{2\,R^2}. \tag{5.15}$$

Here, $g_{ab}^{\rm V}$ is the *rotating* Vaidya metric (5.9) and ℓ_a is geodesic, shear free, expanding and rotating null vector for both $g_{ab}^{\rm V}$ as well as $g_{ab}^{\rm VB}$ and given in (2.6). The Kerr-Schild form (5.14) may be interpreted as the existence of the electromagnetic field on the *rotating* Vaidya null radiating background. If we set $M(u)$ and $e(u)$ are both constant, this Kerr-Schild form may be that of Kerr-Newman black hole. That is, the Kerr-Newman solution itself has the Kerr-Schild form on the Kerr background with the same null vector ℓ_a (2.6).

From this *rotating* Vaidya-Bonnor metric, we could clearly recover the following solutions: (i) *rotating* Vaidya metric (5.9) when $e(u) = 0$, (ii) *rotating* charged Vaidya solution when $e(u)$ becomes constant, (iii) the rotating Kerr-Newman solution (5.4) when $M(u) = e(u) =$ constant and (iv) well-known *non-rotating* Vaidya-Bonnor metric [29] when $a = 0$. It is also noted that when $e = a = 0$, the null density of Vaidya radiating fluid takes the form $\mu^* = -2\,M_{,u}/K\,r^2$. The *non-rotating* Vaidya null radiating metric is of type D in the Petrov classification of spacetime whose one of the repeated principal null vectors, ℓ_a is a geodesic, shear free, non-rotating with non-zero expansion [27], while the rotating one is of *algebraically special* with a null vector ℓ_a which is geodesic, shear free, rotating as well as expanding. It is also noted that when $e = a = 0$, the energy-momentum tensor becomes that of the original *non-rotational null-radiating Vaidya fluid* with $\mu^* = -2\, M_{,u}/K\, r^2$. It is noted that the metric (5.13) may be seen in [10], where after the application of Newman-Janis algorithm, Jing and Wang kept the mass $M(u)$ and the $e(u)$ unchanged as before the application, leading to an easier calculation of NP quantities.

From (5.4) and (5.13) it is observed that the *rotating* Vaidya-Bonnor solution is the 'non-stationary' version of Kerr-Newman black holes. That is, the parameters M and e of Kerr-Newman solution are functions of retarded time coordinate u in *rotating* Vaidya-Bonnor metric. So any known results based on stationary Kerr-Newman black hole may be extended in *non-stationary rotating* Vaidya-Bonnor black hole using these NP quantities. For example, the Hawking's radiation of electrically radiating Kerr-Newman black hole, expressed in classical spacetime metrics in [1] may be extended in this *non-stationary* rotating Vaidya-Bonnor solution. The extended Hawking's radiation process in this *non-stationary* solution might be seen elsewhere.

6 *Rotating solutions with* $M = M(u, r)$, $e(u, r, \theta) = 0$

In this section we discuss the *rotating* spherically symmetric solutions presented in NP quantities (3.3), (3.6) and (3.7) by considering the mass function $M(u, r)$ of two variables u, r only with the vanishing $e(u, r, \theta) = 0$, and derive other *rotating* solutions, which were not able to obtain by the direct applications of Newman-Janis algorithm above. In this case the energy momentum tensor would take the form

$$T_{ab} = \mu^*\, \ell_a\, \ell_b + 2\, \rho^*\, \ell_{(a}\, n_{b)} + 2\, p\, m_{(a}\overline{m}_{b)} + 2\, \omega\, \ell_{(a}\, \overline{m}_{b)} + 2\overline{\omega}\, \ell_{(a}\, m_{b)} \tag{6.1}$$

or in terms of unit vectors

$$\begin{aligned} T_{ab} = {} & \mu^*\, \ell_a\, \ell_b + \rho^*\, (u_a\, u_b - v_a\, v_b) + p\, (w_a\, w_b + z_a\, z_b) \\ & +(\omega + \overline{\omega})\, \{u_{(a}\, w_{b)} + v_{(a}\, w_{b)}\} - i(\omega - \overline{\omega})\, \{u_{(a}\, z_{b)} + v_{(a}\, z_{b)}\}, \end{aligned} \tag{6.2}$$

where

$$\begin{aligned} \mu^* &= -\frac{1}{K\, R^2\, \overline{R^2}}\big\{2r^2 M_{,u} + a^2 r \sin^2\theta\, M_{,uu}\big\}, \\ \rho^* &= \frac{2\, r^2}{K\, R^2\, \overline{R^2}}\, M_{,r}, \\ p &= -\frac{1}{K}\Big\{\frac{2\, a^2 \cos^2\theta}{R^2\, \overline{R^2}}\, M_{,r} + \frac{r}{R^2}\, M_{,r\,r}\Big\}, \\ \omega &= -\frac{i\, a \sin\theta}{\sqrt{2}\, K\, R^2\, \overline{R^2}}\big(R\, M_{,u} - r\, \overline{R}\, M_{,ur}\big). \end{aligned} \tag{6.3}$$

The line element would be of the form

$$ds^2 = \big\{1 - 2rM(u, r)R^{-2}\big\}\, du^2 + 2du\, dr$$

$$+4arM(u,r)R^{-2}\sin^2\theta\,du\,d\phi - 2a\sin^2\theta\,dr\,d\phi$$
$$-R^2d\theta^2 - \Big\{(r^2+a^2)^2 - \Delta^* a^2\sin^2\theta\Big\}R^{-2}\sin^2\theta\,d\phi^2, \qquad (6.4)$$

where $R^2 = r^2 + a^2\cos^2\theta$, $\Delta^* = r^2 - 2rM(u,r) + a^2$ and the Weyl scalars are

$$\begin{aligned}
\psi_2 &= \frac{1}{\overline{R}\,\overline{R}\,R^2}\Big\{-R\,M + \frac{\overline{R}}{6}M_{,r}(4\,r + 2\,i\,a\cos\theta) - \frac{r}{6}\overline{R}\,\overline{R}\,M_{,rr}\Big\},\\
\psi_3 &= -\frac{i\,a\sin\theta}{2\sqrt{2}\overline{R}\,\overline{R}\,R^2}\Big\{(4\,r + \overline{R})\,M_{,u} + r\,\overline{R}\,M_{,ur}\Big\},\\
\psi_4 &= \frac{a^2\,r\sin^2\theta}{2\overline{R}\,\overline{R}\,R^2\,R^2}\Big\{R^2\,M_{,uu} - 2\,r\,M_{,u}\Big\}.
\end{aligned} \qquad (6.5)$$

One may regard this *rotating* metric (6.4) along with the stress-energy momentum tensor (6.1) or (6.2) and the Weyl scalars as the extension of the *non-rotating* solutions discussed by Glass and Krisch [23] and Husain [25].

6.1 *Rotating Husain's solution: $M = M(u,r)$, $a \neq 0$*

Husain [25] has imposed one condition in the equation of state of *non-rotating null fluid* that $p = k\rho^{*b}$ and obtain the solution of the equation of state with $k \geq 1/2$ $b = 1$. However, due the present of the rotating factor a in equation (6.3), one may not be able to get the solution of Husain. So we put $k = 1$ and $b = 1$. Then, the equation to be solved takes a simple form

$$\frac{M_{,r}}{r} = -\frac{M_{,rr}}{2}, \qquad (6.6)$$

which gives the mass function $M(u,r)$

$$M(u,r) = f(u) - \frac{1}{r}\,g(u). \qquad (6.7)$$

It may be treated as *rotating* Husain's solution of $p = k\rho^*$ for $k = 1$. This *rotating* Husain's solution may degenerate to the *rotating* Vaidya-Bonnor solution presented above if one puts $g(u) = e^2(u)/2$ in (6.7).

