

## EQUIPARTITION OF MICROSCOPIC DEGREES OF FREEDOM, SPACE–TIME ENTROPY AND HOLOGRAPHY\*

T. PADMANABHAN

*Inter-University Center for Astronomy  
 and Astrophysics (IUCAA),  
 Post Bag 4, Ganeshkhind,  
 Pune 411 007, India  
 paddy@iucaa.ernet.in*

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One can identify the number density of the microscopic space–time degrees of freedom in any diffeomorphism-invariant theory of gravity by using the principle of equipartition, applied to the area elements of a surface  $\partial\mathcal{V}$  which are at the local Unruh temperature. The entropy associated with these degrees of freedom, which matches with the Wald entropy for the theory, can be used to obtain the field equations of the theory through an extremization principle. When the microscopic degrees of freedom are in local thermal equilibrium, the entropy of a bulk region of space–time resides on its boundary. These facts support an emergent perspective of gravity.

Consider a three-volume  $\mathcal{V}$  with a boundary  $\partial\mathcal{V}$  in a static space–time<sup>a</sup> with metric components  $g_{00} = -N^2$ ,  $g_{0\alpha} = 0$ ,  $g_{\alpha\beta} = h_{\alpha\beta}$ . An observer at rest in this space–time with four-velocity  $u^a = \delta_0^a/N$  will have an acceleration  $a^j = (0, a^\mu)$  where  $a_\mu = (\partial_\mu N/N)$ . In a static space–time, it is easy to show that

$$R_{ab}u^a u^b = \nabla_i a^i = \frac{1}{N} D_\mu (N a^\mu) \quad (1)$$

where  $D_\mu$  is the covariant derivative operator corresponding to the three-space metric  $h_{\alpha\beta}$ . Einstein’s field equations relate the divergence of the acceleration to the source:

$$D_\mu (N a^\mu) \equiv 8\pi N \bar{T}_{ab} u^a u^b \equiv 4\pi \rho_{\text{Komar}} \quad (2)$$

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<sup>a</sup>We use the signature  $(-, +, +, +)$ ; Greek letters go over the spatial coordinates while Latin letters go over the space–time coordinates. Except when otherwise indicated, we use units with  $\hbar = c = k_B = 1$  and  $G = L_P^2 = 1$ .

where  $\bar{T}_{ab} \equiv (T_{ab} - (1/2)g_{ab}T)$  and  $\rho_{\text{Komar}}$  is the Komar mass–energy density. Integrating both sides of Eq. (2) over  $\mathcal{V}$  and using the Gauss theorem gives:

$$E \equiv \int_{\mathcal{V}} d^3x \sqrt{h} \rho_{\text{Komar}} = \frac{1}{2} \int_{\partial\mathcal{V}} \frac{\sqrt{\sigma} d^2x}{L_P^2} \left\{ \frac{N a^\mu n_\mu}{2\pi} \right\} \tag{3}$$

where  $\sigma$  is the determinant of the induced metric on  $\partial\mathcal{V}$  and  $n_\mu$  is the spatial normal to  $\partial\mathcal{V}$ . (We have temporarily restored  $G = L_P^2 \neq 1$  keeping  $\hbar = c = k_B = 1$ .) As we will now describe,<sup>1,2</sup> *this result has a remarkable interpretation.*

To see this, choose  $\partial\mathcal{V}$  to be a  $N = \text{const.}$  surface so that the normal  $n_\mu$  is in the direction of the acceleration and  $a^\mu n_\mu = |\mathbf{a}|$  is the magnitude of the acceleration. We can then introduce an effective Davies–Unruh temperature<sup>3,4</sup>  $T = NT_{\text{loc}} = (N a^\mu n_\mu / 2\pi) = (N |\mathbf{a}| / 2\pi)$  in which the factor  $N$  takes care of the Tolman redshift condition on the temperature. (The reason for choosing the  $N = \text{const.}$  surface for  $\partial\mathcal{V}$  is to ensure a constant redshift factor between the temperatures attributed to different area elements.) Further, we can also attribute  $\Delta n = \sqrt{\sigma} d^2x / L_P^2 = \Delta A / L_P^2$  microscopic degrees of freedom to an element of area  $\Delta A = \sqrt{\sigma} d^2x$  on  $\partial\mathcal{V}$ . If these microscopic degrees of freedom are in equilibrium at the temperature  $T$ , then the equipartition energy contributed by these degrees of freedom is given by  $\Delta\mathcal{E} = (1/2)k_B T \Delta n$  and the total energy contributed by all the degrees of freedom on  $\partial\mathcal{V}$  is given by the integral

