

Quantum stationary geometries and avoidance of singularities

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Abstract. A recent approach to quantum gravity leading to the concept of quantum stationary geometries is reviewed. A stationary states equation is presented for: (a) homogeneous relativistic cosmologies of the various Bianchi types; (b) a Friedmann universe filled with a massless scalar field. The equation is solved near the singularity to show that stationary states avoid the singularity. The result is discussed and compared with other approaches.

1. An approach to quantum gravity

Does quantum gravity avoid the singularities present in the classical theory? This question was answered differently by different people. The divergence in opinion arises mainly from divergent routes to quantum gravity (for a review see MacCallum (1976)).

In the recent past, an attempt was made to discuss quantum cosmology, using Feynman's path integral approach. The approach began with a study of quantum fluctuations around classically singular space-times, and led to the result that quantum fluctuations of the conformal degree of freedom always diverge at the classical singularity (see for details Narlikar 1981, Padmanabhan 1982b, Padmanabhan and Narlikar 1982). Further investigations showed that one can describe the space-time by what may be called 'quantum stationary geometries' (QSG). The structure of QSG was discussed in detail for Friedmann universes (Padmanabhan and Narlikar 1981, Padmanabhan 1982a). It turns out that they are instrumental in stopping the collapse of a system to a singular state, in direct analogy with the standard stationary states of the hydrogen atom, where also a classical collapse is prevented by quantum stationary states. In a later work we have extended the formalism to cover all the Bianchi universes (Padmanabhan 1982c).

While this formalism has led to a consistent picture, there still exist some open questions. In the previous works the effect of the source on QSG was not clear because of conformal invariance (see Padmanabhan (1982a) for a discussion of this point). Further, the source was treated as a classical distribution. In the first part of this paper, we shall consider a Robertson-Walker universe filled with a massless scalar field and will treat the dynamics of both the universe and the scalar field as quantum mechanical. Recently the same problem has been discussed by Isenberg and Gotay (1981), using geometric quantisation, and it was concluded that the singularity is not avoided in the quantum domain. Our approach based on QSG leads to the opposite view. It was noticed before also (Narlikar 1978) that our approach has led to the avoidance of

singularities, in contrast with results based on canonical quantisation. This reflects the basic difference between our approach and the conventional ones. However, within our formalism the results are clear and consistent.

In the latter part of the paper we shall discuss the OSG structure for the Bianchi universes and show that, once again, the singularity is avoided in all these models. This furnishes the logical conclusion of Padmanabhan (1982a, c).

2. Friedmann universe with a scalar source

Consider a space-time described by a metric of the form (in the classical limit)

$$ds^2 = \langle \Omega^2(t) \rangle \left(dt^2 - \frac{dr^2}{1-r^2/a^2} - r^2(\theta^2 + \sin^2 \theta d\phi^2) \right) \quad (2.1)$$

with a massless scalar field $\phi(t)$ as the source. We shall impose the symmetries— isotropy and homogeneity—on the quantum level, treating the variables as functions of time alone. This assumption is made in all standard approaches (lacking a better description!). Classically the physics is determined by the variations in the action,

$$J = (16\pi G)^{-1} \int R \sqrt{-g} d^4x + \frac{1}{2} \int d^4x \sqrt{-g} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi. \quad (2.2)$$

In our case the action takes the form

$$J = -\frac{3v}{8\pi G} \int_{t_1}^{t_2} dt \left(\dot{\Omega}^2 - \frac{\Omega^2}{a^2} - \Omega^2 \dot{\eta}^2 \right) \quad (2.3)$$

where

$$\eta = \left(\frac{4}{3}\pi G \right)^{1/2} \phi \quad (2.4)$$

and

$$v = \int_0^R \frac{4\pi r^2 dr}{(1-r^2/a^2)^{1/2}} \quad (2.5)$$

is the proper volume of the region under consideration and may also be taken to be the total volume of the closed universe. Our main results are independent of the choice made for v . The variation of Ω and η in the action (2.3) will lead to the equations

$$d(\dot{\eta}\Omega^2)/dt = 0, \quad (2.6)$$

$$\ddot{\Omega} = -(\dot{\eta}^2 + 1/a^2)\Omega, \quad (2.7)$$

which leads to the classical solution

$$\Omega_{\text{CL}}^2(t) = (\alpha a^2/4)^{1/2} \sin[(2/a)(t+t_0)] \quad (2.8)$$

where α is a constant. The same solution can be found by solving the Einstein equations with a scalar source. See Padmanabhan (1982b). Clearly the classical evolution has initial and final singularities.