6.2 *Rotating Wang-Wu solutions*

Wang and Wu [30] have expanded the mass function $M(u,r)$ of (6.3) of the non-rotating space in the power of r

$$M(u,r) = \sum_{n=-\infty}^{+\infty} q_n(u)\,r^n, \qquad (6.8)$$

where $q_n(u)$ are arbitrary functions of u. They consider the above sum as an integral when the 'spectrum' index n is continuous. In fact Wang and Wu technique is based on a linear superposition that a linear superposition of mass function of particular solutions is also a solution of Einstein's field equations of *non-rotating* spacetime. Using the expression (6.8) in equations (6.3) we could generate *rotating* solutions with Wang-Wu functions as

$$\begin{aligned}
\mu^* &= \frac{r}{K\,R^2\,\bar{R}^2}\sum_{n=-\infty}^{+\infty}\left\{2\,q_n(u)_{,u}\,r^{n+1} + a^2\sin^2\theta\,q_n(u)_{,uu}\,r^n\right\},\\
\rho^* &= \frac{2\,r^2}{K\,R^2\,\bar{R}^2}\sum_{n=-\infty}^{+\infty} n\,q_n(u)\,r^{n-1},\\
p &= -\frac{1}{K\,R^2}\sum_{n=-\infty}^{+\infty} n\,q_n(u)\,r^{n-1}\left\{\frac{2\,a^2\cos^2\theta}{\bar{R}^2} + (n-1)\right\},\\
\omega &= -\frac{i\,a\sin\theta}{\sqrt{2}K\,R^2\,\bar{R}^2}\sum_{n=-\infty}^{+\infty}\left(R - n\bar{R}\right)q_n(u)_{,u}\,r^n. \qquad (6.9)
\end{aligned}$$

Here one could observe that these *rotating* solutions with functions $q_n(u)$ include many known as well as unknown *rotating* solutions of Einstein's field equations in spherical symmetry as shown by Wang and Wu in *non-rotating* cases [30]. The functions $q_n(u)$ in (6.8) play a great role in generating new solutions whether *rotating* or *non-rotating*. Therefore, we would hereafter refer to these as Wang-Wu functions. Thus, *rotating* solutions could be derived from these solutions as follows.

6.2.1 *Rotating monopole solution*

If one chooses the functions $q_n(u)$ such that

$$q_n(u) = \begin{cases} (b/2), & \text{when } n = 1 \\ 0, & \text{when } n \neq 1 \end{cases} \qquad (6.10)$$

where b is constant, then one might obtain

$$M(u,r) = b\,r/2, \quad \mu* = \omega = 0,$$

$$\rho^* = \frac{r^2\,b}{K\,R^2\,\bar{R}^2}, \quad p = -\frac{b\,a^2\cos^2\theta}{K\,R^2\,\bar{R}^2},$$

with the energy momentum tensor

$$T_{ab} = 2\,\rho^*\,\ell_{(a}\,n_{b)} + 2\,p\,m_{(a}\overline{m}_{b)}.$$

The Weyl scalar takes the form

$$\psi_2 = -\frac{b\,r}{2R\bar{R}\bar{R}}.$$

The *rotating* monopole line element would be of the following form

$$ds^2 = \left\{1 - b\,r^2R^{-2}\right\}du^2 + 2du\,dr$$

$$+2a\,b\,r^2R^{-2}\sin^2\theta\,du\,d\phi - 2a\sin^2\theta\,dr\,d\phi$$
$$-R^2d\theta^2 - \left\{(r^2+a^2)^2 - \Delta^* a^2\sin^2\theta\right\}R^{-2}\sin^2\theta\,d\phi^2, \qquad (6.11)$$

where $R^2 = r^2 + a^2\cos^2\theta$, and $\Delta^* = r^2 - b\,r^2 + a^2$.

Because of the non-vanishing Weyl scalar ψ_2, the *rotating* monopole solution is *stationary* as well as Petrov type D whose one of the repeated principal directions is geodesic, shear free, expanding and rotating null vector ℓ_a. The *rotating* monopole solution has the non-zero pressure p, which leads the difference between the *rotating* and the *non-rotating* monopole solutions. That is, when $a = 0$, one may obtain the *non-rotating* metric [30] with $p = 0$. To study this *rotating* monopole solution (6.11) would be of interest. For example, one could easily embed Kerr-Newman black hole in this *rotating* monopole space to study a different physical nature of the black holes.

6.2.2 *Rotating Kerr-Newman solution*

We may choose the Wang-Wu functions $q_n(u)$ such that

$$q_n(u) = \begin{cases} m, & \text{when } n = 0 \\ -e^2/2, & \text{when } n = -1 \\ 0, & \text{when } n \neq 0, -1 \end{cases} \qquad (6.12)$$

where m and e are constants. Then we could obtain

$$M(u,r) = m - e^2/2r, \quad \mu* = \omega = 0,$$
$$\rho^* = p = \frac{e^2}{K\,R^2\,\bar{R}^2}, \qquad (6.13)$$

and the Weyl scalar is

$$\psi_2 = \frac{1}{\bar{R}\,\bar{R}\,R^2}(e^2 - m\,R),$$

which are the same as given (5.1).

6.2.3 *Rotating Vaidya-Bonnor solution*

The *rotating* Vaidya-Bonnor solution presented above, might also be obtained from these *rotating* Wang-Wu solutions if we choose the functions as

$$q_n(u) = \begin{cases} f(u), & \text{when } n = 0 \\ -h(u)^2/2, & \text{when } n = -1 \\ 0, & \text{when } n \neq 0, -1. \end{cases} \qquad (6.14)$$

Then the corresponding quantities are

$$M(u,r) = f(u) - h^2(u)/2r$$

$$\rho^* = p = \frac{h^2(u)}{K\,R^2\,\bar{R}^2}, \tag{6.15}$$

$$\mu^* = -\frac{1}{K\,R^2\,\bar{R}^2}\big\{2\,r\,(r\,f(u)_{,u} - h\,h_{,u}) + a^2\sin^2\theta\,(r\,f(u)_{,u} - h\,h_{,u})_{,u}\big\},$$

$$\omega = \frac{-i\,a\sin\theta}{\sqrt{2}\,K\,R^2\,\bar{R}^2}\,\big\{R\,f(u)_{,u} - 2h\,h_{,u}\big\}. \tag{6.16}$$

This solution includes the *rotating* Vaidya solution ($h(u) = 0$) obtained above in (5.9). These subsections (6.2.2) and (6.2.3), which are the repetition of the section (5.1) and (5.3) above, show that the general *rotating* solutions with Wang-Wu functions (6.8) could employ to derive the rotating Kerr-Newman as well as *rotating* Vaidya-Bonnor solutions. in a much simpler way.

6.2.4 *Rotating Kerr-Newman-Vaidya solution*

Wang and Wu [30] could combine the three *non-rotating* solutions, namely monopole, de-Sitter and charged Vaidya solution to obtain a new solution representing the *non-rotating* monopole-de Sitter-Vaidya charged solutions. In the same way, we would combine the Kerr-Newman solution with the *rotating* Vaidya solution obtained above in (5.9), if the Wang-Wu functions $q_n(u)$ are chosen such that

$$q_n(u) = \begin{cases} m + f(u), & \text{when } n = 0 \\ -e^2/2, & \text{when } n = -1 \\ 0, & \text{when } n \neq 0, -1, \end{cases} \tag{6.17}$$

where m and e are constants and $f(u)$ is the mass function of *rotating* Vaidya solution (5.9). Thus, we have the mass function

$$M(u,r) = m + f(u) - e^2/2r$$

and using this in (6.9) we obtain other quantities

$$\rho^* = p = \frac{e^2}{K\,R^2\,\bar{R}^2}, \tag{6.18}$$

$$\mu^* = -\frac{r}{K\,R^2\,\bar{R}^2}\big\{2\,r\,f(u)_{,u} + a^2\sin^2\theta\,f(u)_{,uu}\big\},$$

$$\omega = \frac{-i\,a\sin\theta}{\sqrt{2}\,K\,R\,\bar{R}^2}\,f(u)_{,u}. \tag{6.19}$$

