

Quantum fluctuations in gravitational collapse and cosmology

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Summary. It is shown that the conformal degrees of freedom in the metric tensor can be quantized and that this procedure leads to fluctuations around the solutions of the classical Einstein field equations. These fluctuations become progressively more important as the classical solution approaches the space–time singularity. An explicit calculation is given of the quantum mechanical propagator which describes the conformal fluctuations in a collapsing homogeneous ball of dust. As the state of classical singularity is approached the quantum uncertainty diverges. Within the range of quantum uncertainty non-singular final states are possible.

The solution can also be applied to the Friedmann models with the conclusion that the Universe need not have originated in a unique classical big bang. Non-singular models or models without particle horizons are permitted within the range of the quantum uncertainty.

1 Introduction

Although general relativity has provided a good classical theory of gravitation, its marriage to quantum theory has not proved very successful so far. In the classical Einstein equations

$$R_{ik} - \frac{1}{2}g_{ik}R = -\frac{8\pi G}{c^4} T_{ik}, \quad (1)$$

the left-hand side describes the geometry of space–time while the right-hand side describes the physics of the interacting matter and radiation. In a fully quantized version of (1) both sides of the Einstein equations must have their quantum counterparts. Considerable progress has been made in quantizing the linearized version of (1), where the space–time metric g_{ik} differs from the metric η_{ik} of the Minkowski space–time by first-order quantities. However, the linearized approach gives no clue to the subtleties which might exist in the behaviour of strong gravitational fields. The non-linearity of field equations in such cases poses a formidable problem where the standard quantization procedure breaks down. On the other hand, the quantization of physical fields (as represented by T_{ik} above) in a given classical curved space–time leads to interesting results as first pointed out by Hawking (1974).

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In this paper we will consider the problem of quantization of the left-hand side of (1) from a somewhat restricted point of view. It has been noted by DeWitt (1967) that in the quantization problem it is relatively easy to separate out the so called *conformal* degrees of freedom. These represent metric fluctuations of the following type

$$g_{ik} \rightarrow \Omega^2 g_{ik}, \quad (2)$$

where Ω is an arbitrary non-zero function of space–time coordinates. In this paper we will discuss situations in relativistic astrophysics and cosmology where such fluctuations become very important if gravity is treated quantum mechanically.

Specifically, we will consider the space–time singularity which develops at the end of gravitational collapse or which is supposed to mark the beginning of a big bang universe. The singularity is an outcome of several powerful theorems on classical general relativity [Penrose (1965); Hawking & Ellis (1973)]. However, close to singularity the action functional S describing the physical system usually becomes small compared to \hbar (= Planck's constant/ 2π). Under such circumstances the system must be treated quantum mechanically rather than classically. Does this result in a change from classical conclusions? We will try to answer this question within the realm of quantum conformal fluctuations.

2 The path integral formalism

The relativistic equations are derivable from the Hilbert action principle with the action given by

$$S = \frac{c^4}{16\pi G} \int_{\mathcal{V}} R \sqrt{-g} d^4x + S_m, \quad (3)$$

where R = scalar curvature, g = determinant of g_{ik} and \mathcal{V} is the 4-volume of space–time under consideration. S_m denotes the action for matter (and radiation) which gives rise to the energy tensor T_{ik} in (1).

In the classical variation $\delta S = 0$ for $g_{ik} \rightarrow g_{ik} + \delta g_{ik}$, with the metric variations vanishing on the boundary $\partial \mathcal{V}$. What about the metric derivatives $g_{ik,l}$? In the usual derivation it is tacitly assumed that their variations also vanish on the boundary $\partial \mathcal{V}$. Hawking (1976) has pointed out that the inclusion of a suitable surface integral in (3) will remove this restriction on $\delta g_{ik,l}$ which arises essentially because R contains second derivatives of the metric tensor. This difference apart, the general character of Einstein's is believed to be similar to that in the rest of classical field theory. Given suitable initial data it is usually possible to work out the subsequent dynamical behaviour of space–time geometry.