We shall approach the quantum theory by postulating that the probability amplitude for the universe to make a transition from a state with (Ω_1, η_1) at a time t to a state (Ω_2, η_2) at time t_2 is given by the path integral kernel,

$$K[\Omega_2 \eta_2 t_2; \Omega_1 \eta_1 t_1] = \iint \mathcal{D}\Omega(t) \mathcal{D}\eta(t) \exp\{(i/\hbar)J[\Omega, \eta]\}. \quad (2.9)$$

The state may be described by the wavefunction $\psi(\Omega, \eta, t)$ whose propagation is via the equation

$$\psi(\Omega_2 \eta_2; t_2) = \int d\Omega_1 d\eta_1 K(\Omega_2 \eta_2 t_2; \Omega_1 \eta_1 t_1) \psi(\Omega_1 \eta_1 t_1). \quad (2.10)$$

Some aspects of the kernel (when the quantum effects of the source are neglected) are treated in Padmanabhan (1982b). Here, however, we are interested in the stationary wavefunctions of the system. For this, it is better to consider the ‘Schrödinger equation’ corresponding to the action in (2.3) rather than the kernel. The Hamiltonian for the system is

$$\hat{H} = \left(\frac{\hbar^2}{2m} \frac{\partial^2}{\partial \Omega^2} - \frac{\hbar^2}{2m} \frac{1}{\Omega^2} \frac{\partial^2}{\partial \eta^2} - \frac{1}{2} \frac{m}{a^2} \Omega^2 \right), \quad m = \frac{3v}{4\pi G}. \quad (2.11)$$

The stationary states of the quantum geometry can be found from the equation

$$\hat{H}\psi = E\psi, \quad (2.12)$$

or

$$\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial \Omega^2} - \frac{\hbar^2}{2m} \frac{1}{\Omega^2} \frac{\partial^2 \psi}{\partial \eta^2} - \frac{1}{2} \frac{m}{a^2} \Omega^2 \psi = E\psi. \quad (2.13)$$

Any other state—and the kernel—can be found from suitable superpositions of the stationary states. (This approach to quantum gravity is somewhat unconventional; the details of the methodology and logistics are presented in detail in the references given previously and will not be repeated here.)

We are interested in the behaviour of ψ as a function of Ω . The η dependence of ψ can be separated out by

$$\psi_\alpha(\Omega, \eta) = e^{-\alpha\eta} g_\alpha(\Omega) \quad (2.14)$$

where α is a (positive) separation constant.

The $g_\alpha(\Omega)$ satisfies the equation

$$\Omega^2 \frac{d^2 g}{d\Omega^2} - (\alpha + \beta\Omega^4 + \varepsilon\Omega^2)g = 0, \quad \beta = \frac{m^2}{\hbar^2 a^2}, \quad \varepsilon = \frac{2mE}{\hbar^2}. \quad (2.15)$$

By making a series of substitutions,

$$x = \beta^{1/2} \Omega^2, \quad p = \frac{1}{16}(1 + 4\alpha^2), \quad a = \frac{1}{2} + p + \varepsilon/4\beta^{1/2}, \quad b = 1 + 2p, \quad (2.16)$$

$$g(x) = x^{(1/4+p)} e^{-x/2} y(x), \quad (2.17)$$

one can reduce (2.15) to Kummer’s equation,

$$x d^2 y/dx^2 + (b-x)dy/dx - ay = 0. \quad (2.18)$$

Thus the normalisable solution can be expressed in terms of the confluent hypergeometric function, ${}_1F_1(x)$, as

$$g(x) = x^{(1/4+p)} e^{-1/2x} {}_1F_1(x). \quad (2.19)$$

The function ${}_1F_1(x)$ is well studied, and near the origin has a series expansion (Slater 1960)

$${}_1F_1(x) = 1 + \frac{a}{b} x + \frac{a(a+1)}{b(b+1)} \frac{x^2}{2!} + \dots \quad (2.20)$$