The Weyl scalars for this *rotating* solution are

$$\psi_2 = \frac{1}{R\,\bar{R}\,\bar{R}^2}\big[e^2 - R\{m + f(u)\}\big]$$

$$\psi_3 = \frac{-i\,a\sin\theta}{2\sqrt{2}R\,\overline{R}\,R^2}\big\{4\,r\,f(u)_{,u} + \overline{R}\,f(u)_{,u}\big\}, \tag{6.20}$$

$$\psi_4 = \frac{a^2 r\sin^2\theta}{2R\,\overline{R}\,R^2\,R^2}\big\{R^2 f(u)_{,uu} - 2r f(u)_{,u}\big\}.$$

This would represent a *rotating non-stationary* Kerr-Newman-Vaidya solution with the line element

$$\begin{aligned} ds^2 \;=\;& [1 - R^{-2}\{2r(m + f(u)) - e^2\}]\,du^2 + 2du\,dr \\ &+2aR^{-2}\{2r(m + f(u)) - e^2\}\sin^2\theta\,du\,d\phi - 2a\sin^2\theta\,dr\,d\phi \\ &-R^2 d\theta^2 - \{(r^2 + a^2)^2 - \Delta^* a^2\sin^2\theta\}\,R^{-2}\sin^2\theta\,d\phi^2, \end{aligned} \tag{6.21}$$

where $\Delta^* = r^2 - 2r\{m + f(u)\} + a^2 + e^2$. Here m and e are the mass and the charge of rotating Kerr-Newman solution, a is the rotation per unit mass and $f(u)$ represents the mass function of rotating Vaidya null radiating fluid. The solution (6.21) would describe a black hole if $m + f(u) > a^2 + e^2$ with external event horizon at $r_+ = \{m + f(u)\} + \sqrt{[\{m + f(u)\}^2 - a^2 - e^2]}$, an internal Cauchy horizon at $r_- = \{m + f(u)\} - \sqrt{[\{m + f(u)\}^2 - a^2 - e^2]}$ and the *non-stationary* limit surface $r \equiv r_e(u, \theta) = \{m + f(u)\} + \sqrt{[\{m + f(u)\}^2 - a^2\cos^2\theta - e^2]}$. When we set $f(u) = 0$, the metric (6.21) recovers the usual Kerr-Newman black hole, and if $m = 0$, then it is the 'rotating' charged Vaidya null radiating black hole (5.9).

In this *rotating* solution, the Vaidya *null fluid* is interacting with the *non-null electromagnetic field* whose Maxwell scalar ϕ_1 can be obtained from (6.18). Thus, we could write the total energy momentum tensor (EMT) for the *rotating* solution (6.21) as follows:

$$T_{ab} = T^{(n)}_{ab} + T^{(e)}_{ab}, \tag{6.22}$$

where the EMTs for the *rotating null fluid* as well as that of the *electromagnetic field* might be given respectively

$$T^{(n)}_{ab} = \mu^*\,\ell_a\,\ell_b + 2\,\omega\,\ell_{(a}\,\overline{m}_{b)} + 2\,\overline{\omega}\,\ell_{(a}\,m_{b)}, \tag{6.23}$$

$$T^{(e)}_{ab} = 2\,\rho^*\,\{\ell_{(a}\,n_{b)} + m_{(a}\,\overline{m}_{b)}\}. \tag{6.24}$$

The appearance of non-vanishing ω shows the null fluid is *rotating* as the expression (6.19) of ω involves the rotating constant a coupling with $\partial f(u)/\partial u$ - both are non-zero quantities for a *rotating* Vaidya null radiating universe (5.9).

This *rotating* Kerr-Newman-Vaidya metric (6.21) could be written in Kerr-Schild form on the Kerr-Newman background as

$$g^{\rm KNV}_{ab} = g^{\rm KN}_{ab} + 2Q(u, r, \theta)\ell_a\ell_b \tag{6.25}$$

where

$$Q(u,r,\theta) = -rf(u)R^{-2}, \tag{6.26}$$

and the vector ℓ_a is a geodesic, shear free, expanding as well as rotating null vector of both $g_{ab}^{\rm KN}$ as well as $g_{ab}^{\rm KNV}$ and given in (2.6) and $g_{ab}^{\rm KN}$ is the Kerr-Newman metric (5.4) with $m = e = constant$. This null vector ℓ_a is one of the double repeated principal null vectors of the Weyl tensor of $g_{ab}^{\rm KN}$. This completes the proof of theorem 1 stated above.

It appears that the *rotating* Kerr-Newman geometry might be thought of joining smoothly to the rotating Vaidya geometry at its null radiative boundary, as shown by Glass and Krisch [23] in the case of Schwarzschild geometry joining to the non-rotating Vaidya space-time. The Kerr-Schild form (6.23) would recover that of Xanthopoulos [31] $g'_{ab} = g_{ab} + \ell_a\ell_b$, when $Q(u,r,\theta) \to 1/2$ and that of Glass and Krisch [23] $g'_{ab} = g_{ab}^{\rm Sch} - \{2f(u)/r\}\ell_a\ell_b$ when $e = a = 0$ for *non-rotating* Schwarzschild background space.

Thus, one could consider that the Kerr-Schild form presented in (6.25) above would be the extension of those of Xanthopoulos as well as Glass and Krisch. When we set $a = 0$, this *rotating* Kerr-Newman-Vaidya solution (6.21) would recover to *non-rotating* Reissner-Nordstrom-Vaidya solution with the Kerr-Schild form $g_{ab}^{\rm RNV} = g_{ab}^{\rm RN} - \{2f(u)/r\}\ell_a\ell_b$ when $a = 0$ which is still a generalization of Xanthopoulos and Glass and Krisch in the charged Reissner-Nordstrom solution. It is worth to mention that our new solution (6.21) could not be considered as a bimetric theory as $g_{ab}^{\rm KNV} \neq \frac{1}{2}(g_{ab}^{\rm KN} + g_{ab}^{\rm V})$.

To interpret the Kerr-Newman-Vaidya solution as a black hole during the early inflationary phase of *rotating* Vaidya null radiating universe *i.e.*, the Kerr-Newman black hole embedded in *rotating* Vaidya null radiating background space, we may write our Kerr-Schild form (6.25) as

$$g_{ab}^{\rm KNV} = g_{ab}^{\rm V} + 2Q(r,\theta)\ell_a\ell_b \tag{6.27}$$

where

$$Q(r,\theta) = -(rm - e^2/2)R^{-2}, \tag{6.28}$$

Here, the constants m and e are the mass and the charge of Kerr-Newman black hole, $g_{ab}^{\rm V}$ is the *rotating* Vaidya null radiating black hole (5.9) and ℓ_a is the geodesic null vector given in (2.6) for both $g_{ab}^{\rm KNV}$ and $g_{ab}^{\rm V}$. When we set $f(u) = a = 0$, $g_{ab}^{\rm V}$ will recover the flat metric, then $g_{ab}^{\rm KNV}$ becomes the original Kerr-Schild form written in spherical symmetric flat background.

These two Kerr-Schild forms (6.25) and (6.27) certainly confirm that our metric $g_{ab}^{\rm KNV}$ is a solution of Einstein's field equations since the background rotating metrics $g_{ab}^{\rm KN}$ and $g_{ab}^{\rm V}$ are solutions of Einstein's equations. They both have different stress-energy tensors $T_{ab}^{\rm (e)}$ and $T_{ab}^{\rm (n)}$ given in (6.24) and (6.23) respectively. Looking to the Kerr-Schild form (6.27), the Kerr-Newman-Vaidya black hole could be treated as a generalization of Kerr-Newman black hole by incorporating Visser's suggestion [32] that *Kerr-Newman black hole embedded in an axisymmetric cloud of matter would be of interest.* Hawking et al [33] have also mentioned the possibility to embed the rotating black hole solutions with a theory for which they know the corresponding conformal field theory.

