

# ON THE EQUATION FOR STABILITY OF A STAR AGAINST RADIAL OSCILLATIONS

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## ABSTRACT

With regard to the equations governing stability of a star against the radial oscillations it is shown mathematically that there exists a space invariant  $K$  such that  $(dK/dr) = 0$ . It is argued that in addition to the normal boundary conditions used at the star's origin  $r = 0$  and at its surface  $r = R$ , the invariance of  $K$  for all  $r$  will offer a sufficiency condition in the computation of results.

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## 1. Introduction

With a view to obtaining the eigenfrequencies and eigenfunctions of radial modes of a massive star, one carries out the analysis [1,2] on the basis of Einstein's field equations for a metric well known for its form, namely

$$ds^2 = e^{2\nu(r)} dt^2 - e^{2\lambda(r)} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (1)$$

The adiabatic motion of the star in its  $n$ -th normal mode ( $n = 0$  is the fundamental mode) is expressed in terms of the amplitude  $u_n(r)$  and described by the small perturbation in  $r$  as

$$\delta r(r, t) = e^{\nu(r)} u_n(r) e^{i\omega_n t} / r^2. \quad (2)$$

Here  $\omega_n$  is the star's oscillation frequency which one normally computes, and the function  $u_n(r)$  governs the normal modes. The latter satisfies the Sturm - Liouville's form

$$\frac{d(P(r) \frac{du_n(r)}{dr})}{dr} + [Q(r) + \omega_n^2 W(r)] u_n(r) = 0, \quad (3)$$

or

$$P(r) \frac{d^2 u_n(r)}{dr^2} + \frac{dP}{dr} \frac{du_n}{dr} + [Q(r) + \omega_n^2 W(r)] u_n(r) = 0. \quad (4)$$

Further, the functions  $P(r)$ ,  $Q(r)$  and  $W(r)$  are expressed in terms of the equilibrium configuration of the star by the following expressions

$$P(r) = e^{(\lambda+3\nu)} \Gamma p / r^2, \quad (5)$$

$$Q(r) = -4e^{(\lambda+3\nu)} \frac{1}{r^3} \frac{dp}{dr} - \frac{8\pi}{r^2} e^{(3(\lambda+\nu))} p(\epsilon + p) + e^{(\lambda+3\nu)} \frac{1}{r^2(\epsilon + p)} \left(\frac{dp}{dr}\right)^2, \quad (6)$$

$$W(r) = e^{(3\lambda+\nu)} (\epsilon + p) / r^2. \quad (7)$$

The quantities  $\epsilon$  and  $p$  in eqs.(5)-(7), respectively are the energy density ( or total mass energy) and the pressure of the stellar equilibrium configuration as measured by a local observer. The pressure gradient ( $dp/dr$ ) is obtained from the Tolman-Oppenheimer-Volkoff (TOV) equations [3],

$$\frac{dm}{dr} = 4\pi r^2 \rho, \quad (8)$$

$$\frac{dp}{dr} = -\frac{\rho m}{r^2} \left(1 + \frac{p}{r}\right) \left(1 + \frac{4\pi p r^3}{m}\right) \left(1 - \frac{2m}{r}\right)^{-1}, \quad (9)$$

where  $\rho = \rho(r)$  and other symbols have their usual meanings. The symbol  $\Gamma$  denotes the varying adiabatic index at constant entropy, given by

$$\Gamma = \left(\frac{\epsilon + p}{p}\right) \frac{dp}{dr}. \quad (10)$$

The boundary conditions satisfied by  $u_n(r)$  in eq.(3) are [4]<sup>2</sup>

$$u_n(r) \sim r^3, \quad (11)$$

at star's origin  $r = 0$ , and

$$(du_n(r)/dr) = 0, \quad (12)$$

at star's surface  $r = R$ . In fact, the solution of (4) subject to the boundary conditions (11) and (12) leads to the frequency spectrum  $\omega_n^2$  ( $n = 0, 1, 2, \dots$ ) of normal radial modes for a given stellar model. As a result, the eigenfrequencies  $\omega_n^2$  form an infinite discrete sequence

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<sup>2</sup>In the literature (see, for example, Ref.(1) p.138) other limiting forms of  $u_n(r)$  at  $r = 0$ , say  $u_n(0) = 0$ , are also used, but we choose the one used recently by Kettner et al, [4] for the charm star.

, i.e.  $\omega_n^2 < \omega_{n+1}^2$ . Such a prescription when used say for a charm star has shown [4] that these equilibrium configurations possess a characteristic mode of vibration i.e. of zero frequency ( $\omega_n^2 = 0$ ) when and only when the star's mass attains an extremum value.

Recently, by converting eq.(9) into a Riccati form in  $p(r)$ , and thereafter using the Riccati transformation a Schrodinger-like (or in general Sturm-Liouville type) equation is obtained [5]. As a result the existence of a space invariant in the TOV theory is shown in rigorous mathematical terms. The existence of this invariant in the TOV theory while suggests a consistency check of the equation of state appearing in the underlying model, however, offers a deep insight into the computational aspect of eqs.(8) and (9). Here, in the same vein (since the Schrodinger-like eq.(3) is appearing at the starting point itself ), we wish to exploit eq.(4) with a view to obtaining an amplitude ( $u_n(r)$ )-pressure ( $p(r)$ ) (or for that matter an amplitude-frequency ( $\omega_n$ )) connection vis-a-vis the spatial invariance of their appropriate combination during the process of computation of (4). It turns out that the space invariant derived in this case is not only independent of the frequencies as such but also of a function  $q_n^2(r)$ - a measure of frequencies and a combination of the functions P, Q, W of eq.(4), viz.,

$$q_n^2(r) = [P'^2 + 2P(2Q - P'' + 2\omega_n^2 W)]/4P^2. \quad (13)$$

In other words, the derived space invariant fixes the geometry in terms of the amplitude and pressure for the computation of (4) for any given  $r$  and thereby offers a consistency condition ( in addition to the boundary conditions (11) and (12)) for this purpose for all the prescribed values of  $r$  i.e. for  $0 < r < R$ .