From this (since ${}_1F_1(x)$ is finite at the origin for this normalisable solution) one sees that

$$g(x) \rightarrow 0 \quad \text{as } x \rightarrow 0. \quad (2.21)$$

Thus the singular state $\Omega^2 = 0$ has zero probability in all quantum stationary states. Since any other state can be built out of the stationary states, the singularity has zero probability to exist in the quantum domain.

Thus in this approach to quantum gravity the singularity is avoided by the existence of stationary states. One is reminded of the analogy with the hydrogen atom—the classical collapse of the electron is prevented by the quantum stationary states (Wheeler 1964).

Just as in any quantum system, the good behaviour of the wavefunction at infinity imposes some restrictions on the parameters. The study of Kummer's equation shows that (Slater 1960) if the wavefunction has to die out at infinity, one must satisfy

$$\frac{1}{2} + p + \varepsilon/4\beta^{1/2} = -n \quad (2.22)$$

where n is a positive integer or zero. (This replaces a considerably simpler condition imposed on the radiation source in Padmanabhan (1982a).) The physical meaning of (2.22) is under investigation.

We have considered the case of a closed model. But the conclusions remain valid, i.e. the stationary states avoid the singularity, for the open model as well. When the sign of a^2 changes, β changes sign in (2.15). Thus an exact solution involves Kummer's equation with imaginary argument. But from (2.15) one can see that, as far as behaviour near $\Omega^2 = 0$ is concerned, the $\beta\Omega^4$ term can be neglected. Thus the sign of the β term is immaterial.

We shall next study the stationary states for the Bianchi universes.

3. Stationary states of Bianchi models

The Bianchi cosmologies were the 'test areas' for quantum gravity right from the start. We shall consider the diagonal Bianchi models which can be written in the form

$$ds^2 = dt^2 - g_{ik}(t)\sigma^i\sigma^k \quad (3.1)$$

with

$$[\sigma_i, \sigma_k] = c_{ik}^j \sigma_j, \quad (3.2)$$

$$g_{ik} = e^{2\lambda(t)}(e^{-2\beta})_{ik}, \quad (3.3)$$

$$\beta_{ik} = \text{diag}[\beta_1, -\frac{1}{2}\beta_1 + \frac{1}{2}\sqrt{3}\beta_2, -\frac{1}{2}\beta_1 - \frac{1}{2}\sqrt{3}\beta_2]. \quad (3.4)$$

Here σ_i are suitably chosen one-forms and c_{jk}^i are the structure constants which determine the Bianchi type. The classical Einstein action can be reduced to the form

$$J = (16\pi G)^{-1} \int R\sqrt{-g} d^4x = (V/16\pi G) \int L dt \quad (3.5)$$

with

$$L = -e^{3\lambda}[6\dot{\lambda}^2 - \frac{3}{2}(\dot{\beta}_1^2 + \dot{\beta}_2^2)] + R^* e^{3\lambda}. \quad (3.6)$$

Here $R^* = e^{-2\lambda}f(\beta_1, \beta_2)$ depends on the Bianchi type chosen. (This can be derived by a direct but tedious computation; see MacCallum (1979); we have used correct signs in (3.6) consistent with our convention.)

Classically a homogeneous diagonal cosmology is described by the functions $\lambda(t)$, $\beta_1(t)$ and $\beta_2(t)$. Their evolutions are deterministic and in some cases one reaches a singularity characterised by ($e^{2\lambda} = 0$), at sometime $t = t_1$.