$$\mathcal{E} \equiv \frac{1}{2} k_B \int_{\partial\mathcal{V}} dn T = E \tag{4}$$

where we have used Eq. (3). *The equipartition energy attributed to the surface degrees of freedom keeps count of the total energy contained in the bulk volume enclosed by the surface* which could be thought of as a realization of the holographic principle. Alternatively, if we assume that the law of equipartition holds on  $\partial\mathcal{V}$ , then we can read off the density of surface degrees of freedom  $\Delta n = 2\Delta E / T$  from the known expression for  $T$ . (One can, of course, rescale  $\Delta n \rightarrow \Delta n / f$  replacing  $1/2$  in Eq. (4) by  $f/2$  with some numerical factor  $f$ . We will stick with  $f = 1$  for simplicity; our results do not depend on this choice.)

Why should such an equipartition law for bits of area elements be embedded within the field equations of gravity?

To understand its origin, let us go back to the study of the thermodynamics of, say, an ordinary gas. When a gaseous body interacts with another mechanical system, it can exchange energy with the latter (e.g. by moving a piston in a heat engine). To describe these processes, one needs to introduce a different category of variables like temperature, heat content, entropy *etc.* The nature of these variables, as well as the ability of macroscopic systems to store and supply energy, was a mystery in the early days of thermodynamics when the real nature of heat was not understood. It was Boltzmann who demystified these processes by *inferring* the existence of microscopic degrees of freedom in, e.g. a gas and relating the *thermodynamical* variables to the *mechanical* variables associated with the microscopic

degrees of freedom. The key to Boltzmann’s description is the fact that *a gas can be heated* — which would have been impossible if it did not possess microscopic degrees of freedom. In such a context, the equipartition of energy amongst the microscopic degrees of freedom arises quite naturally.

But we now know that the *space-times can also be heated!* An observer at rest around a spherical body will feel the space-time to be hot if the body collapses to form a black hole. Even *flat* space-time will appear to be hot to an accelerating observer. It follows from the Boltzmann dictum (“If you can heat it, it has microstructure”) that the space-time must possess microscopic degrees of freedom which will come into equilibrium in any static space-time at the temperature perceived by a class of observers. The resulting equipartition law should be related to the dynamics of the space-time — which is precisely what we saw above.

Remarkably enough, such an equipartition law exists for *any* diffeomorphism invariant theory of gravity and allows us to identify the corresponding surface density  $\Delta n/\sqrt{\sigma}d^2x$  of microscopic states. Consider a theory based on the Lagrangian  $L(R_{cd}^{ab}, g^{ab}) + L_{\text{matt}}$  in  $D$  dimensions. This Lagrangian leads to the field equations (see e.g. Sec. 3.5 of Ref. 5):

$$G_{ab} = P_a^{cde} R_{bcde} - 2\nabla^c \nabla^d P_{acdb} - \frac{1}{2} L g_{ab} \equiv \mathcal{R}_{ab} - \frac{1}{2} L g_{ab} = \frac{1}{2} T_{ab} \tag{5}$$

where  $P^{abcd} \equiv (\partial L/\partial R_{abcd})$ . Any such theory of gravity, which is invariant under the diffeomorphism  $x^a \rightarrow x^a + q^a$ , has a conserved Noether current  $J^a$  which depends on the vector field  $q^a$ . In static space-times, if we take  $q^a = \xi^a$ , the Killing vector corresponding to time-translation invariance, the expression for the Noether current is remarkably simple:  $J^a = 2\mathcal{R}_b^a \xi^b$ . Introducing the anti-symmetric Noether potential  $J^{ab}$  by  $J^a \equiv \nabla_b J^{ab}$  one can easily show that, in any static space-time,  $2\mathcal{R}_b^a u_a u^b = \nabla_a (J^{ba} u_b N^{-1})$  or, equivalently  $2N\mathcal{R}_b^a u_a u^b = D_\alpha (J^{b\alpha} u_b)$  [which is the generalization of Eq. (1)]. The source for gravity in a general theory (analogous to Komar mass density) is defined through  $\rho \equiv 4N\mathcal{R}_{ab} u^a u^b$ . On integrating  $\rho$  over a region bounded by an  $N = \text{const.}$  surface and using

$$\int_{\mathcal{V}} 2N\mathcal{R}_{ab} u^a u^b \sqrt{h} d^{D-1}x = - \int_{\partial\mathcal{V}} d^{D-2}x \sqrt{\sigma} (N n_\alpha J^{0\alpha}) \tag{6}$$

we get

$$E \equiv \int_{\mathcal{V}} \sqrt{h} d^{D-1}x \rho = -2 \int_{\partial\mathcal{V}} d^{D-2}x \sqrt{\sigma} (N n_\alpha J^{0\alpha}). \tag{7}$$

This is the analog of the equipartition law in Eq. (3). [In Einsteinian gravity,  $\mathcal{R}_{ab} = R_{ab}/16\pi$  and  $J_{ab} = (16\pi)^{-1} \partial_{[a} \xi_{b]}$  giving  $J^{0\mu} = -a^\mu/8\pi$  which will reduce Eq. (7) to Eq. (3).] In a general theory, the expression for  $\Delta n$  is not just proportional to area and  $J^{0\alpha}$  encodes this difference. This expression simplifies to an interesting form in two contexts.