## 2. Derivation of the Space Invariant

To derive the space invariant , we first eliminate the  $(du_n(r)/dr)$ -term in (4) by using the transformation,

$$u_n(r) = Av_n(r)/(P(r))^{1/2} \quad (14)$$

where  $A$  is an arbitrary constant. This leads to a Schrodinger- like equation for  $v_n(r)$ , viz.,

$$v_n''(r) + q_n^2(r)v_n(r) = 0, \quad (15)$$

where  $q_n^2(r)$  is given by eq.(13) . Now, following the method of Ref.(5), we make an ansatz for the solution of (15) as

$$v_n(r) = Nw_n(r)\sin(\phi(r)), \quad (16)$$

where  $\phi(r)$  is the phase function to be determined later and  $w_n(r)$  is a real function. Using (16) in (15) and subsequently equating the coefficients of (mutually orthogonal) sine and cosine functions in the resultant expression to zero, one obtains

$$w_n'' + [q_n^2(r) - \phi'^2]w_n = 0, \quad (17)$$

$$\phi''w_n + 2\phi'w_n' = 0. \quad (18)$$

Eq. (18) immediately provides the phase function  $\phi(r)$  as

$$\phi(r) = k \int w_n^{-2}(r)dr + \delta, \quad (19)$$

where  $k$  and  $\delta$  are arbitrary constants, and eq.(17) yields a nonlinear equation,

$$w_n''(r) + q_n^2(r)w_n(r) = k^2/w_n^3(r), \quad (20)$$

which is known as Milne's equation in the literature. Further, the elimination of  $q_n^2(r)$  from eqs. (15) and (20) ( as before in Ref.(5)) immediately yields the invariant,

$$K = k^2(v_n/w_n)^2 + (w_n v_n' - w_n' v_n)^2, \quad (21)$$

with respect to the space variable  $r$ . Note that the structure (21) can also be derived easily if one makes the ansatz

$$v_n(r) = N_1 w_n(r) \exp(\phi(r)), \quad (22)$$

in stead of (16) for the solution of (15). In this case , however, one would equate the real and imaginary parts separately to zero in the resultant expression obtained by substituting (22) in (15). Thus, the form (21) is the desired space invariant satisfying  $(dK/dr) = 0$  and valid for all  $r$  and for all  $n$ .

### 3. Discussion and Concluding Remarks

It may be mentioned that an invariant like (21) while appears [6] to be the characteristic feature of a second-order ordinary differential equation in general and of Sturm-Liouville form in particular with respect to the independent variable, the same has been studied in a variety of contexts[5-13]. No doubt such an invariant ( however with respect to time) has been the subject of study [7,8] for the last three decades or so in the context of time-dependent classical harmonic oscillator, an invariant similar to (21) was speculated [9] in solving the Dirac equation in one dimension. Recently, the analogies in the quantum domain have been carried[10] through and a common basis in the Riccati equation is sought [11] for both Schrodinger

quantum mechanics and supersymmetric quantum mechanics within this framework . The invariant like (21) while has suggested [5] a check on the use of an equation of state in studying the gravitational equilibrium of a star through the TOV eqs.(8) and (9) , here it is going to offer an additional consistency condition towards the computation of eq.(4). In other words, the existence of  $K$  in the present context will provide [14] a predictor-corrector kind of test of the numerical results in the process of computation of (4); since  $K$  in (21) as a whole is a constant for all  $r$  ,but it depends on the functions of  $r$ . For this purpose, conditions (11) and (12) while are necessary, (21) will be a kind of sufficiency and consistency check on the results for any value of  $r$ .

Note that the invariance of  $K$  in (21) remains a mathematical but important requirement and as such no physical meaning can be assigned for it except for a few available already in the literature ( see, for example, Refs. [5] and [10-13]. In spite of the several manifestations of  $K$  mentioned above, the role of the space invariant of Ref.[5] and that given by (21) here is more or less the same (in the sense that both translate the nature of the solution of the second- order differential equation into the connection between various physical quantities and set the geometrical constraint thereof). In Ref.[5] it happens to be the pressure- density connection (equation of state ) whereas here it is the pressure- amplitude connection for normal modes of radial oscillations of a star ( cf. eqs. (5) and (14)).

The invariant  $K$  as such is not sensitive to the  $r$ -dependence of  $u_n(r)$ . For the trivial solution  $u_n(r) = 0$  of eq.(4) ( which ,in fact, accommodates the boundary condition  $u_n(0) = 0$ , of Ref.[1]), the space invariant  $K$  vanishes ; otherwise for any  $r$  ( not equal to zero)  $K$  is nonzero finite and depends on the constant  $k$  of eq.(19). Further,  $K$  also becomes zero for the complex version of the ansatz (16) i.e. for

eq.(22). While the last fact can be attributed to the occurrence of real functions in the problem to some extent,  $K$  can easily be computed at the origin and at the surface of the star for the forms (11) and (12) of  $u_n(r)$ . Interestingly, it turns out that  $K$  attains a constant value  $N^2 k^2$  not only for these values of  $r$ , namely for  $r = 0$  and  $r = R$ , but also for all  $r$ .

To summarize, we mention that while studying the stability of a star against radial oscillations one should take cognizance of the space invariant appearing in the theory. At the same time its role in the computation of normal modes of a star can not be ignored. This invariant is expected to play the role of a geometrical constraint or a consistency check for all  $r$  in addition to the two boundary conditions, known already in the literature.

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