Quantum gravity described the space-time geometry through the wavefunction $\psi(\lambda, \beta_1, \beta_2, t)$. The propagation kernel for this can be represented by the path integral

$$K[\bar{\lambda}\bar{\beta}_2\bar{\beta}_1\bar{t}; \lambda\beta_2\beta_1t] = \int \mathcal{D}\lambda\mathcal{D}\beta_1\mathcal{D}\beta_2 \exp(iJ/\hbar). \quad (3.7)$$

However, it is easier to work with the stationary states (as in § 2). The Hamiltonian can be written down, once the form of R^* is known. This form is different for the class A and class B Bianchi models (MacCallum 1979), which we shall now treat separately.

3.1. Class A models

Here R^* has the form

$$R^* = e^{-2\lambda} \left(-\frac{2(1-3h)}{h} \right) A^2 \exp \frac{2}{(1-3h)^{1/2}} \beta, \quad (3.8)$$

$$\beta = (1-3h)^{-1/2} (-\beta_1 - \sqrt{-3h}\beta_2). \quad (3.9)$$

(see equation (11.32) of MacCallum (1979)). Here h is the Bianchi parameter and A is a constant. The Hamiltonian has the form

$$H = -\frac{\hbar^2}{4q^2} \left(\frac{\partial^2}{\partial u_1^2} + \frac{\partial^2}{\partial u_2^2} \right) + \frac{\hbar^2}{4} \frac{\partial^2}{\partial q^2} - c_1 q^{2/3} \exp(-c_2 u_1) \quad (3.10)$$

where we have made a series of transformations,

$$u_1 = \left(\frac{3}{8}\right)^{1/2} [3/2(1-3h)]^{1/2} (\beta_1 + \sqrt{-3h}\beta_2), \quad (3.11)$$

$$u_2 = \left(\frac{3}{8}\right)^{1/2} [3/2(1-3h)]^{1/2} (\sqrt{-3h}\beta_1 - \beta_2), \quad (3.12)$$

$$q = \left(\frac{8}{3}\right)^{1/2} e^{3/2\lambda}, \quad (3.13)$$

$$c_1 = \left(\frac{3}{8}\right)^{1/2} \frac{2A^2(3h-1)}{h}, \quad c_2 = \left(\frac{8}{3}\right)^{1/2} \left(\frac{8}{3(1-3h)}\right)^{1/2}. \quad (3.14)$$

The stationary states equation reads as

$$-\frac{\hbar^2}{4q^2} \left(\frac{\partial^2}{\partial u_1^2} + \frac{\partial^2}{\partial u_2^2} \right) \psi + \frac{\hbar^2}{4} \frac{\partial^2 \psi}{\partial q^2} - c_1 q^{2/3} \exp(-c_2 u_1) \psi = E\psi. \quad (3.15)$$

One can separate out the u_2 dependence by assuming $\psi(u_1 u_2 q) = \exp(iku_2)\Phi(u_1, q)$. Then $\Phi(u_1, q)$ satisfies the equation

$$\left(-\frac{\partial^2 \Phi}{\partial u_1^2} + q^2 \frac{\partial^2 \Phi}{\partial q^2} \right) - \frac{4c_1}{\hbar^2} q^{8/3} \exp(-c_2 u_1) \Phi = \left(\frac{4E}{\hbar^2} q^2 - k^2 \right) \Phi. \quad (3.16)$$

It is difficult to solve this equation exactly. But, we can use the fact that we are only interested in the solution near $q = 0$. More specifically we are interested in the well behaved, normalisable solution of this equation near $q = 0$, in order to find out whether $\psi(q = 0)$ is zero or otherwise. Near $q = 0$, the equation can be approximated to

$$(-\partial^2 \Phi / \partial u_1^2 + q^2 \partial^2 \Phi / \partial q^2) = -k^2 \Phi. \quad (3.17)$$

Now putting $\Phi(u_1, q) = \exp(ipu_1) f(q)$ we get the equation

$$q^2 \frac{d^2 f}{dq^2} = -(p^2 + k^2)f. \quad (3.18)$$

The normalisable solution (which is finite at the origin) behaves as

$$f(q) \sim q^n, \quad n > 0 \quad (\text{for } q \sim 0). \quad (3.19)$$

In other words,

$$\psi(q, u_1 u_2) \rightarrow 0 \quad \text{as } q \rightarrow 0. \quad (3.20)$$

We once again get the result that the stationary states avoid the singularity (zero probability for existence).