This data may be specified in the form of a 3-geometry ${}^{(3)}\mathcal{G}_1$, and its time derivative ${}^{(3)}\dot{\mathcal{G}}_1$ on an initial space-like hypersurface Σ_1 . The difficulties and restrictions in such a specification have been extensively discussed (see Misner, Thorne & Wheeler 1973). Assuming that the space–time volume \mathcal{V} is sandwiched between Σ_1 and a later space-like hypersurface Σ_2 , the aim is to use (1) to calculate the 3-geometry ${}^{(3)}\mathcal{G}_2$ on Σ_2 . This is the essence of the classical problem.

In the quantum analogue of this situation, which becomes important for $S \lesssim \hbar$, we may state the problem somewhat differently. We are no longer restricted to geometries obtained from (3) by $\delta S = 0$. Instead, all geometries are possible. Following the path integral formulation of Feynman (1948) we associate a probability amplitude proportional to

$$\exp(iS/\hbar) \quad (4)$$

for any given geometry. The total probability amplitude to have a space-time with a 3-geometry ${}^{(3)}\mathcal{G}_1$ on Σ_1 and ${}^{(3)}\mathcal{G}_2$ on Σ_2 is given by

$$K [{}^{(3)}\mathcal{G}_2, \Sigma_2; {}^{(3)}\mathcal{G}_1, \Sigma_1] = \sum \exp \left(\frac{iS}{\hbar} \right). \quad (5)$$

The left-hand side is the Feynman propagator for geometrodynamics. The right-hand side involves the sum over 'histories', i.e. over the various different geometries consistent with the specified initial and final states. This sum over geometries is a vaguely defined concept and poses numerous difficulties when attempts are made to do an explicit calculation. The difficulty of defining measure which is present in other path integral problems, is present here too.

It is, however, possible to make progress by confining ourselves to conformal fluctuations only. To this end we will specify the classical solution of (1) under a given set of boundary conditions by barred quantities, e.g. \bar{g}_{ik} , \bar{R}_{ik} , \bar{T}_{ik} etc. A non-classical solution is obtained from this by a conformal transformation:

$$g_{ik} = \Omega^2 \bar{g}_{ik} \quad (6)$$

where Ω is a general function of space and time. For the metric g_{ik} we have

$$\int_{\mathcal{V}} R \sqrt{-g} d^4x = \int_{\mathcal{V}} (\Omega^2 \bar{R} - 6\Omega_i \Omega^i) \sqrt{-\bar{g}} d^4x + \int_{\partial \mathcal{V}} 6\Omega \Omega^i \sqrt{-\bar{g}} d\Sigma_i. \quad (7)$$

Here $\Omega_i = \partial\Omega/\partial x^i$, x^i being the space-time coordinates ($i = 1, 2, 3, 4$; 4-time like). The indices are raised or lowered with the metric \bar{g}_{ik} . The surface-integral is cancelled if one uses the Hawking prescription of adding a surface integral to the action. Alternatively, if one uses the restrictions $\delta g_{ik,l} = 0$ on the surface, the surface integral turns out to be unimportant for the computation of K . Henceforth we shall ignore it

The problem of computation of K can now be restated in the following form:

$$K [\Omega_2, \Sigma_2; \Omega_1, \Sigma_1] = \sum \exp \left\{ \frac{i}{\hbar} \left(S_m + \frac{c^4}{16\pi G} \int_{\mathcal{V}} (\Omega^2 \bar{R} - 6\Omega_i \Omega^i) \sqrt{-\bar{g}} d^4x \right) \right\} \quad (8)$$

where the summation is only over the histories of Ω . \bar{R} , \bar{g} etc. are supposed to be *known* functions of x^i . The matter part of the action may also depend on Ω , as is the case in the explicit example considered in the following section. The problem now is considerably simplified.