6.2.5 *Rotating Kerr-Newman-Vaidya-Bonnor solution*

Similarly, one may combine the *rotating* Vaidya-Bonnor solution obtained above in (5.13) with the Kerr-Newman solution (5.4) to generate another *rotating* solution with the mass function

$$M(u,r) = m + f(u) - (e^2 + h^2(u))/2r, \tag{6.29}$$

representing a *rotating Kerr-Newman-Vaidya-Bonnor solution*:

$$\begin{aligned} ds^2 \; = \; & [1 - R^{-2}\{2r(m+f(u)) - e^2 - h^2(u)\}]\, du^2 + 2du\, dr \\ & + 2aR^{-2}\{2r(m+f(u)) - e^2 - h^2(u)\} \sin^2\theta \, du\, d\phi - 2a\sin^2\theta\, dr\, d\phi \\ & - R^2 d\theta^2 - \{(r^2+a^2)^2 - \Delta^* a^2 \sin^2\theta\}\, R^{-2} \sin^2\theta\, d\phi^2, \end{aligned} \tag{6.30}$$

where $\Delta^* = r^2 - 2r\{m + f(u)\} + a^2 + e^2 + h^2(u)$. This *rotating* solution could also be written in Kerr-Schild form (6.25) with the function:

$$Q(u,r,\theta) = -\{r\, f(u) - h^2(u)/2\}\, R^{-2},$$

When the charge e of the Kerr-Newman solution vanishes, this *rotating* solution (6.30) would reduce to a *rotating Kerr-Vaidya-Bonnor* solution with the mass function:

$$M(u,r) = m + f(u) - h^2(u)/2r. \tag{6.31}$$

It suggests that by choosing the Wang-Wu functions $q_n(u)$ properly one could generate as many *rotating* solutions as required. However, the generation of these types of rotating solutions would be restricted that the energy-momentum tensor of the fluids must be of the form given in (4.1).

6.2.6 *Rotating Kerr-Newman-de Sitter metrics*

Here we present the *rotating* de Sitter as well as *rotating* Kerr-Newman-de Sitter metrics in NP formalism.

A. *Rotating de Sitter solution*:

First we derive a *rotating* de Sitter solution of Einstein's equations. For this we choose the Wang-Wu functions as

$$q_n(u) = \begin{cases} \Lambda^*/6, & \text{when } n = 3 \\ 0, & \text{when } n \neq 3 \end{cases} \tag{6.32}$$

to obtain the mass function

$$M(u,r) = \frac{\Lambda^* r^3}{6}, \tag{6.33}$$

The line element for the *rotating* de Sitter metric is

$$\begin{aligned} ds^2 &= \Big\{1 - \frac{\Lambda^* r^4}{3\,R^2}\Big\}\,du^2 + 2du\,dr \\ &+2a\frac{\Lambda^* r^4}{3}R^{-2}\sin^2\theta\,du\,d\phi - 2a\sin^2\theta\,dr\,d\phi \\ &-R^2 d\theta^2 - \Big\{(r^2+a^2)^2 - \Delta^* a^2\sin^2\theta\Big\}\,R^{-2}\sin^2\theta\,d\phi^2, \end{aligned} \tag{6.34}$$

where $R^2 = r^2 + a^2\cos^2\theta$, $\Delta^* = r^2 - \Lambda^* r^4/3 + a^2$. This corresponds to the *rotating* de Sitter solution for $\Lambda^* > 0$, and to the anti-de Sitter solution for $\Lambda^* < 0$. In general Λ^* denotes the cosmological constant of the de Sitter space. Then the changed NP quantities are

$$\gamma = -\frac{1}{2\overline{R}\,R^2}\Big\{(1 - \frac{1}{3}\Lambda^* r^2) r\,\overline{R} + \Delta^*\Big\},$$

$$\phi_{11} = -\frac{1}{2\,R^2\,R^2}\Lambda^* r^2 a^2\cos^2\theta, \tag{6.35}$$

$$\psi_2 = \frac{1}{3\overline{R}\,\overline{R}\,R^2}\Lambda^* r^2 a^2\cos^2\theta, \tag{6.36}$$

$$\Lambda = \frac{\Lambda^* r^2}{6R^2}. \tag{6.37}$$

This means that in *rotating* de Sitter cosmological universe, the Λ^* is coupling with the rotational parameter a. From these NP quantities we could clearly observe that the *rotating* de Sitter cosmological metric is a Petrov type D gravitational field whose one of the repeated principal null vectors, ℓ_a is geodesic, shear free, expanding as well as non-zero rotation. The *rotating* cosmological space possesses an energy-momentum tensor

$$T_{ab} = 2\,\rho^*\,\ell_{(a}\,n_{b)} + 2\,p\,m_{(a}\overline{m}_{b)}, \tag{6.38}$$

where $K\,\rho^* = 2\,\phi_{11} + 6\,\Lambda$ and $K\,p = 2\,\phi_{11} - 6\,\Lambda$ are related to the density and the pressure of the cosmological matter which is, however not a perfect fluid. If we set the rotational parameter a, we would recover the non-rotating de Sitter metric [34], which is a solution of the Einstein's equations for an empty space with $\Lambda \equiv g_{ab}R^{ab} = \Lambda^*/6$ or constant curvature. However, it is observed that the *rotating* de Sitter metric (6.34) is neither *empty* nor *constant curvature*. It certainly describes a *stationary rotating* spherical symmetric solution representing Petrov type D spacetime. So it is noted that to the best of the present author's knowledge, this *rotating* de Sitter metric has not been seen discussed before.

B. *Kerr-Newman-de Sitter solution:*

By choosing the Wang-Wu function as

$$q_n(u) = \begin{cases} m, & \text{when } n = 0 \\ -e^2/2, & \text{when } n = -1 \\ \Lambda^*/6, & \text{when } n = 3 \\ 0, & \text{when } n \neq 0, -1, 3 \end{cases} \tag{6.39}$$

we could obtain the mass function

$$M(u,r) = m - \frac{e^2}{2r} + \frac{\Lambda^* r^3}{6}, \tag{6.40}$$

where m and e are constants and are the mass and the charge of the Kerr-Newman solution. The line element with this mass function is

$$\begin{aligned} ds^2 &= \Big\{1 - R^{-2}\Big(2mr - e^2 + \frac{\Lambda^* r^4}{3}\Big)\Big\}\, du^2 + 2du\, dr \\ &+2aR^{-2}\Big(2mr - e^2 + \frac{\Lambda^* r^4}{3}\Big) \sin^2\theta\, du\, d\phi - 2a\sin^2\theta\, dr\, d\phi \\ &-R^2 d\theta^2 - \Big\{(r^2 + a^2)^2 - \Delta^* a^2 \sin^2\theta\Big\}\, R^{-2} \sin^2\theta\, d\phi^2, \end{aligned} \tag{6.41}$$

where $R^2 = r^2 + a^2\cos^2\theta$, $\Delta^* = r^2 - 2mr - \Lambda^* r^4/3 + a^2 + e^2$. Then the changed NP quantities are

$$\begin{aligned} \gamma &= \frac{1}{2\overline{R}\, R^2}\Big[\Big(r - m - \frac{2}{3}\Lambda^* r^3\Big)\overline{R} - \Delta^*\Big], \\ \phi_{11} &= \frac{1}{2\, R^2\, R^2}\Big(e^2 - \Lambda^* r^2 a^2 \cos^2\theta\Big), \\ \psi_2 &= \frac{1}{\overline{R}\, \overline{R}\, R^2}\Big\{e^2 - m\, R + \frac{\Lambda^* r^2}{3} a^2 \cos^2\theta\Big\} \\ \Lambda &= \frac{\Lambda^* r^2}{6R^2}. \end{aligned} \tag{6.42}$$

We have seen from the above that in each expression of ϕ_{11} and ψ_2, there is a cosmological Λ^* coupling with the rotational parameter a. This means that the rotating cosmological parameter Λ^* has the effect of its presence in the curvature of the embedded Kerr-Newman black hole. The metric (6.41) admits the following energy momentum tensor

$$T_{ab} = 2\,\rho^*\,\ell_{(a}\, n_{b)} + 2\,p\, m_{(a}\overline{m}_{b)},$$

with the density and the pressure of the matter field

$$\rho^* = \frac{1}{K\, R^2\, R^2}\Big(e^2 + \Lambda^* r^4\Big),$$

$$p = \frac{1}{K\, R^2\, R^2}\Big\{e^2 - \Lambda^* r^2\Big(r^2 + 2a^2 \cos^2\theta\Big)\Big\}.$$