3.2. Class B models

Here R^* has a general form,

$$R^* = \frac{1}{2} e^{-2\lambda} f(\beta_1, \beta_2) \quad (3.21)$$

(see equation (11.31) of MacCallum (1979)). The particular form of $f(\beta_1, \beta_2)$ depends on the Bianchi type. The Hamiltonian has the form ($q = \exp(\frac{3}{2}\lambda)$)

$$\hat{H} = \frac{3\hbar^2}{32} \frac{\partial^2}{\partial q^2} - \frac{\hbar^2}{6q^2} \left(\frac{\partial^2}{\partial \beta_1^2} + \frac{\partial^2}{\partial \beta_2^2} \right) + \frac{1}{2} q^{2/3} f(\beta_1, \beta_2). \quad (3.22)$$

The stationary state equation again reads

$$\hat{H}\psi = E\psi. \quad (3.23)$$

We shall assume that ψ has a non-trivial dependence on β_1 or β_2 , i.e. we put

$$(\partial^2/\partial \beta_1^2 + \partial^2/\partial \beta_2^2)\psi = -\alpha^2 \psi \quad (3.24)$$

with $\alpha \neq 0$. Then near $q \sim 0$, the equation can be approximated by

$$\frac{3}{32} \hbar^2 q^2 \frac{\partial^2 \psi}{\partial q^2} + \frac{1}{6} \hbar^2 \alpha^2 \psi = 0. \quad (3.25)$$

Once again the solution that is normalisable (finite at origin) has the behaviour

$$\psi \sim q^n, \quad n > 0 \quad (\text{near } q = 0). \quad (3.26)$$

Thus once again, the wavefunction vanishes at the classical singularity. (Notice that since any state can be represented as a linear superposition of basis stationary states our result holds for all states.)

From these examples one can see that quantum gravity mimics the hydrogen atom nicely, as regards the stopping of collapse. Actually the analogy can be pushed further. The hydrogen atom wavefunctions go as r^l near the origin, where l is the angular momentum value. Thus all anisotropic states ($l \neq 0$) have zero probability at the origin while the isotropic case ($\alpha = 0$) has a constant value at the origin. We have seen above that for quantum stationary geometries which are anisotropic ($\alpha \neq 0$), the wavefunction vanishes at the origin. When $\alpha = 0$, we have the isotropic Friedmann universe. An analysis similar to the above will show that the probability density does not vanish at the origin—as is known from the previous analysis (Padmanabhan 1982a). Nevertheless, the collapse is halted just as in hydrogen atom s-states (Padmanabhan and Narlikar 1981, Padmanabhan 1982a).

Before concluding, we would like to comment about a detail. In a previous work (Padmanabhan 1982c) we established the stationary state structure for Bianchi cosmologies using a path integral in superspace. At that time, it was shown that for FRW models the superspace approach is the same as the ordinary space-time approach. A more detailed investigation will show that this equivalence exists for all Bianchi models—as it should. Thus our results here can also be derived from a manifestly coordinate independent superspace method, introduced in the previous paper. This gives the results a sense of credibility regarding general covariance.

5. Conclusion

Our approach to quantum gravity, though based on an unconventional use of path integrals, does lead to a consistent picture. Following a general result on the divergence of conformal fluctuations (Padmanabhan and Narlikar 1982), we have now demonstrated the ‘mechanism’ of avoidance of singularity in a fairly general context. We have examined non-trivial sources and relaxed the condition of isotropy. It seems very plausible that the stationary states of quantum geometry are an important physical concept. Earlier we had attempted an investigation of the singularity avoidance using the concept of ‘effective metric’ (see Padmanabhan 1981). However, this semiclassical approach could not lead to a consistent picture. Further investigations, including the one presented in this paper, show that a fully quantum gravitational concept of stationary states avoids the difficulties encountered before.

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