Consider the application of (8) to the empty Minkowski space-time, with Ω a function of t only. Then (8) becomes

$$K [\Omega_2, t_2; \Omega_1, t_1] = \sum \exp \left\{ \frac{iVc^4}{16\pi G\hbar} \int -6\Omega^2 dt \right\}. \quad (9)$$

where $V = 3$ -volume of the subspace under question and the Σ_1, Σ_2 are given by $t = t_1, t_2$. The action on the right-hand side of (9) is remarkably similar to the action of a free particle moving in a one-dimensional space denoted by Ω . For this the solution is already known and is given by

$$K [\Omega_2, t_2; \Omega_1, t_1] = \left(\frac{3Vc^4}{8\pi^2 G(t_2 - t_1) i\hbar} \right) \exp \left\{ \frac{-3iVc^4(\Omega_2 - \Omega_1)^2}{8\pi G\hbar(t_2 - t_1)} \right\}. \quad (10)$$

We will next apply (8) to the problem of gravitational collapse.

3 The collapsing dust ball

The discovery of quasi-stellar objects regenerated interest in the problem of relativistic gravitational collapse. The quasi-stellar objects are bright but compact which suggested the localization of enormous quantities of matter ($\gtrsim 10^6 M_\odot$) into single coherent units. Thermo-nuclear energy is generally found to be insufficient to provide equilibrium of such objects against their self-gravity, with the result that they undergo gravitational contraction which becomes progressively more violent. Even ordinary stars, if they are sufficiently massive ($M \gtrsim 3M_\odot$) are believed to share this fate after they have exhausted their stores of nuclear energy. The general solution of a collapsing object with pressure is not known, although many qualitative features are relatively better understood. In any case, once the collapsing object has crossed into the event-horizon, it encounters a space-time singularity provided the equation of state is not too esoteric (e.g. with negative densities or with pressure exceeding density).

To see how quantum uncertainty develops in the final stages of the collapse it is instructive therefore to use the classical case of zero pressure which has been solved explicitly. Oppenheimer & Snyder (1939) had solved the problem of a homogeneous dust ball collapsing from infinity. In this paper we will use the notation of Hoyle & Narlikar (1964) in describing the case of a dust ball collapsing from an initial state of rest and uniform density in an otherwise empty space-time.

3.1 THE CLASSICAL SOLUTION

The external solution in this case is given by the familiar Schwarzschild line element

$$d\bar{s}^2 = c^2 dT^2 \left(1 - \frac{2GM}{c^2 R}\right) - dR^2 \left(1 - \frac{2GM}{c^2 R}\right)^{-1} - R^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (11)$$

where M is the gravitational mass of the collapsing dust ball. The internal solution is best expressed in comoving coordinates; with the line element

$$d\bar{s}^2 = c^2 dt^2 - Q^2(t) \left\{ \frac{dr^2}{1 - \alpha r^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right\}. \quad (12)$$

Here r, θ, ϕ are the (constant) comoving coordinates of a typical dust particle and t its proper time. The object is confined to $r \leq r_b$ and its coordinate volume is given by

$$V = \int_0^{r_b} \frac{4\pi r^2 dr}{\sqrt{1 - \alpha r^2}}. \quad (13)$$

The parameter α is related to the starting density ρ_0 of the object by

$$\alpha = \frac{8\pi G \rho_0}{3c^2}. \quad (14)$$

The scale-function $Q(t)$ monotonically decreases to zero from an initial state of $Q = 1$, $\dot{Q} = 0$, and satisfies the differential equation

$$\dot{Q}^2 = \alpha c^2 \left(\frac{1 - Q}{Q} \right). \quad (15)$$

Thus the time of collapse to the final singularity is given by $t_0 = \pi/2c\sqrt{\alpha}$.

It will be convenient to choose the zero of the t scale to be at the classical singularity so that the initial instant is denoted by $-t_0$. We will be interested mainly in the epochs close to $t = 0$ where $Q \ll 1$ for which

$$Q \sim (-^{3/2} \sqrt{\alpha} c t)^{2/3} = (^{3/2} \sqrt{\alpha} c)^{2/3} \tau^2, \quad (16)$$

where $\tau = (-t)^{1/3}$.