Note that the field equations arising from the expression in Eq. (5) will contain higher-than-second-order derivatives of the metric. This can be avoided by

restricting to the class of theories for which  $\nabla_a P^{abcd} = 0$ , which are essentially the Lanczos–Lovelock models.<sup>6–8</sup> In that case,  $J^{ab} = 2P^{abcd}\nabla_c \xi_d$  and we get  $J^{0\alpha} = -4|\mathbf{a}|P_{0\beta}^{0\alpha}n^\beta$  giving

$$E = \int_{\partial\mathcal{V}} d^{D-2}x \sqrt{\sigma} (16\pi P_{0\beta}^{0\alpha} n^\beta n_\alpha) \left( \frac{N|\mathbf{a}|}{2\pi} \right) \equiv \frac{1}{2} \int dnT \tag{8}$$

where  $T = N|\mathbf{a}|/2\pi$  as before but the number of microscopic degrees of freedom  $\Delta n$  associated with an area element  $\sqrt{\sigma}d^{D-2}x$  is

$$\Delta n = 32\pi P_{0\beta}^{0\alpha} n^\beta n_\alpha \sqrt{\sigma} d^{D-2}x = 32\pi P_{cd}^{ab} \epsilon_{ab} \epsilon^{cd} \sqrt{\sigma} d^{D-2}x \tag{9}$$

where  $\epsilon_{ab} \equiv (1/2)(u_a n_b - u_b n_a)$  is the binormal to  $\partial\mathcal{V}$ .

The second context in which the above analysis remains valid is when one deals with a general theory but evaluates the surface integral on the bifurcation horizon. It can be shown that in this case, one can again use  $J^{ab} = 2P^{abcd}\nabla_c \xi_d$  because an additional term in  $J^{ab}$  vanishes on the horizon (see Eq. (86) of Ref. 5). We again get the same equipartition law in Eq. (8) on the horizon of any diffeomorphism-invariant theory of gravity.

In either context, Eq. (8) shows that the surface density of microscopic degrees of freedom is given by

$$\frac{dn}{\sqrt{\sigma}d^{D-2}x} = 32\pi P_{cd}^{ab} \epsilon_{ab} \epsilon^{cd}. \tag{10}$$

This suggests that the entropy associated with a surface will be given by an integral over  $P_{cd}^{ab} \epsilon_{ab} \epsilon^{cd}$ . This is precisely the expression for Wald entropy<sup>9,10</sup> *but we have obtained it using only the equipartition law!*

In fact, one can do better. For a macroscopic system like a gas, one can obtain the dynamical equations from maximizing an entropy functional  $S[q^A]$  expressed in terms of appropriate variables  $q^A$ . Analogously, one should be able to derive the field equations of gravity from maximizing a suitable entropy functional of space–time.<sup>11,12</sup> There is, however, one crucial difference between the thermodynamics of gases and the thermodynamics of space–time. We know that the same space–time can exhibit different thermal behavior to different observers and hence one would expect the entropy functional *etc.* to take different forms in different contexts. We will need to apply the maximization principle to a *class* of observers to obtain the dynamical equations. We will now describe how one can obtain a suitable form of entropy functional in different contexts.

To motivate this, consider a static space–time with a bifurcation horizon  $\mathcal{H}$  described by the surface  $N^2 \equiv -\xi^a \xi_a = 0$ . The horizon temperature  $T \equiv \beta^{-1} = \kappa/2\pi$  where  $\kappa$  is the surface gravity. We attribute an entropy density  $\beta J_b u^b$  to the space–time as perceived by the static observers with the four-velocity  $u^a = \xi^a/N$ ,

so that the total entropy is

$$S_{\text{grav}}[u^i] = \beta \int_{\mathcal{V}} J_b u^b \sqrt{h} d^{D-1}x. \tag{11}$$

Using  $J^a = 2\mathcal{R}_b^a \xi^b$  and Eq. (6) and integrating the expression over a region bounded by the  $N = \text{const.}$  surface, it is easy to see that  $S = (1/2)\beta E$  which is a statement of equipartition.<sup>1,2,13</sup> Further, if we take  $\partial\mathcal{V}$  to be the horizon  $\mathcal{H}$  and use  $\beta T = 1$ , we get the horizon entropy to be

$$S = \frac{1}{4} \int_{\mathcal{H}} dn = \frac{1}{4} \int_{\mathcal{H}} 32\pi P_{cd}^{ab} \epsilon_{ab} \epsilon^{cd} \sqrt{\sigma} d^{D-2}x \tag{12}$$

which is the standard expression for the Wald entropy<sup>9</sup> in a general theory, thereby justifying the choice in Eq. (11).