Without loss of generality, we could write this T_{ab} with the decomposition of $\rho^* = \rho^{*(\mathrm{E})} + \rho^{*(\mathrm{C})}$ and $p = p^{(\mathrm{E})} + p^{(\mathrm{C})}$ that

$$T_{ab} = 4\,\rho^{*(\mathrm{E})}\{\ell_{(a}\,n_{b)} + m_{(a}\overline{m}_{b)}\} + 2\{\rho^{*(\mathrm{C})}\,\ell_{(a}\,n_{b)} + 2\,p^{(\mathrm{C})}\,m_{(a}\overline{m}_{b)}\}, \tag{6.43}$$

where

$$\begin{aligned}\rho^{*(\mathrm{E})} &= p^{(\mathrm{E})} = \frac{e^2}{K\,R^2\,\bar{R}^2},\\ \rho^{*(\mathrm{C})} &= \frac{\Lambda^* r^4}{K\,R^2\,\bar{R}^2},\ p^{(\mathrm{C})} = \frac{-\Lambda^* r^2}{K\,R^2\,\bar{R}^2}\left(r^2 + 2a^2\cos^2\theta\right).\end{aligned}$$

The advantage of writing T_{ab} in the form (6.43) is that, for *non-rotating* Reissner-Nordstrom-de Sitter metric ($a = 0$), the energy-momentum tensor could be written in the form of Guth's modification of T_{ab} [35] as

$$T_{ab} = T_{ab}^{(\mathrm{E})} + \Lambda^* g_{ab} \tag{6.44}$$

where $T_{ab}^{(\mathrm{E})}$ is the energy-momentum tensor for non-null electromagnetic field and g_{ab} is the Reissner-Nordstrom metric. This indicates that Guth's modification of T_{ab} may be acceptable only in the case of *non-rotating* metrics, and it may not be possible to extend to the *rotating* solutions as seen from (6.43). Otherwise the energy-momentum tensor (6.43) itself may consider the Guth's modification of T_{ab} in the case of *rotating* embedded spaces.

The metric (6.41) describes a *rotating stationary* spherically symmetric solution and is Petrov type D, whose one of the repeated principal null directions is ℓ_a. That is, the metric could be written in Kerr-Schild form on the de Sitter background as

$$g_{ab}^{\mathrm{KNdS}} = g_{ab}^{\mathrm{dS}} + 2Q(r,\theta)\ell_a\ell_b \tag{6.45}$$

where $Q(r,\theta) = -(rm - e^2/2)R^{-2}$, and the vector ℓ_a is a geodesic, shear free, expanding as well as rotating null vector of both g_{ab}^{KNdS} as well as g_{ab}^{dS} and given in (2.6). and g_{ab}^{KN} is the Kerr-Newman metric (5.4) with $m = e = constant$. This null vector ℓ_a is one of the double repeated principal null vectors of the Weyl tensor of g_{ab}^{KNdS} and g_{ab}^{dS}. This completes the proof of theorem 3 stated above. We could also write the Kerr-Schild form (6.45) on the Kerr-Newman background as

$$g_{ab}^{\mathrm{KNdS}} = g_{ab}^{\mathrm{KN}} + 2Q(r,\theta)\,\ell_a\,\ell_b, \tag{6.46}$$

where $Q(r,\theta) = -(\Lambda^* r^4/6)\,R^{-2}$, and Λ^* is the de Sitter cosmological constant and g_{ab}^{KN} is the Kerr-Newman metric (5.4) with $m = e = constant$. This gives the proof of the theorem 4 cited in the introduction.

It is quite natural to recover *rotating* de Sitter ($m = 0, a \neq 0, e = 0$) and non-rotating Reissner-Nordstrom-de Sitter ($m \neq 0, a = o, e \neq 0$) metrics from the rotating Kerr-Newman-de Sitter solution ($m \neq 0, a \neq 0, e \neq 0$). It is also worth mentioning that from the rotating Kerr-Newman-de Sitter metric, we could recover a *rotating* charged de Sitter cosmological universe ($m = 0, a \neq 0, e \neq 0$), which may be seen in [36] as an 'instantaneous' naked singularity, where the Hawking's evaporation of black holes embedded into the de Sitter space has been expressed in classical spacetime metrics by considering the charge e as a function of radial coordinate r.

It is noted that one could find the difference between this Kerr-Newman-de Sitter metric (6.41) and that of Mallett [8] used by Koberlin [37] with the comments made by Xu [9]. Mallett's derivation of Kerr-Newman-de Sitter metric is based on the direct application of Newman-Janis algorithm to the Reissner-Nordstrom-de Sitter 'seed' solution. That makes the difference between our metric (6.41) and his solution. One may also observe that our Kerr-Newman-de Sitter solution (6.41) is different, in the terms involving the cosmological constant Λ^*, from the one derived by Carter [38] and used by Gibbon and Hawking [39], Khanal [40], Hawking, et al [33], and others.

6.2.7 *Rotating Kerr-Newman-Vaidya-de Sitter solution*

In the same way, we would combine the Kerr-Newman-de Sitter solution with the *rotating* Vaidya solution obtained above in (5.9), if the Wang-Wu functions $q_n(u)$ are chosen such that

$$q_n(u) = \begin{cases} m + f(u), & \text{when } n = 0 \\ -e^2/2, & \text{when } n = -1 \\ \Lambda^*/6, & \text{when } n = 3 \\ 0, & \text{when } n \neq 0, -1, 3, \end{cases} \tag{6.47}$$

where m and e are constants and $f(u)$ is related with the mass of *rotating* Vaidya solution (5.9). Thus, we have the mass function

$$M(u, r) = m + f(u) - \frac{e^2}{2r} + \frac{\Lambda^* r^3}{6}$$

and other quantities are

$$\rho^* = \frac{1}{K\, R^2\, \bar{R}^2}(e^2 + \Lambda^* r^4),$$

$$p = \frac{1}{K\, R^2\, \bar{R}^2}\big\{e^2 - \Lambda^* r^2(r^2 + 2a^2\cos\theta)\big\}, \tag{6.48}$$

$$\begin{aligned} \mu^* &= -\frac{r}{K\, R^2\, \bar{R}^2}\big\{2\, r\, f(u)_{,u} + a^2\sin^2\theta\, f(u)_{,uu}\big\}, \\ \omega &= \frac{-i\, a \sin\theta}{\sqrt{2}\, K\, \bar{R}\, R^2}\, f(u)_{,u}, \\ \Lambda &\equiv g^{ab}\, R_{ab} = \frac{\Lambda^* r^2}{6\, R^2}, \end{aligned} \tag{6.49}$$

and ϕ_{11}, ϕ_{12}, ϕ_{22} could be obtained from equations (6.48) with (3.10). The Weyl scalars are given below

$$\begin{aligned}
\psi_2 &= \frac{1}{\overline{R}\,\overline{R}\,R^2}\Big[e^2 - R\{m + f(u)\} + \frac{\Lambda^* r^2}{3}a^2\cos^2\theta\Big], \\
\psi_3 &= \frac{-i\,a\sin\theta}{2\sqrt{2}\overline{R}\,\overline{R}\,R^2}\big\{\big(4r + \overline{R}\big)f(u)_{,u}\big\}, \\
\psi_4 &= \frac{a^2 r\sin^2\theta}{2\overline{R}\,\overline{R}\,R^2\,R^2}\big\{R^2 f(u)_{,uu} - 2rf(u)_{,u}\big\}.
\end{aligned} \tag{6.50}$$