3.2 THE PROPAGATOR FOR CONFORMAL HOMOGENEOUS FLUCTUATIONS

We will look for such conformal fluctuations of the above classical solution, as preserve the symmetries of the classical problem. It is easy to see that the symmetries, i.e. homogeneity and isotropy of the line element (12) are preserved by a conformal factor Ω which depends on t only. Writing

$$\Omega dt = d\tilde{t}, \quad \tilde{Q} = Q\Omega, \quad (17)$$

we can transform (12) to a similar form in $(r, \theta, \phi, \tilde{t})$ coordinates with Q replaced by \tilde{Q} . We will now consider the quantum mechanical problem posed in the last section, with Ω a function of t only.

We take the surfaces Σ_μ to be those given by $t = t_\mu < 0$ ($\tau = \tau_\mu > 0$) respectively, for $\mu = 1, 2$. Also, we will confine our attention to the interior of the dust ball only. Outside $r = r_b$ we can choose Ω to be unity asymptotically as $R \rightarrow \infty$. Using (8) and noting that for the classical solution the scalar curvature is given by $\bar{R} = 8\pi G\rho_0/c^2 Q^3$, we get

$$K[\Omega_2, t_2; \Omega_1, t_1] = \sum \exp \left\{ \frac{iV\rho_0 c^2}{\hbar} \int_{t_1}^{t_2} \left(\frac{1}{2} \Omega^2 - \frac{Q^3}{\alpha c^2} \dot{\Omega}^2 - \Omega \right) dt \right\}. \quad (18)$$

Here the last term in the integral arises from S_m for the system of dust particles a, b, c, \dots :

$$S_m = - \sum_a m_a da = - \int \bar{\rho} \sqrt{-\bar{g}} \Omega d^4x = - \rho_0 V \int_{t_1}^{t_2} \Omega dt. \quad (19)$$

It has been assumed as is usual in relativity that masses do not change under a conformal transformation. [If masses transform as Ω^{-1} as in the theory of Hoyle & Narlikar (1974) then S_m in (19) would not depend on Ω .]

In (18) we now have the following path-integral problem: 'Given that Ω -paths start at $\Omega(t_1) = \Omega_1$ and end at $\Omega(t_2) = \Omega_2$ what is the value of K ?' If Q were a constant the problem would be almost similar to that of a harmonic oscillator. Since the integrand is at most a quadratic in Ω and $\dot{\Omega}$, the following simplification is possible:

$$K[\Omega_2, t_2; \Omega_1, t_1] = F(t_1, t_2) \exp \left\{ \frac{iV\rho_0 c^2}{\hbar} \int_{t_1}^{t_2} \left(\frac{1}{2} \Omega_c^2 - \frac{Q^3}{\alpha c^2} \dot{\Omega}_c^2 - \Omega_c \right) dt \right\} \quad (20)$$

where $\Omega_c(t)$ is the 'classical' solution of the variational problem

$$\delta \int_{t_1}^{t_2} \left(\frac{1}{2} \Omega^2 - \frac{Q^3}{\alpha c^2} \dot{\Omega}^2 - \Omega \right) dt = 0, \quad \text{for } \delta\Omega(t_1) = \delta\Omega(t_2) = 0. \quad (21)$$

For a proof of this general result see Feynman & Hibbs (1965). The function F has to be determined from indirect methods.

The solution of (21) is expressed more conveniently in terms of the departure from the Einstein solution $\Omega \equiv 1$; i.e. by

$$\Phi = \Omega - 1. \quad (22)$$

A tedious but straightforward calculation gives

$$\Phi_c = A\tau^{-1} + B\tau^{-2}, \quad (23)$$

where

$$A = \frac{\Phi_1\tau_1^2 - \Phi_2\tau_2^2}{\tau_1 - \tau_2}, \quad B = \frac{\tau_1\tau_2(\Phi_2\tau_2 - \Phi_1\tau_1)}{\tau_1 - \tau_2}. \quad (24)$$

Substitution in (20) gives

$$K[\Phi_2, \tau_2; \Phi_1, \tau_1] = F(\tau_1, \tau_2) \exp \left[\frac{3iV\rho_0c^2}{4(\tau_1 - \tau_2)} \{ \tau_1^3(\tau_1 - 2\tau_2)\Phi_1^2 + 2\tau_1^2\tau_2^2\Phi_1\Phi_2 + \tau_2^3(\tau_2 - 2\tau_1)\Phi_2^2 \} \right]. \quad (25)$$