If the thermodynamic interpretation of gravity is correct, one should be able to obtain the field equations of gravity from extremizing the space–time entropy. To do this, we have to recast the expression for  $S$  in Eq. (11) as a *space–time* integral and generalize to a context in which the space–time has no special attributes. The first task is easy. The space–time entropy in Eq. (11) can also be expressed in the form

$$S_{\text{grav}}[u^i] = \beta \int_{\mathcal{V}} J_b u^b \sqrt{h} d^{D-1}x = \int 2\mathcal{R}_{ab} u^a u^b \sqrt{-g} d^Dx \tag{13}$$

by restricting the time integration to the range  $(0, \beta)$  which can be justified by using the Euclidean continuation of the space–time in which the time-coordinate is periodic with period  $\beta$ . But since Eq. (13) does not involve any  $\beta$ , we are no longer confined to a space–time with a horizon. Further, in a general nonstatic space–time, we do not have any special class of observers who can be used to define the vector field  $u^a$ . Instead, one has to attribute the entropy to the displacements of null surfaces and define the entropy density in terms of the set of all null vectors  $k^a$ . (A more detailed justification for this approach can be given in terms of local Rindler observers; see e.g. Ref. 5.) This suggests that the expression for gravitational entropy of space–time can be taken, in the general context, to be

$$S_{\text{grav}}[k^i] \propto \int d^Dx \mathcal{R}_{ab} k^a k^b. \tag{14}$$

It can be shown<sup>5,11,12</sup> that maximizing  $(S_{\text{grav}} + S_{\text{matter}})$  for all null vectors  $k^a$  simultaneously leads to the field equations in Eq. (5), that is, the field equations in any diffeomorphism-invariant theory can be obtained from an entropy maximization principle.

The expression for  $S_{\text{grav}}$  in a general theory can be expressed in an alternate form by separating out a total divergence. Direct computation shows that

$$2\mathcal{R}_{ab} k^a k^b = 4\nabla_c [P_{ab}^{cd} k^a \nabla_d k^b] + \mathcal{S}_{\text{grav}}[k^i] \tag{15}$$

where

$$\mathcal{S}_{\text{grav}}[k^i] = 4[P_{ab}^{cd}\nabla_c k^a\nabla_d k^b + (k^b\nabla_c k^a)\nabla_d P_{ab}^{cd} + k_a k_c\nabla_b\nabla_d P^{abcd}] \quad (16)$$

is a quadratic expression in  $k^a$  and its derivatives. One can obtain<sup>5,14</sup> the same field equations in Eq. (5) by using  $\mathcal{S}_{\text{grav}}$  as the entropy density instead of  $2\mathcal{R}_{ab}k^ak^b$ . (In the case of Lanczos–Lovelock models, the expression in Eq. (16) simplifies considerably and we find that the gravitational entropy density of space–time is a quadratic expression  $\mathcal{S}_{\text{grav}} \propto P_{ab}^{cd}\nabla_c k^a\nabla_d k^b$  which was investigated earlier in Refs. 11 and 12.) When the field equations hold, the entropy  $\mathcal{S}_{\text{grav}}[k^i]$  of a volume  $\mathcal{V}$  will reside on its surface  $\partial\mathcal{V}$ . Further, in any static space–time  $\mathcal{S}_{\text{grav}}[u^i]$  (defined using the four-velocity of static observers) will be a total divergence because  $\mathcal{R}_{ab}u^au^b$  is also a total divergence in this case. This shows that the entropy perceived by static observers also resides on the surface. Both these results, again, generalizes the notion of holography beyond Einstein’s theory.

We see that the equipartition law allows one to identify the number density of microscopic degrees of freedom on a constant redshift surface, using which one can define the entropy of the horizon in a general theory of gravity (which agrees with the Wald entropy). This definition of entropy, recast in terms of null vector fields, allows us to obtain the field equation by an entropy extremization principle. These facts further strengthen the idea that gravity is an emergent phenomenon and space–time thermodynamics is more fundamental.

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