This may represent a *rotating* Kerr-Newman-Vaidya-de Sitter solution with the line element

$$\begin{aligned}
ds^2 &= \Big[1 - R^{-2}\big\{2r\big(m + f(u) + \frac{\Lambda^* r^4}{3}\big) - e^2\big\}\Big]\,du^2 + 2du\,dr \\
&\quad + 2aR^{-2}\big\{2r\big(m + f(u) + \frac{\Lambda^* r^4}{3}\big) - e^2\big\}\sin^2\theta\,du\,d\phi - 2a\sin^2\theta\,dr\,d\phi \\
&\quad - R^2 d\theta^2 - \big\{(r^2 + a^2)^2 - \Delta^* a^2\sin^2\theta\big\}\,R^{-2}\sin^2\theta\,d\phi^2,
\end{aligned} \tag{6.51}$$

where $\Delta^* = r^2 - 2r\{m + f(u) - \Lambda^* r^4/3\} + a^2 + e^2$. Here m and e are the mass and the charge of rotating Kerr-Newman solution, a is the non-zero rotation parameter per unit mass and $f(u)$ represents the mass function of rotating Vaidya null radiating fluid. When we set $f(u) = 0$, the metric (6.51) recovers the Kerr-Newman-de Sitter black hole (6.41), and if $m = 0$, then it is the 'rotating' charged Vaidya null radiating black hole (5.13). When one sets $\Lambda^* = 0$, this metric would recover the Kerr-Newman-Vaidya metric ((6.21). In this *rotating* solution, the Vaidya *null fluid* is interacting with the *non-null electromagnetic field* whose Maxwell scalar ϕ_1 can be obtained from (6.48). Thus, we could write the total energy momentum tensor (EMT) for the *rotating* solution (6.51) as follows:

$$T_{ab} = T^{(n)}_{ab} + T^{(\mathrm{E})}_{ab} + T^{(\mathrm{C})}_{ab}, \tag{6.52}$$

where the EMTs for the *rotating null fluid*, the *electromagnetic field* and cosmological matter field are given respectively

$$\begin{aligned}
T^{(n)}_{ab} &= \mu^*\,\ell_a\,\ell_b + 2\,\omega\,\ell_{(a}\,\overline{m}_{b)} + 2\,\overline{\omega}\,\ell_{(a}\,m_{b)}, \\
T^{(\mathrm{E})}_{ab} &= 4\,\rho^{*(\mathrm{E})}\{\ell_{(a}\,n_{b)} + m_{(a}\overline{m}_{b)}\}, \\
T^{(\mathrm{C})}_{ab} &= 2\{\rho^{*(\mathrm{C})}\,\ell_{(a}\,n_{b)} + 2\,p^{(\mathrm{C})}\,m_{(a}\overline{m}_{b)}\},
\end{aligned} \tag{6.53}$$

where μ^* and ω are given in (6.49) and

$$\begin{aligned}
\rho^{*(\mathrm{E})} &= p^{(\mathrm{E})} = \frac{e^2}{K\,R^2\,R^2}, \\
\rho^{*(\mathrm{C})} &= \frac{\Lambda^* r^4}{K\,R^2\,R^2}, \quad p^{(\mathrm{C})} = -\frac{\Lambda^* r^2}{K\,R^2\,R^2}\big(r^2 + 2a^2\cos^2\theta\big).
\end{aligned} \tag{6.54}$$

The appearance of non-vanishing ω shows the null fluid is *rotating* as the expression (6.53) of ω involves the rotating constant a coupling with $\partial f(u)/\partial u$ – both are non-zero quantities for a *rotating* Vaidya null radiating universe.

This *rotating* Kerr-Newman-Vaidya-de Sitter metric (6.51) could be written in Kerr-Schild form on the de Sitter background as

$$g_{ab}^{\rm KNVdS} = g_{ab}^{\rm dS} + 2Q(u,r,\theta)\ell_a\ell_b \tag{6.55}$$

where

$$Q(u,r,\theta) = -[r\{m+f(u)\} - e^2/2]R^{-2}, \tag{6.56}$$

and the vector ℓ_a is a geodesic, shear free, expanding as well as rotating null vector of both $g_{ab}^{\rm dS}$ as well as $g_{ab}^{\rm KNVdS}$ and given in (2.6). We could also write this solution (6.51) in another Kerr-Schild form on the Kerr-Newman background as

$$g_{ab}^{\rm KNVdS} = g_{ab}^{\rm KN} + 2Q(u,r,\theta)\ell_a\ell_b \tag{6.57}$$

where $Q(u,r,\theta) = -\{rf(u) + \Lambda^* r^4/6\}R^{-2}$. These two Kerr-Schild forms (6.55) and (6.57) certainly confirm that our metric $g_{ab}^{\rm KNVdS}$ is a solution of Einstein's field equations since the background rotating metrics $g_{ab}^{\rm KN}$ and $g_{ab}^{\rm dS}$ are both solutions of Einstein's field equations. They both have different stress-energy tensors $T_{ab}^{(\rm E)}$ and $T_{ab}^{(\rm C)}$ given in (6.53).

6.2.8 *Rotating Vaidya-Bonnor-de Sitter solution*

In the same way, we would combine the *rotating* Vaidya-Bonnor solution (5.13) with the *rotating* de Sitter solution obtained above in (6.34), if the Wang-Wu functions $q_n(u)$ are chosen such that

$$q_n(u) = \begin{cases} f(u), & \text{when } n=0 \\ -e^2(u)/2, & \text{when } n=-1 \\ \Lambda^*/6, & \text{when } n=3 \\ 0, & \text{when } n\neq 0,-1,3, \end{cases} \tag{6.58}$$

where $f(u)$ and $e(u)$ are related with the mass and the charge of *rotating* Vaidya-Bonnor solution (5.13). Thus, we have the mass function

$$M(u,r) = f(u) - \frac{e^2(u)}{2r} + \frac{\Lambda^* r^3}{6}, \tag{6.59}$$

and other quantities are

$$\rho^* = \frac{1}{K\,R^2\,R^2}\{e^2(u) + \Lambda^* r^4\},$$

$$p = \frac{1}{K\,R^2\,R^2}\{e^2(u) - \Lambda^* r^2(r^2 + 2a^2\cos\theta)\}, \tag{6.60}$$

$$\mu^* = -\frac{1}{K\,R^2\,\overline{R}^2}\Big[2\,r^2\Big\{f(u)_{,u} - \frac{1}{r}e(u)\,e(u)_{,u}\Big\} + a^2\sin^2\theta\,\Big\{f(u)_{,u} - \frac{1}{r}e(u)\,e(u)_{,u}\Big\}_{,u}\Big],$$

$$\omega = \frac{-i\,a\sin\theta}{\sqrt{2}\,K\,R^2\overline{R}^2}\Big\{R\,f(u)_{,u} - 2e(u)\,e(u)_{,u}\Big\},$$

and ϕ_{11}, ϕ_{12}, ϕ_{22} could be obtained from equations (6.60) with (3.10). The Weyl scalars are given below

$$\begin{aligned}
\psi_2 &= \frac{1}{R\overline{R}\,\overline{R}^2}\Big[e^2(u) - R\,f(u) + \frac{\Lambda^* r^2}{3}a^2\cos^2\theta\Big],\\
\psi_3 &= \frac{-i\,a\sin\theta}{2\sqrt{2}R\,\overline{R}\,R^2}\Big[\big(4\,r + \overline{R}\big)f(u)_{,u} - 4e(u)\,e(u)_{,u}\Big],\\
\psi_4 &= \frac{a^2\sin^2\theta}{2R\,\overline{R}\,R^2\,\overline{R}^2}\Big[R^2\Big\{r\,f(u)_{,u} - e(u)e(u)_{,u}\Big\}_{,u}\\
&\quad -2r\Big\{r\,f(u)_{,u} - e(u)e(u)_{,u}\Big\}\Big].
\end{aligned} \tag{6.61}$$