The 'reproducing' property of the kernel is sufficient to fix $F(\tau_1, \tau_2)$ in the form:

$$F = \left(\frac{3iV\rho_0c^2\tau_1^2\tau_2^2}{4\pi\hbar(\tau_1 - \tau_2)} \right)^{1/2}. \quad (26)$$

Hence we have

$$K[\Phi_2, \tau_2; \Phi_1, \tau_1] = \left(\frac{3iV\rho_0c^2\tau_1^2\tau_2^2}{4\pi\hbar(\tau_1 - \tau_2)} \right)^{1/2} \exp \left[\frac{3iV\rho_0c^2}{4\hbar(\tau_1 - \tau_2)} \{ \tau_1^3(\tau_1 - 2\tau_2)\Phi_1^2 + 2\tau_1^2\tau_2^2\Phi_1\Phi_2 + (\tau_2 - 2\tau_1)\tau_2^3\Phi_2^2 \} \right]. \quad (27)$$

Within the simplifying assumptions of the problem this result is *exact*. We will now apply it to study the effect of quantum uncertainty in the limiting case of $t_2 \rightarrow 0$, $\tau_2 \rightarrow 0$, i.e. near the epoch where the classical solution predicts a space-time singularity.

3.3 THE GROWTH OF QUANTUM UNCERTAINTY

Although it is interesting that a definite expression for the propagator has emerged, the form (27) by itself does not readily convey the information about the effect of quantum uncertainty. It is more instructive to apply it to 'wave functions' describing the states of the collapsing system. Generally speaking, we have the following result:

$$\psi(\Phi_2, \tau_2) = \int K[\Phi_2, \tau_2; \Phi_1, \tau_1] \psi(\Phi_1, \tau_1) d\Phi_1, \quad (28)$$

which connects the wave function at epoch τ_1 to the wave function at epoch $\tau_2 (< \tau_1)$. In a purely classical situation $\Phi_1 \equiv 0$ and $|\psi(\Phi_1, \tau_1)|^2$ behaves like the delta function $\delta(\Phi_1)$. However, we will assume that $|\psi(\Phi_1, \tau_1)|^2$ is not strictly a delta function, but is a function strongly peaked at $\Phi_1 = 0$. A convenient expression for $\psi(\Phi_1, \tau_1)$ is the following:

$$\psi(\Phi_1, \tau_1) = (2\pi\Delta^2)^{-1/4} \exp\left(-\frac{\Phi_1^2}{4\Delta_1^2}\right), \quad (29)$$

where $\Delta_1 \ll 1$. It is easy to check that $|\psi|^2$ has the integral equal to unity over the entire Φ_1 range, although the smallness of Δ_1 implies that with a high degree of probability the solution at τ_1 is close to the classical one, which is the mean of this probability distribution ($\langle \Phi_1 \rangle = 0$).

Using (29) and (27) to compute $\psi(\Phi_2, \tau_2)$ we find that $|\psi(\Phi_2, \tau_2)|^2$ has again the form of the normal distribution whose mean is still the classical solution $\Phi_2 = 0$; however, the standard deviation has increased to

$$\Delta_2 = \frac{\hbar}{3V\rho_0 c^2 \tau_1 \Delta_1} \cdot \left\{ 1 + \left(\frac{3V\rho_0 \Delta_1^2 c^2}{\hbar} \tau_1^3 \right)^2 \right\}^{1/2} \tau_2^{-2}. \quad (30)$$

Note that however small Δ_1 may be, Δ_2 diverges to infinity as $\tau_2 \rightarrow 0$. Thus even though the solution was initially highly classical, the uncertainty does eventually dominate the situation. In quantum mechanical language, the solution (29) describes a stationary wave packet in the (Φ_1, τ_1) plane. At τ_2 the wave packet while still stationary, has diffused considerably, making in the limit any value of Φ_2 possible.