This may represent a *rotating* Vaidya-Bonnor-de Sitter solution with the line element

$$\begin{aligned}
ds^2 = &\Big[1 - R^{-2}\Big\{2r\Big(f(u) + \frac{\Lambda^* r^3}{3}\Big) - e^2(u)\Big\}\Big]\,du^2 + 2du\,dr\\
&+2aR^{-2}\Big\{2r\Big(f(u) + \frac{\Lambda^* r^4}{3}\Big) - e^2(u)\Big\}\sin^2\theta\,du\,d\phi - 2a\sin^2\theta\,dr\,d\phi\\
&-R^2 d\theta^2 - \Big\{(r^2 + a^2)^2 - \Delta^* a^2\sin^2\theta\Big\}\,R^{-2}\sin^2\theta\,d\phi^2,
\end{aligned} \tag{6.62}$$

where $\Delta^* = r^2 - 2r\{f(u) - \Lambda^* r^3/3\} + a^2 + e^2(u)$. Here, a is the non-zero rotational parameter per unit mass and $f(u)$ represents the mass function of rotating Vaidya null radiating fluid. We could write the total energy momentum tensor (EMT) for the *rotating* solution (6.62) as follows:

$$T_{ab} = T_{ab}^{(n)} + T_{ab}^{(\mathrm{E})} + T_{ab}^{(\mathrm{C})}, \tag{6.63}$$

where the EMTs for the *rotating null fluid*, the *electromagnetic field* and cosmological matter field might be given respectively

$$T_{ab}^{(n)} = \mu^*\,\ell_a\,\ell_b + 2\,\omega\,\ell_{(a}\,\overline{m}_{b)} + 2\,\overline{\omega}\,\ell_{(a}\,m_{b)}, \tag{6.64}$$

$$T_{ab}^{(\mathrm{E})} = 4\,\rho^{*(\mathrm{E})}\big\{\ell_{(a}\,n_{b)} + m_{(a}\overline{m}_{b)}\big\}, \tag{6.65}$$

$$T_{ab}^{(\mathrm{C})} = 2\big\{\rho^{*(\mathrm{C})}\,\ell_{(a}\,n_{b)} + 2\,p^{(\mathrm{C})}\,m_{(a}\overline{m}_{b)}\big\}, \tag{6.66}$$

where

$$\rho^{*(\mathrm{E})} = p^{(\mathrm{E})} = \frac{e^2(u)}{K\,R^2\,\overline{R}^2},$$

$$\rho^{*(\mathrm{C})} = \frac{\Lambda^* r^4}{K\,R^2\,R^2}, \quad p^{(\mathrm{C})} = -\frac{\Lambda^* r^2}{K\,R^2\,R^2}\big(r^2 + 2a^2\cos^2\theta\big).$$

If we set $a = 0$, we would recover the *non-rotating* Vaidya-Bonnor-de Sitter solution and then the energy-momentum tensor (6.63) could be written in the form of Guth's modification of T_{ab} [35] as

$$T_{ab} = T^{(\mathrm{n})}_{ab} + T^{(\mathrm{E})}_{ab} + \Lambda^* g_{ab} \tag{6.67}$$

where $T^{(\mathrm{E})}_{ab}$ is the energy-momentum tensor for non-null electromagnetic field and g_{ab} is the *non-rotating* Vaidya-Bonnor-de Sitter metric tensor. From this, without loss of generality, one could regard the EMT (6.63) as the extension of Guth's modification [35] of energy-momentum tensor in *rotating* space.

The Vaidya-Bonnor-de Sitter metric could be written in the following Kerr-Schild form

$$g^{\mathrm{VBdS}}_{ab} = g^{\mathrm{dS}}_{ab} + 2Q(u,r,\theta)\ell_a\ell_b \tag{6.68}$$

where $Q(u,r,\theta) = -\{rf(u) - e^2(u)/2\}\,R^{-2}$. Here, g^{dS}_{ab} is the *rotating* de Sitter metric (6.34) and ℓ_a is geodesic, shear free, expanding and rotating null vector for both g^{dS}_{ab} as well as g^{VBdS}_{ab} and given in (2.6). The above Kerr-Schild form could be written on the *rotating* Vaidya-Bonnor background given in (5.13).

$$g^{\mathrm{VBdS}}_{ab} = g^{\mathrm{VB}}_{ab} + 2Q(r,\theta)\ell_a\ell_b \tag{6.69}$$

where $Q(r,\theta) = -(\Lambda^* r^4/6)R^{-2}$. These two Kerr-Schild forms (6.68) and (6.69) would prove the *non-stationary* version of theorem 3 and 4 in the case of *non-stationary rotating* Vaidya-Bonnor-de Sitter solution. If we set $f(u)$ and $e(u)$ are both constant, this Kerr-Schild form may be that of Kerr-Newman-de Sitter black hole (6.45). The *rotating* Vaidya-Bonnor-de Sitter metric may describe a *non-stationary* spherically symmetric solution whose Weyl curvature tensor is algebraically special in Petrov classification possessing a geodesic, shear free, expanding and non-zero rotation null vector ℓ_a given in (2.6). One could easily recover a *rotating* Vaidya-de Sitter metric from this Vaidya-Bonnor-de Sitter solution by setting the charge $e(u) = 0$. If one sets $a = 0$, $e(u) = 0$ in (6.62), one may obtain the standard non-rotating Vaidya-de Sitter solution [41]. Ghosh and Dadhich [42] have studied the gravitational collapse problem in *non-rotating* Vaidya-de Sitter space by identifying the de Sitter cosmological constant Λ^* with the bag constant of the null strange quark fluid. Also if one sets $a = 0$ in (6.62) one could recover the *non-rotating* Vaidya-Bonnor-de Sitter black hole [43]. It certainly indicates that all embedded solutions (6.21), (6.30), (6.41), (6.62) could derive by using Wang-Wu functions (6.8) in the *rotating* solutions (6.4), otherwise it might not be possible to obtain them by direct application of Newman-Janis algorithm as seen above in section 5.

7 Conclusion

In summary this paper has exhibited general solutions in terms of NP quantities for generating *rotating* solutions of Einstein's equations of spherically symmetric metric within the limitation of Newman-Janis algorithm. We could observe that the general solutions in NP quantities are so powerful to study the nature of the spacetime geometries. For example, we could conclude that the metric (2.8) with three variables, in general possesses a geodesic, shear free, rotating and expanding null vector ℓ_a as shown by equation (3.3)–(3.5). The non-vanishing ψ_2, ψ_3, ψ_4, suggest that the space is *algebraically special* in the Petrov classification. Chandrasekhar [20] has established a relation of spin coefficients ρ, μ, τ, π in the case of an affinely parameterized geodesic vector, generating an integral which is constant along the geodesic in a *vacuum* Petrov type D space-time

$$\frac{\rho}{\bar{\rho}} = \frac{\mu}{\bar{\mu}} = \frac{\tau}{\bar{\pi}} = \frac{\pi}{\bar{\tau}}. \tag{7.1}$$

This relation is being derived on the basis of the vacuum Petrov type D space-time with $\psi_2 \neq 0$, $\psi_0 = \psi_1 = \psi_3 = \psi_4 = 0$ and $\phi_{01} = \phi_{02} = \phi_{10} = \phi_{20} = \phi_{12} = \phi_{21} = \phi_{00} = \phi_{22} = \phi_{11} = \Lambda = 0$. However, it has been shown in [22] that the *non-vacuum* Petrov type D spacetimes *i.e.* Kerr-Newman solution possessing electromagnetic field and Kantowski-Sachs metric with dust energy-momentum tensor, satisfy the Chandrasekhar's relation (7.1). We would also like to show that the very general metric (2.8) with three variables, which is of *algebraically special* in Petrov classification having non-zero ψ_2, ψ_3, ψ_4 and stress-energy tensor (4.1), still satisfies the relation (7.1) as follows

$$\frac{\rho}{\bar{\rho}} = \frac{\mu}{\bar{\mu}} = \frac{\tau}{\bar{\pi}} = \frac{\pi}{\bar{\tau}} = \frac{R}{\bar{R}}. \tag{7.2}$$

This relation (7.2) shows that all *rotating* solutions, stationary and non-stationary, discussed here satisfy the relation (7.1). Thus, it seems reasonable to refer to the relation (7.1) as *Chandrasekhar's identity* as mentioned by Fermandes and Lun [44]. This certainly indicates that the NP spin coefficients (3.3) could be used to extend the known *vacuum* results like the relation (7.1) to the *non-vacuum* ones. Further, we also easily observe from the energy-momentum tensor (4.1) that the metric (2.8) with three variables does not include the perfect fluid. From the above discussion, it certainly indicates that the Newman-Janis algorithm is a powerful tool to generate rotating solutions from non-rotating 'seed' solutions, However such generated *rotating* solutions from the application of Newman-Janis algorithm have some limitations that these *rotating* solutions do not include spacetimes admitting rotating perfect fluid. To have a spacetime admitting a rotating perfect fluid one may look for another algorithm rather than that of Newman and Janis.

From the above results presented in this paper, it suggests that rotating spacetime geometry must also, in general have rotating matter fields, like the one represented by the stress-energy tensor T_{ab} in (4.1) with rotation function ω. It means that the matching of a rotating non-stationary space-time geometry with the usual non-rotating perfect fluid may not have a reasonably good sense. So one might need to look for a rotating perfect fluid to match with a *rotating non-stationary* spacetime geometry. Some of the rotating solutions discussed above include rotating *non-stationary* solutions, like Kerr-Newman-Vaidya black hole, Kerr-Newman-Vaidya-de Sitter black hole, Vaidya-Bonnor-de Sitter black hole. To study the nature of these *rotating* black holes would certainly be a new area of interest in classical General Relativity, since all known black hole theorems, like 'no hair theorem', Penrose's theorems are based on *stationary* black holes, rotating or non-rotating.

It is found that the Wang-Wu functions (6.8) in rotating spacetime geometry is also powerful to generate rotating solutions as shown above. By choosing a suitable Wang-Wu function $q(u)$, we obtain a rotating de Sitter space-time model. We could also recover the widely used (i) Schwarzschild-de Sitter solution, (ii) Reissner-Nordstrom-de Sitter black hole solution, (iii) Kerr-de Sitter solution, (iv) Kerr-Newman-de Sitter solution for early inflation scenarios from the rotating Vaidya-Bonnor-de Sitter solution (6.62). These embedded de Sitter space-times could generate by using Wang-Wu functions in rotating solutions given in (6.9). This shows that Wang-Wu functions in rotating space-time geometry could be applied to generate Kerr-de Sitter and Kerr-Newman-de Sitter solutions in a simple way – not directly applying Newman-Janis algorithm to Schwarzschild-de Sitter as well as Reissner-Nordstrom-de Sitter 'seed' solutions to derive Kerr-de Sitter as well as Kerr-Newman-de Sitter solutions respectively. Thus, our rotating solutions (6.9) with Wang-Wu functions could avoid the difficulties suggested by Xu [9] that Newman-Janis algorithm cannot be used to derive rotating Kerr-Newman-de Sitter solution from non-rotating Reissner-Nordstrom-de Sitter 'seed' metric. The definitions of embedded spaces used here are in agreement with the one defined by Cai et al. [45]. It is worth mentioning that our *rotating* embedded solutions into the *rotating* de Sitter universe, namely Kerr-Newman-de Sitter solutions (6.41) is found different from the ones discussed in [8,9,38]. Hence one could conclude that, to the best of the author's knowledge, the *rotating* embedded solutions (6.41), (6.51), (6.62) and other reducible solutions from (6.62), and also (6.21), (6.30) have not been seen derived before. Other *non-rotating* embedded solutions of Einstein's equations might be found in Kramer, et al. [46] (and references there in), Hodgkinson [47]. One may also have rotating solutions describing rotating black holes embedded into the *rotating* monopole solutions (6.11) if one chooses the Wang-Wu functions according to his choice, for examble, *rotating* Kerr-Newman-monopole, *rotating* Vaidya-Bonnor-monopole etc. However, one has to remember the suggestions made by Lee, et al [48] and Visser [32] that 'matters of such embedded spacetimes may become more complicated if horizons are present.' The situation of this statement could be seen in equations (6.22), (6.43), (6.44), (6.63), (6.67) above of different matter fields located at the same radial coordinate of the embedded spacetime geometries. This certainly suggests that one has *either* to confine oneself within the standard spacetimes with well-defined horizons *or* to forget the arbitrary horizons while looking for new investigations with such embedded classical solutions presented here. Therefore, it is emphasized here that, if we do not investigate these rotating embedded de Sitter solutions, we might have not come accross the natural existence of Guth's modification of energy-momentum tensors (6.44), (6.67) and their extensions in *rotating* cases (6.43), (6.52), (6.63).

It is also believed that these *rotating* solutions (5.9) and (5.13) discussed above could be able to avoid the undesirable comments on (i) the interpretation of variable mass Kerr solution [4] made by Herrera and Martinez [5], Herrera et.at [6], (ii) Mallett,s result [8] on de Sitter space mentioned by Xu [9] and (iii) the assumptions of Jing and Wang [10] on the mass $M(u)$ and the charge $e(u)$ after the application of Newman-Janis algorithm to the non-rotating Vaidya-Bonnor 'seed' solution with mass $M(u)$ and charge $e(u)$ as mentioned in the introduction. The *rotating* solutions (5.13) would strongly complement the spirit of the effort to generate *rotating* non-stationary metrics discussed in [4,10,27].

Looking to these overall *rotating* solutions derived above one could conclude that

1. all *stationary rotating* spherically symmetric solutions including (a) Kerr-Newman, (b) rotating monopole, (c) rotating de Sitter, (d) Kerr-Newman de Sitter solutions which are derivable from the application of Newman-Janis algorithm, are Petrov type D and each spacetime has the same repeated principal null vector ℓ_a, which is geodesic, shear free, expanding as well as rotating. This completes the proof of theorem 5.

2. rotating Vaidya (5.9), rotating Vaidya-Bonnor (5.13), rotating Kerr-Newman-Vaidya (6.21), rotating Kerr-Newman-Vaidya-Bonnor (6.26), rotating Vaidya-de Sitter (6.62) when $e(u) = 0$ and rotating Vaidya-Bonnor-de Sitter (6.62) are all *non-stationary* spherically symmetric solutions. Their Weyl curvature tensors are *algebraically special* in the Petrov classification with null vector ℓ_a given in (2.6). This leads the proof of the theorem 6 stated in the introduction.

The remarkable feature of the analysis of *rotating* solutions in this paper is that all the *rotating* solutions, *stationary* Petrov type D and *non-stationary* algebraically special, presented here possess the same null vector ℓ_a, which is geodesic, shear free, expanding as well as non-zero rotation. From the analysis of these *rotating* solutions one could observe that some solutions after making *rotating* spacetimes have disturbed their gravitational structure. For example, the *rotating* monopole solution possesses a different energy-momentum tensor by introducing the monopole pressure p in which the monopole constant couples with the rotating parameter a . Similarly, the *rotating* de Sitter solution becomes Petrov type D spacetime metric, where the rotating parameter a is coupled with the de Sitter cosmological constant and so on. We have shown that all the *rotating* embedded solutions presented here could be written in Kerr-Schild forms, indicating the extension of those of Xanthopoulos [31] and of Glass and Krisch [23] as mentioned before. Also our new *rotating* embedded solutions could not be considered as bimetric theories because $g_{ab}^{\rm KNV} \neq \frac{1}{2}(g_{ab}^{\rm KN} + g_{ab}^{\rm V})$ as an example.

As a part of the further discussion of physical properties of *rotating* solutions appeared in section 5 and 6 above, the paper II [36] deals with the *variable-charged, non-rotating* Reissner-Nordstrom-de Sitter and *rotating* Kerr-Newman-de Sitter black holes to study the Hawking's radiation effect in relativistic viewpoint. One might regard the results of paper II as an extension in embedded de Sitter spaces of the earlier results discussed in [1]. These two papers [1,36] are based on *stationary* black holes. In another paper, we might discuss the extension of earlier works of relativistic aspect of Hawking radiation in *non-stationary* variable-charged black holes, *rotating* and *non-rotating*. Therefore, it may conclude that all these *rotating non-stationary* solutions presented here might serve the need for the extension of *stationary* solutions of Einstein's field equations to *non-stationary* ones for future investigation.

Acknowledgement

The author acknowledges his appreciation for hospitality received from Inter-University Centre for Astronomy and Astrophysics (IUCAA), Pune during the preparation of this paper.

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