Does this uncertainty include non-singular states at $\tau_2 \rightarrow 0$? From (17) we see that the scale factor $\tilde{Q} = Q\Omega$ determines whether the line element represents a singularity or not. Since as $\tau_2 \rightarrow 0$, $\Delta_2 \sim \tau_2^{-2}$ we have $\Omega \sim \tau_2^{-2}$. Also, the function Q behaves as τ_2^2 so that

$$\tilde{Q} \cong \text{constant}. \quad (31)$$

Thus the full range of uncertainty permits non-singular solutions. With $\Omega \sim \tau^{-2} \sim (-t)^{-2/3}$ (17) gives

$$\tilde{t} \sim t^{1/3} + \text{constant}. \quad (32)$$

Thus, for the comoving dust particles the zero of the effective proper-time \tilde{t} is at a finite affine length. The object may therefore pass through $\tilde{t} = 0$ with a non-singular state.

3.4 SOME ORDERS OF MAGNITUDE

We will apply (30) to two situations of astrophysical orders of magnitude.

Suppose we have a dust ball of mass M with an initial density ρ_0 . At what stage during its collapse will quantum uncertainty take over? To answer this question set

$$\left(\frac{3V\rho_0 \Delta_1^2 c^2}{\hbar} \right) \tau_1^3 \equiv \eta \ll 1, \quad (33)$$

to indicate that the initial state at τ_1 is highly classical. Then from (30) we get

$$\Delta_2 \sim \eta^{-1/2} \left(\frac{\tau_1}{\tau_2} \right)^2 \left(\frac{\hbar}{3Mc^2} \sqrt{\frac{8\pi G\rho_0}{3}} \right)^{1/2}, \quad (34)$$

where we have set $V\rho_0 \sim M$, $\tau_1^3 = -t_1 \sim t_0$. For $M \sim M_\odot$ and $\rho_0 \sim 1 \text{ gm cm}^{-3}$ the expression in the square brackets is $\sim 10^{-43}$. The factor $(\tau_1/\tau_2)^2$ denotes the linear shrinkage since $Q \propto \tau^2$. For quantum uncertainty to be important, $\Delta_2 \gtrsim 1$, so that the linear shrinkage must be by a factor

$$f(\eta) = 10^{-43} \eta^{-1/2}. \quad (35)$$

Thus, the less the uncertainty in the early stages the sooner will the quantum effects dominate.

Apply (34) next to the dust balls which begin their collapse from an initial state close to that of a black hole. In this case we have for (34)

$$\Delta_2 \sim \eta^{-1/2} \left(\frac{\tau_1}{\tau_2} \right)^2 \left(\frac{R_C}{R_S} \right)^{1/2}, \quad (36)$$

where R_C = Compton wavelength associated with the black hole and R_S = Schwarzschild radius of the black hole. Setting $R_C = R_S$ gives

$$M = \sqrt{\frac{c\hbar}{2G}} \sim 10^{-5} \text{ g}. \quad (37)$$

For black holes of masses smaller than this, quantum uncertainty begins to dominate straight away: their collapse cannot be treated classically with any degree of confidence.

The limiting black hole mass for evaporation by the Hawking process is $\sim 10^{15}$ g, for which

$$\left(\frac{R_C}{R_S} \right)^{1/2} \sim 10^{-20}. \quad (38)$$

For the collapse of such masses quantum effects become important for a linear shrinkage of

$$f(\eta) \cong 10^{-20} \eta^{-1/2}. \quad (39)$$

The validity of the assumption of a classical background metric in the computations of particle production by black holes is to be judged by the above expression. A strictly classical metric would imply a very small η (ideally $\eta = 0$) which would mean that the quantum effects within the black hole become important very soon, i.e. for $f(\eta)$ not very small compared to unity. A large η (i.e. $\eta \sim 1$) will put off the development of quantum uncertainty to a much smaller value of $f(\eta)$ but it will imply moderate quantum fluctuations of the metric even at $\tau = \tau_1$. Thus it appears that in dealing with black holes of masses $\leq 10^{15}$ g the quantum fluctuations of the metric cannot be ignored. This conclusion, arrived at for dust-ball black holes, needs to be confirmed for more general cases of gravitational collapse.

4 Big bang cosmology

The above result can be applied to the big bang models of general relativity. The Friedmann models which are the homogeneous and isotropic dust solutions of the Einstein equations are described by the line element

$$d\bar{s}^2 = c^2 dt^2 - Q^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right], \quad (40)$$

where the parameter $k = 0, -1$ or $+1$. For $k = +1$ the line element is a scaled version of the dust-ball line element (12), and so the results can be taken over. V in this case could represent the coordinate volume of the *entire* universe. For the $k = 0$ or -1 models the volume V is infinite and we consider a bounded coordinate region of the entire universe. For such a region V is finite. For example, in the $k = 0$ Einstein–deSitter model, the present particle horizon is given by $r = 2cH^{-1}$, where H = present Hubble's constant. If we confine our attention to this region we have $V = 32\pi c^3/3H^3$.

So far as the dynamical calculation is concerned, all three types of the models given by (40) behave similarly in the classical solution near singularity. The effect of the k -term is not felt when Q is small.

It is worth emphasizing that the time sense in the expanding universe solution is reverse to that of the collapsing object. Hence we have to argue that the present (nearly classical) state of the universe could have arisen from a wide range of non-classical states close to the singular epoch. This was emphasized by Hoyle & Narlikar (1970) in a qualitative way. They had pointed out that some of the non-classical solutions might not have particle horizons.

Particle horizons play the role of restricting the flow of information from one part of the universe to another and their existence raises the question of how the primordial fireball radiation became so homogeneous. Misner (1969) had looked to a mix-master type solution which at any instant has no particle horizon in one direction (which keeps changing). However, this approach is now known to be ineffective in achieving the desired degree of homogeneity [see Chitre 1972].

If we take into account the early uncertainty of states of the universe, the issue would be resolved if we could show that there is a finite probability for the universe to have come out of a range of states with no particle horizons. In the discussion by Hoyle & Narlikar (1970) the model with no particle horizons was taken to be the one with

$$\tilde{Q} \propto \tilde{r}. \quad (41)$$

With the help of (20) we see that this requires the conformal factor

$$\Omega \propto t^{-2/3} \exp(at^{1/3}) \propto \tau^{-2} \exp(a\tau), \quad (42)$$

where a is a constant. (To represent expanding solutions the sign of t is now reversed from that in Section 3.) Since for small τ , $\exp(a\tau) \sim 1$, $\Omega_2 \sim \Delta_2$ as $\tau_2 \rightarrow 0$. Thus there is a finite probability for the universe to have emerged from such a state. [In these estimates we would conclude a zero probability if $\Omega/\Delta_2 \rightarrow \infty$.]

5 Conclusion

The approach described here should be looked upon as one of the many attempts to understand the nature of the space–time singularity that lies at the end of a gravitational collapse or at the beginning of the universe according to classical general relativity. It is generally agreed that the existence of the singularity reflects some incompleteness in our understanding of gravitation through classical general relativity. It may well be that a better understanding or removal of the singularity could be achieved entirely at the classical level by a future more complete theory than relativity. In the present approach we have shown that going over to the quantum theory opens up a wider range of final states in the collapse problem (or a wider range of initial states in the problem of the Universe). Even the simple problem of conformal fluctuations shows how the quantum uncertainty rapidly grows near a space–time singularity.

There are numerous ways in which this approach could be extended to cover more general cases and to bring in more degrees of freedom. For example, homogeneous anisotropically expanding universes with line elements

$$d\bar{s}^2 = c^2 dt^2 - X^2(t) dx^2 - Y^2(t) dy^2 - Z^2(t) dz^2$$

would show quantum fluctuations in $X(t)$, $Y(t)$ and $Z(t)$. An inhomogeneous but spherically symmetric gravitational collapse will require a conformal function $\Omega(t, r)$. The solution of such simplified specific problems will lead to a better understanding of quantum gravity.

Finally, the present paper has been devoted to the demonstration of the onset of uncertainty close to the classical singularity. It is hoped to extend the approach in a later paper to the discussion of possible ‘tunnelling’ through the singularity.

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